Skolkovo Institute of Science and Technology

BETHE VECTORS AND THEIR SCALAR PRODUCTSIN QUANTUM INTEGRABLE MODELS

Doctoral Thesis

by

ANDRII LIASHYK

CENTER FOR ADVANCED STUDIES

Supervisor
Professor Anton Zabrodin

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Abstract

Quantum integrable models are a special class of physical models. These models describe non trivial systems of interacting particles and at the same time they can be studied accuracy using mathematical tools. They offer us a unique training ground for a deep study of non-trivial physical phenomena explicitly.

A wide class of quantum integrable models is associated with higher rank algebras. Integrable models with symmetries of high rank appear in condensed matter physics, in particular in the gl(m|n)-invariant XXX Heisenberg spin chain, in multi-component Bose/Fermi gas [37], and in the study of models of cold atoms (the Hubbard model [33], the t-J model [34–36]). Also spin chains of higher rank are interesting in the context of computing correlation functions in N= 4 supersymmetric Yang-Mills theory [8, 9].

The role of the scalar product of Bethe vectors is extremely important in the study of correlation functions of local operators of the underlying quantum models [4, 13, 61]. One can reduce the problem of calculation of the form factors and the correlation functions of local operators to the calculation of the scalar products of the Bethe vectors [15, 16].

The study of integrable systems with high rank symmetry is still a challenging task. Until recently, such models have either not been studied at all, or have been studied under various simplifying hypotheses. The results presented in the thesis are the first in this direction.

List of publications

1. A. Liashyk, S. Pakuliak, E. Ragoucy, N. Slavnov, *New symmetries of $gl(N)$–invariant Bethe vectors*, J. Stat. Mech.: Theory Exp. 2019 (4), 044001
2. A. Hutsalyuk, A. Liashyk, S. Pakuliak, E. Ragoucy, N. Slavnov, *Scalar products and norm of Bethe vectors for integrable models based on $U_q(\hat{gl}_m)$*, SciPost Phys. 4, 006 (2018)
3. A. Hutsalyuk, A. Liashyk, S. Pakuliak, E. Ragoucy, N. Slavnov, *Norm of Bethe vectors in $gl(m|n)$ based models*, Nucl. Phys. B926 (2018) 256-278
4. A. Hutsalyuk, A. Liashyk, S. Pakuliak, E. Ragoucy, N. Slavnov, *Scalar products of Bethe vectors in the models with $gl(m|n)$ symmetry*, Nucl. Phys. B, 923 (2017) 277-311
5. A. Hutsalyuk, A. Liashyk, S. Pakuliak, E. Ragoucy, N. Slavnov, *Current presentation for the super-Yangian double $DY(gl(m|n))$ and Bethe vectors*, Russian Mathematical Surveys 72 (1), 33, 2017

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Chapter 1

Introduction

My thesis presents the results of five articles in which I am one of the co-authors. The articles are devoted to the study of Bethe vectors and their scalar products in quantum integrable models with high rank symmetry. This research is the development of mathematical apparatus of the study of correlation functions of these systems. In fact, this thesis is completely devoted to the description of Bethe vectors and to the study of their properties.

In this Chapter I give an overview of the results of my thesis. To simplify explanation in this Chapter we consider Yangian $Y(\mathfrak{gl}_N)$ case [1–4] instead of super-Yangian $Y(\mathfrak{gl}_{n|m})$ and quantum affine algebra $U_q(\hat{\mathfrak{gl}}_N)$ cases [2–5] considered in the rest Chapters.

1.1 Quantum R-matrix structure

Quantum integrability was discovered in 1931 by Hans Bethe [6] for Heisenberg spin chain (1.11). He discovered an exact solution to the spectral problem

$$H|\psi_j\rangle = E_j|\psi_j\rangle \tag{1.1}$$

considering eigenstate $|\psi_j\rangle$ as a clever linear superposition of plane waves. We call a system *integrable* if its spectral problem can be solved exactly. This method now known as Coordinate Ansatz Bethe. It continues to be relevant to a multitude of widely differing problems.

One of the continuation of this method is the Algebraic Bethe Ansatz which is the basis of this work. The most fundamental structure of the algebraic Bethe ansatz is R-matrix. Depending on point of view one can perceive it a scattering matrix of some $2 \to 2$ scattering process [7–9] or as a set of structure functions of bilinear algebra which depends on spectral parameter [1–3]. This algebra is called RTT-algebra. Elements of the algebra can be encoded in $N \times N$ matrix $T(u)$ which is called monodromy matrix. Usually relations in the RTT-algebra (this algebra was introduced in [13]) are formulated as an equation in the tensor product of two finite-dimensional spaces $V_1 \otimes V_2$:

$$R_{12}(u,v)\, T_1(u)T_2(v) = T_2(v)T_1(u)\, R_{12}(u,v). \tag{1.2}$$

Here subscripts mean tensor multiplier in which the operator acts. Here $T_1(u) = T(u) \otimes 1$ and $T_2(u) = 1 \otimes T(u)$, their elements act in some space $\mathcal{H}$ called physical space. The arguments u, v of the monodromy matrix are called spectral parameters. The spectral parameter is a complex number. The R-matrix acts in both spaces. In this Chapter we use R-matrix associated with $Y(\mathfrak{gl}_N)$ [24]

$$R_{12}(u) = u\,\mathbf{1} + c\,P_{12}, \tag{1.3}$$

where **1** is the unity operator, P_{12} is permutation operator, and parameter c is a complex number. *Yangian* is RTT-algebra (1.2) with rational R-matrix (1.3) (such representation of quantum algebras was obtained in [3, 4]).

Let us multiply (1.2) by the inverse matrix to $R_{12}(u,v)$ and take the trace over space $V_1 \otimes V_2$. Using the property of the trace one can obtain commutativity relation

$$[t(u), t(v)] = 0, \tag{1.4}$$

for the transfer matrix

$$t(u) = \sum_i T_{ii}(u). \tag{1.5}$$

Due to equation (1.4) the coefficients in a series expansion at some point u_0 of the transfer matrix $t(u) = \sum_k (u-u_0)^k H_k$ commute

$$[H_n, H_m] = 0. \tag{1.6}$$

These coefficients are called Hamiltonians. One can say that the transfer matrix is a generating function of the commuting Hamiltonians of some integrable system.

Thus, the presence of R-matrix structure implies the presence of a large number of conservation laws in the system and indicates the integrability of this system.

To use algebraic Bethe ansatz approach, besides quantum R-matrix structure one needs an existence of a special vector $|0\rangle \in \mathcal{H}$ called vacuum. This vector must have several properties

$$\begin{aligned} &T_{ji}(u)|0\rangle = 0, \qquad \text{with } \ i<j \\ &T_{ii}(u)|0\rangle = \lambda_i(u)|0\rangle, \end{aligned} \tag{1.7}$$

where $\lambda_i(u)$ are some functions depending on the concrete quantum integrable model. The action of $T_{ij}(u)$ with $i<j$ onto vacuum $|0\rangle$ is nontrivial. In the quantum integrable models the multiple action of upper-triangular elements of monodromy matrix onto $|0\rangle$ generates a basis in the physical space $\mathcal{H}$.

Generalized model. In the framework of **Chapter 1** we assume that $\lambda_i(u)$ are free functional parameters and we do not specify any of their concrete dependencies [13, 19, 61]. It means that one can find concrete quantum integrable model for any specific choice of $\lambda_i(u)$.

1.2 Spin chain as basic example

In the past, the structure of the R-matrix was discovered in a large number of quantum systems [33–37]. Usually it is a very non-trivial problem to find

R-matrix structure. One of the simplest examples is a spin chain. One can construct quantum integrable system inductively using general properties of R-matrix.

To construct a spin chain we use the rational $Y(\mathfrak{gl}_N)$ R-matrix (1.3). In this case the monodromy matrix of the spin chain is

$$T_0(u) = R_{01}(u-\xi_1)R_{02}(u-\xi_2)\dots R_{0n}(u-\xi_n). \tag{1.8}$$

Here R_{0i}-matrix acts non-trivially in the space $V_0 \otimes V_i$, and as unity in the rest spaces V_j (with $j \neq i$). The monodromy matrix acts in the space $V_0 \otimes V_1 \otimes V_2 \otimes \dots \otimes V_n$. This space is divided into two parts: physical space $\mathcal{H} = V_1 \otimes V_2 \otimes \dots \otimes V_n$ and auxiliary space V_0. We consider the monodromy matrix as matrix acting in the N-dimensional auxiliary space with noncommutative elements acting in the physical space $\mathcal{H}$. The parameters ξ_i are called inhomogeneities. The monodromy matrix satisfies the RTT-relation (1.2).

The model described by monodromy matrix (1.8) is called the inhomogeneous $\mathfrak{gl}_N$ XXX spin chain. It is the most typical example of quantum integrable model with quantum R-matrix structure. It exists for any R-matrix.

One can set all parameters $\xi_i = 0$. Then, the model becomes homogeneous spin chain. To describe the quantum integrable system obtained from this monodromy matrix let us consider one very special Hamiltonian in the expansion of transfer matrix of homogeneous spin chain

$$H = (t(0))^{-1}t'(0). \tag{1.9}$$

From (1.9) one can obtain that Hamiltonian is a sum of permutations [26]

$$H = c\sum_i P_{i,i+1}. \tag{1.10}$$

This Hamiltonian is the sum of operators, each of them acting in two adjacent spaces. This property is called ultra locality.

If size of the monodromy matrix is $N = 2$, this Hamiltonian coincides with XXX Heisenberg spin chain [55, 56]

$$H^{XXX} = \sum_i \sigma_i^x\,\sigma_{i+1}^x + \sigma_i^y\,\sigma_{i+1}^y + \sigma_i^z\,\sigma_{i+1}^z, \tag{1.11}$$

where σ_i's are usual Pauli matrices acting in the space V_i.

In the case of spin chain vacuum vector is $|0\rangle = \mathbf{e}_1^{(1)} \otimes \dots \otimes \mathbf{e}_1^{(N)}$, where $\mathbf{e}_1^{(i)}$ is a vector $(1,0,0,\dots,0)^T$ from the space V_i. According to (1.7) the lower triangular elements of the monodromy matrix annihilate the vacuum. The vacuum is eigenvector for the diagonal elements with eigenvalues

$$\begin{aligned}\lambda_1(u) &= \prod_{k=1}^{n}(u-\xi_k+c)\,,\\ \lambda_i(u) &= \prod_{k=1}^{n}(u-\xi_k)\,, \quad i=2,\ldots,N.\end{aligned} \tag{1.12}$$

The monodromy matrix of the inhomogeneous XXX spin chain (1.8) satisfies all the necessary properties for the application of the algebraic Bethe ansatz approach.

1.3 Algebraic Bethe ansatz for $\mathfrak{gl}_2$

Let us consider how algebraic Bethe ansatz works in the most simple case of $N=2$ [3, 44]. In this case the monodromy matrix is 2×2 matrix

$$T(u)=\begin{pmatrix}A(u) & B(u)\\ C(u) & D(u)\end{pmatrix}. \tag{1.13}$$

To apply algebraic Bethe ansatz we need a vector $|0\rangle\in\mathcal{H}$ called vacuum. Vacuum should have the following properties:

$$\begin{aligned}A(u)|0\rangle &= a(u)|0\rangle,\\ D(u)|0\rangle &= d(u)|0\rangle,\\ C(u)|0\rangle &= 0,\end{aligned} \tag{1.14}$$

where $a(u)$ and $d(u)$ are the eigenvalues of corresponding operators on vacuum.

To simplify all the next expressions let us introduce shorthand notation [41]. The symbol "bar" in $\bar{u}$ means that it is a set of variables $\bar{u}=\{u_1,u_2,\ldots,u_n\}$. The subscript i in $\bar{u}$ means that one element of the set is excluded $\bar{u}_i=\bar{u}\backslash\{u_i\}$. We also use superscripts to denote the different sets $\bar{t}^1,\bar{t}^2$ and so on. If some function depends on a set instead of a variable then one should understand that this expression is a product of this function over all elements in this set. One can use also this notation for function depending on two sets of variables. For example

$$a(\bar{u})=\prod_{u_i\in\bar{u}}a(u_i),\quad f(\bar{u},\bar{v}_i)=\prod_{u_k\in\bar{u}}\prod_{v_j\in\bar{v},j\neq i}f(u_k,v_j). \tag{1.15}$$

Using RTT-relation (1.2) with R-matrix (1.3) one can show that

$$[T_{ij}(u), T_{ij}(v)] = 0. \tag{1.16}$$

So, we can also extend the shorthand notation to the product of operators

$$T_{ij}(\bar{u}) = T_{ij}(u_1)T_{ij}(u_2)\dots T_{ij}(u_n). \tag{1.17}$$

In the case of $\mathfrak{gl}_2$ there is only one monodromy matrix elements acting nontrivial onto vacuum $|0\rangle$. It is upper triangular element $B(u)$. One can introduce a Bethe vector associated with set $\bar{u} = \{u_1, u_2, \dots, u_n\}$

$$\mathbb{B}(\bar{u}) = B(\bar{u})|0\rangle = B(u_1)B(u_2)\dots B(u_n)|0\rangle. \tag{1.18}$$

Due to (1.16) Bethe vector is symmetric in elements of set $\bar{u}$. We suppose that Bethe vector can become eigenvector of transfer matrix $t(u) = A(u) + D(u)$. To find it out we need the commutation relations of the diagonal elements with $B(u)$. These commutation relations follow from the RTT relation (1.2):

$$\begin{aligned} A(u)B(v) &= f(v,u)B(v)A(u) + g(u,v)B(u)A(v), \\ D(u)B(v) &= f(u,v)B(v)D(u) + g(v,u)B(u)D(v). \end{aligned} \tag{1.19}$$

where

$$f(v,u) = \frac{v-u+c}{v-u}, \quad g(v,u) = \frac{c}{v-u}. \tag{1.20}$$

The action of the transfer matrix $t(u) = A(u)+D(u)$ on the Bethe vector (1.18) gives us the equation

$$t(z)\mathbb{B}(\bar{u}) = \tau(z|\bar{u})\,\mathbb{B}(\bar{u}) + \sum_{i=1}^{n} g(z,u_i)\Lambda_i\,\mathbb{B}(\bar{u}_i \cup \{z\}), \tag{1.21}$$

where

$$\tau(z|\bar{u}) = a(z)f(\bar{u},z) + d(z)f(z,\bar{u}), \tag{1.22}$$

and

$$\Lambda_i = a(u_i)f(\bar{u}_i,u_i) - d(u_i)f(u_i,\bar{u}_i). \tag{1.23}$$

If we set all $\Lambda_i = 0$ then Bethe vector $\mathbb{B}(\bar{u})$ becomes eigenvector with eigenvalue $\tau(z|\bar{u})$ (1.22). The conditions $\Lambda_i = 0$ are called the system of Bethe equations.

Unfortunately, generalization of this scheme to algebras of higher rank is not so simple.

In the first time formula for the Bethe vector in the $\mathfrak{gl}_3$ case was proposed by P. P. Kulish and N. Yu. Reshetikhin [9]. Later this formula was reformulated [18] in the following way

$$\mathbb{B}(\bar{u}, \bar{v}) = \sum \frac{\mathrm{K}(\bar{v}_{\mathrm{I}}|\bar{u}_{\mathrm{I}})}{\lambda_2(\bar{v}_{\mathrm{II}})\lambda_2(\bar{u})} \frac{f(\bar{v}_{\mathrm{II}}, \bar{v}_{\mathrm{I}})f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}})}{f(\bar{v}, \bar{u})} T_{13}(\bar{u}_{\mathrm{I}})T_{12}(\bar{u}_{\mathrm{II}})T_{23}(\bar{v}_{\mathrm{II}})|0\rangle. \quad (1.24)$$

Here sets of the Bethe parameters $\bar{u}$ and $\bar{v}$ are divided into two subsets $\bar{u} \Rightarrow \{\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}\}$ and $\bar{v} \Rightarrow \{\bar{v}_{\mathrm{I}}, \bar{v}_{\mathrm{II}}\}$, such that $\#\bar{u}_{\mathrm{I}} = \#\bar{v}_{\mathrm{I}}$. The sum is taken over all possible partitions of this type. The function K is Izergin determinant [29] (partition function of six-vertex model with domain wall boundary conditions)

$$\mathrm{K}(\bar{v}|\bar{u}) = \prod_{i<j} \frac{c^2}{(u_i - u_j)(v_j - v_i)} \prod_{i,j} \frac{v_i - u_j + c}{c} \det_{ij} \left(\frac{c^2}{(v_i - u_j + c)(v_i - u_j)} \right). \quad (1.25)$$

Let us notice that this Bethe vector depends on two sets of variables $\bar{u}, \bar{v}$. All the upper triangular elements of the monodromy matrix are involved in a construction of Bethe vector. Now it is not a monomial in the elements of the monodromy matrix and the number of terms grows exponentially with the sizes of sets $\bar{u}, \bar{v}$. A the coefficients are extremely nontrivial.

Chapter 2 contains a generalization of the Bethe vector and its properties (like co-product property and recurrence equation for Bethe vectors) that help to apply Algebraic Bethe ansatz scheme to models with super-Yangian $Y(\mathfrak{gl}_{n|m})$ symmetries.

1.4 Bethe vector and Gauss decomposition

One of the most important notions of Algebraic Bethe Ansatz is a Bethe vector. It depends on a set of complex variables called Bethe parameters. The distinguishing feature of these vectors is that they become eigenvectors of the transfer matrix provided the Bethe parameters satisfy a special system of equations (Bethe equations) (1.34). In this case we call it on-shell Bethe vector or eigenvector . Otherwise, if the Bethe parameters are generic complex numbers, then the corresponding vectors are called off-shell Bethe vector, or simply Bethe vector. In this section we deal with the universal monodromy matrix. This means that it depends only on the underlying algebra generators. For models related to higher rank symmetries, in addition to our construction there are also method based on the so-called Nested Bethe Ansatz, which was elaborated in the pioneering papers [10, 19, 20], and method based on the trace formula [30].

To define construction of the Bethe vector, that we use, one needs to consider Yangian double algebra $DY(\mathfrak{gl}_N)$ [2]. This algebra can be constructed as two copies of RTT-algebra (1.2) with cross RTT-relation between these copies [21] . It is algebra generated by two monodromy matrices $T^{\pm}$ with relations

$$R_{12}(u-v)\, T_1^{\mu}(u)T_2^{\nu}(v) = T_2^{\nu}(v)T_1^{\mu}(u)\, R_{12}(u-v), \quad \mu,\nu=\pm. \tag{1.26}$$

We identify the monodromy matrix $T^+(u)$ with the previous monodromy matrix $T(u)$ (1.8) and look for the eigenvector only of the transfer matrix $t^+(z) = \mathrm{tr}\ T^+(z)$, but to express the Bethe vector we need to use both of monodromy matrices $T^{\pm}$. At the very end the Bethe vector depends only on the entries of the monodromy matrix T^+ and does not depend on the entries of the monodromy matrix T^-. Within particular integrable system (for example, the spin chain from the section 1.2) there is no second monodromy matrix T^-. It seems that T^- can not be constructed in the framework of the integrable system in a regular way, but T^- arises naturally when we consider quantum algebras. One can consider T^- as external algebra of symetries of the integrable system.

There are another ways to describe quantum algebras [1–4]. One of them is an approach based on Drinfeld currents [2]. In order to establish a relation between two representation of Yangian double algebra one has to use the following Gauss decomposition of the monodromy matrices [31]

$$T^{\pm}(u) = \mathbf{F}^{\pm}(u)\cdot \mathbf{K}^{\pm}(u)\cdot \mathbf{E}^{\pm}(u). \tag{1.27}$$

In the above formula $\mathbf{F}^{\pm}(u)$ are upper-triangular matrices with unities $\mathbf{1}$ on the diagonal, $\mathbf{K}^{\pm}(u) = \mathrm{diag}(k_1^{\pm}(u), k_2^{\pm}(u), \ldots, k_N^{\pm}(u))$ are diagonal matrices, and $\mathbf{E}^{\pm}(u)$ are lower-triangular matrices, again with unities on the diagonal.

The elements of matrices $\mathbf{F}^{\pm}, \mathbf{K}^{\pm}, \mathbf{E}^{\pm}$ should be considered as other basis elements of Yangian double algebra $DY(\mathfrak{gl}_N)$ (1.26) instead of $T^{\pm}_{ij}$. The commutation relations for the elements of matrices $\mathbf{F}^{\pm}, \mathbf{K}^{\pm}, \mathbf{E}^{\pm}$ follow from RTT-relations (1.26). Details of this connection can be found in [19, 31]. These commutation relations are given in the **Chapter 2**.

It turns out that the Bethe vector has a simpler presentation in terms of currents $\mathbf{F}^{\pm}, \mathbf{K}^{\pm}, \mathbf{E}^{\pm}$, despite the fact that the integrable system is usually formulated in terms of T^+_{ij}.

We formulate a construction of the Bethe vector in terms of the full currents

$$\mathcal{F}_i(u) = F^+_{i,i+1}(u) - F^-_{i,i+1}(u). \tag{1.28}$$

We emphasize that the full currents depend on both parts of the Yangian double algebra. Our construction of Bethe vector depends on the elements of $F^{\pm}$ only.

Bethe vector depends on $N-1$ sets (of the size r_i) of parameters $\bar{t}^i = \{t_1^i, t_2^i, \dots, t_{r_i}^i\}$ associated with the simple roots of the algebra $\mathfrak{gl}_N$. The Bethe vector is symmetric with respect to permutations of Bethe parameters from the same sets. For brevity, we unite all the sets $\bar{t}^i$ by one set $\bar{t}$.

Then, the construction of the Bethe vector associated with the set $\bar{t}$ is given by

$$\mathbb{B}(\bar{t}) = \mathcal{N}(\bar{t})\ \mathrm{P}^+ \left(\mathcal{F}_1(t_1^1)\dots\mathcal{F}_1(t_{r_1}^1)\dots\mathcal{F}_{N-1}(t_1^{N-1})\dots\mathcal{F}_{N-1}(t_{r_{N-1}}^{N-1})\right)|0\rangle, \tag{1.29}$$

where the normalisation

$$\mathcal{N}(\bar{t}) = \frac{\prod_{i=1}^{N-1}\lambda_i(\bar{t}^i)}{\prod_{i=1}^{N-2} f(\bar{t}^{i+1},\bar{t}^i)} \prod_{i=1}^{N-1}\prod_{1\le k<l\le r_i} f(t_l^i, t_k^i). \tag{1.30}$$

Here the symbol P^+ means projection, which annihilates all the terms with F_i^- on the left

$$\mathrm{P}^+(F_{i,j}^-(z)\ Q(\mathbf{F}^\pm)) = 0, \tag{1.31}$$

where $Q(\mathbf{F}^\pm))$ means any polynomial in the elements of the matrices $\mathbf{F}^\pm$.

Using commutation relation for the full current one can prove that the Bethe vector (1.29) is symmetric under permutations of the elements $t_k^i \leftrightarrow t_l^i$ of the same set $\bar{t}^i$. In the **Chapter 2** we proof that this construction satisfy all required properties to be Bethe vector.

To get the formula in term of the monodromy matrix (1.8) entries one should substitute all full currents in (1.29) using equation (1.28) and using commutation relations for the entries of the matrices $\mathbf{F}^\pm$ reorder in the way to put all the entries of the matrix $\mathbf{F}^-$) on the left

$$\mathbb{B}(\bar{t}) = \mathrm{P}^+\left(\sum_i Q_i^-(\mathbf{F}^-)Q_i^+(\mathbf{F}^+)\right)|0\rangle, \tag{1.32}$$

where $Q_i^\pm(\mathbf{F}^\pm)$ are polynomials in the elements of the matrices $\mathbf{F}^\pm$ respectively. Then we drop all the terms nonconstant $Q_i^-(\mathbf{F}^-)$ and express all the rest F^+ in term of T_{ij}^+ using formulas inverse to the Guasse decomposition (1.27).

One can find some properties of the P^+ in the **Chapter 2**. In [24] one can find the motivation and details of introducing this projection at the level of the Hopf algebra. Details of calculation of this projection in the simplest cases of $U_q(\hat{\mathfrak{gl}}_2)$ and $U_q(\hat{\mathfrak{gl}}_3)$ one can find in [25].

In the **Chapter 2** we give the proof of construction (1.29) in more general case of $Y(\mathfrak{gl}(n|m))$. This result is based on the study of the q-deformed case [26–28].

We find the formulas for action of $T_{ij}(z)$ onto Bethe vector (1.29) as linear expansion in Bethe vectors in the **Chapter 2**. One can find action formulas for the $\mathfrak{gl}_3$ case in [31] and for the $\mathfrak{gl}_{2|1}$ case in [32]. We use action formulas of upper triangular $T_{ij}(z)$ to find the recursion equation for Bethe vector.

The diagonal elements $T_{ii}(z)$ are included in the definition of the transfer matrix (1.5) $t(z) = \sum_i T_{ii}(z)$. It is proven [10, 30] that Bethe vector becomes eigenvector for transfer matrix

$$t(z)\, \mathbb{B}(\bar t) = \tau(z|\bar t)\, \mathbb{B}(\bar t), \tag{1.33}$$

if Bethe parameters satisfy the system of equations

$$\frac{\lambda_k(t^k_i)}{\lambda_{k+1}(t^k_i)} = \frac{f(t^k_i, \bar t^k_i)}{f(\bar t^k_i, t^k_i)} \frac{f(\bar t^{k+1}, t^k_i)}{f(t^k_i, \bar t^{k-1})}. \tag{1.34}$$

This system is called Bethe equations. In principle, a system of the equations for the Bethe parameters $\bar t$ called the Bethe equations if the condition for their satisfaction implies that the Bethe vector becomes the eigenvector of the transfer matrix.

Then the eigenvalue is

$$\tau(z|\bar t) = \sum_{i=1}^{N} \lambda_i(z) f(\bar t^i, z) f(z, \bar t^{i-1}), \tag{1.35}$$

where sets $\bar t^0 = \bar t^N = \emptyset$.

We use action formulas of lower triangular $T_{ij}(z)$ to find the recursion equation (1.43) for the highest coefficient (1.42).

An important property of the Bethe vector is the co-product property. It is also known as the composite model introduced in [33]. Assume that the monodromy matrix (1.8) can be represented as product of two other $T(u) = T^{(2)}(u)T^{(1)}(u)$ (such that $[T^{(2)}(u), T^{(1)}(v)] = 0$). Then the relation which expresses the Bethe vector $\mathbb{B}(\bar t)$ associated with $T(u)$ in term of Bethe vectors $\mathbb{B}^{(i)}(\bar t)$ associated with $T^{(i)}(u)$ is called co-product formula:

$$\mathbb{B}(\bar t) = \sum \frac{\prod_{s=1}^{N-1} f(\bar t^s_{\rm II}, \bar t^s_{\rm I})}{\prod_{s=1}^{N-2} f(\bar t^{s+1}_{\rm II}, \bar t^s_{\rm I})}\, \mathbb{B}^{(1)}(\bar t_{\rm I}) \prod_{s=1}^{N-1} \lambda^{(1)}_{s+1}(\bar t^s_{\rm II}) \otimes \mathbb{B}^{(2)}(\bar t_{\rm II}) \prod_{s=1}^{N-1} \lambda^{(2)}_{s}(\bar t^s_{\rm I}). \tag{1.36}$$

Here the sum is taken over all possible partitions of all the sets of the Bethe parameters $\bar t^k$ into pairs of subsets $\bar t^k \Rightarrow \{\bar t^k_{\rm I}, t^k_{\rm II}\}$.

The composed model was introduced for the calculation of the form factors of local operators in $\mathfrak{gl}_2$ models [33]. The same idea was used in [34, 35] for $\mathfrak{gl}_3$ case and in [36] for $\mathfrak{gl}_{2|1}$ case. We find another application of the co-product property described in [37, 38]. Our method based on the co-product formula directly leads to the sum formula, in which the scalar product is given as a sum over partitions of Bethe parameters. The structure of the scalar product of the Bethe vectors is encoded in the co-product formula for the Bethe vector.

1.5 Scalar product of Bethe vectors

Scalar products of Bethe vectors play a very important role in the Algebraic Bethe ansatz. They are a necessary tool for calculating form factors and correlation functions within this framework.

To define a scalar product of Bethe vectors we need a dual Bethe vector. The dual Bethe vector belongs to dual physical space $\mathcal{H}^*$. We suppose that the dual physical space $\mathcal{H}^*$ contains a dual vacuum $\langle 0|$ (such that $\langle 0|0\rangle = 1$) with properties

$$\begin{aligned} \langle 0|T_{ij}(u) &= 0, \qquad \text{with } \; i<j \\ \langle 0|T_{ii}(u) &= \lambda_i(u)\langle 0|, \end{aligned} \tag{1.37}$$

where functions λ_i are the same as in (1.7). Then the dual Bethe vector $\mathbb{C}(\bar{t})$ can be obtained from Bethe vector $\mathbb{B}(\bar{t})$ using "transposition" antimorphism Ψ (the supersymmeric analog of this antimorphism is described in [40]) defined by

$$\begin{aligned} \Psi(AB) &= \Psi(B)\Psi(A), \\ \Psi(T_{ij}(u)) &= T_{ji}(u), \\ \Psi(|0\rangle) &= \langle 0|. \end{aligned} \tag{1.38}$$

The dual Bethe vector is

$$\mathbb{C}(\bar{t}) = \Psi(\mathbb{B}(\bar{t})). \tag{1.39}$$

Now we can define the scalar product of the Bethe vectors

$$S(\bar{s}|\bar{t}) = \mathbb{C}(\bar{s})\mathbb{B}(\bar{t}). \tag{1.40}$$

One can prove that scalar product is symmetric $S(\bar{s}|\bar{t}) = S(\bar{t}|\bar{s})$ applying antimorphism Ψ and taking into account $\Psi^2 = 1$.

The sum formula for the scalar product was obtained in the $\mathfrak{gl}_2$ case [13], in the $\mathfrak{gl}_3$ case [39] and in the $\mathfrak{gl}_{2|1}$ case [41].

In the **Chapter 3** using the co-product formula (1.36) and the idea of the generalized model we prove that the scalar product (1.40) of Bethe vectors has the following bilinear form

$$S(\bar{s}|\bar{t}) = \sum \prod_{k=1}^{N-1} \lambda_k(\bar{s}^k_{\rm I})\lambda_{k+1}(\bar{s}^k_{\rm II})\ \lambda_{k+1}(\bar{t}^k_{\rm I})\lambda_k(\bar{t}^k_{\rm II})$$
$$\times\ Z(\bar{s}_{\rm I}|\bar{t}_{\rm I})Z(\bar{t}_{\rm II}|\bar{s}_{\rm II})\frac{\prod_{k=1}^{N-1} f(\bar{s}^k_{\rm II},\bar{s}^k_{\rm I})f(\bar{t}^k_{\rm I},\bar{t}^k_{\rm II})}{\prod_{j=1}^{N-2} f(\bar{s}^{j+1}_{\rm II},\bar{s}^j_{\rm I})f(\bar{t}^{j+1}_{\rm I},\bar{t}^j_{\rm II})}. \qquad (1.41)$$

Here all the sets of the Bethe parameters $\bar{t}^k$ and $\bar{s}^k$ are divided into two subsets $\bar{t}^k \Rightarrow \{\bar{t}^k_{\rm I},\bar{t}^k_{\rm II}\}$ and $\bar{s}^k \Rightarrow \{\bar{s}^k_{\rm I},\bar{s}^k_{\rm II}\}$, such that $\#\bar{t}^k_{\rm I} = \#\bar{s}^k_{\rm I}$. The sum is taken over all possible partitions of this type.

The function $Z(\bar{s}|\bar{t})$ is called the highest coefficient. It appears in the scalar product (1.41) in the term associated with extreme partition $\bar{s}^k_{\rm I} = \bar{s}^k$, $\bar{s}^k_{\rm II} = \emptyset$, and $\bar{t}^k_{\rm I} = \bar{t}^k$, $\bar{t}^k_{\rm II} = \emptyset$

$$S(\bar{s}|\bar{t}) = Z(\bar{s}|\bar{t}) \prod_{k=1}^{N-1} \lambda_k(\bar{s}^k)\lambda_{k+1}(\bar{t}^k) + \dots . \qquad (1.42)$$

The highest coefficient were obtained in the $\mathfrak{gl}_2$ [29] and $\mathfrak{gl}_{2|1}$ [41] cases explicitly in the determinant form, and in the $\mathfrak{gl}_3$ case [40] as sum.

The highest coefficient $Z(\bar{s}|\bar{t})$ can be determined recursively using the action formulas and recursion for Bethe vectors, which is given in **Chapter 3**. The highest coefficient $Z(\bar{s}|\bar{t})$ possesses the following recursions:

$$Z(\bar{s}|\bar{t}) = \sum_{p=2}^{N} \sum_{\substack{\mathrm{part}(\bar{s}^2,\dots,\bar{s}^{p-1})\\ \mathrm{part}(\bar{t}^1,\dots,\bar{t}^{p-1})}} \frac{g(\bar{t}^1_{\rm I},\bar{s}^1_{\rm I})f(\bar{t}^1_{\rm I},\bar{t}^1_{\rm II})f(\bar{t}^1_{\rm II},\bar{s}^1_{\rm I})}{f(\bar{s}^p,\bar{s}^{p-1}_{\rm I})}$$
$$\times \prod_{\nu=2}^{p-1} \frac{g(\bar{s}^\nu_{\rm I},\bar{s}^{\nu-1}_{\rm I})g(\bar{t}^\nu_{\rm I},\bar{t}^{\nu-1}_{\rm I})f(\bar{s}^\nu_{\rm II},\bar{s}^\nu_{\rm I})f(\bar{t}^\nu_{\rm I},\bar{t}^\nu_{\rm II})}{f(\bar{s}^\nu,\bar{s}^{\nu-1}_{\rm I})f(\bar{t}^\nu_{\rm I},\bar{t}^{\nu-1})}$$
$$\times\ Z\left(\{\bar{s}^1_{\rm II},\bar{s}^2_{\rm II},\dots,\bar{s}^{p-1}_{\rm II}\},\{\bar{s}^p,\dots,\bar{s}^{N-1}\}\big|\{\bar{t}^1_{\rm II},\bar{t}^2_{\rm II},\dots,\bar{t}^{p-1}_{\rm II}\},\{\bar{t}^p,\dots,\bar{t}^{N-1}\}\right), \qquad (1.43)$$

Here for every fixed $p \in \{2,\dots,N\}$ the sums are taken over partitions $\bar{t}^k \Rightarrow \{\bar{t}^k_{\rm I},\bar{t}^k_{\rm II}\}$ with $k = 1,\dots,p-1$ and $\bar{s}^k \Rightarrow \{\bar{s}^k_{\rm I},\bar{s}^k_{\rm II}\}$ with $k = 2,\dots,p-1$, such that $\#\bar{t}^k_{\rm I} = \#\bar{s}^k_{\rm I} = 1$ for $k = 2,...,p-1$. The subset $\bar{s}^1_{\rm I}$ is a fixed Bethe parameter from the set $\bar{s}^1$. There is no sum over partitions of the set $\bar{s}^1$ in (1.43).

Chapter 5 contains generalization of equations (1.41) and (1.43) to the case of quantum affine algebra $U_q(\hat{\mathfrak{gl}}_N)$.

1.6 Norm of eigenvector

One can prove that a norm of eigenvector of transfer matrix (1.5) has a determinant form. **Chapter 4** contains proof of this statement.

Let us describe the idea of proof. There is a list of axioms that is given in the **Chapter 4**. This set of axioms defines a function in a unique way. We proved that the norm and some combination with determinant satisfy this list of axioms at the same time. Thus, they coincide.

Finally, the result of this statement is

$$S(\bar{t}|\bar{t}) = \prod_{\nu=1}^{N-1} \prod_{\substack{p,q=1 \\ p\neq q}}^{r_\nu} f(t_p^\nu, t_q^\nu) \left(\prod_{\nu=1}^{N-2} f(\bar{t}^{\nu+1}, \bar{t}^\nu) \right)^{-1} \det G, \tag{1.44}$$

where matrix G is $(N-1) \times (N-1)$ block-matrix. The size of the block $G^{(\mu,\nu)}$ is $r_\mu \times r_\nu$. To describe the elements of $G^{(\mu,\nu)}$ we introduce a function Φ

$$\Phi_j^{(\mu)} = \frac{\lambda_\mu(t_j^\mu)}{\lambda_{\mu+1}(t_j^\mu)} \frac{f(\bar{t}_j^\mu, t_j^\mu)}{f(t_j^\mu, \bar{t}_j^\mu)} \frac{f(t_j^\mu, \bar{t}^{\mu-1})}{f(\bar{t}^{\mu+1}, t_j^\mu)}. \tag{1.45}$$

It is easy to see that Bethe equations (1.34) can be written as

$$\Phi_j^{(\mu)} = 1, \quad \mu = 1, \dots, N-1, \quad j = 1, \dots, r_\mu. \tag{1.46}$$

The entries of matrix G are defined as

$$G_{jk}^{(\mu,\nu)} = -c \frac{\partial \log \Phi_j^{(\mu)}}{\partial t_k^\nu}. \tag{1.47}$$

This statement generalizes Gaudin formula in $\mathfrak{gl}_2$ case [1] and Reshetikhin result in $\mathfrak{gl}_3$ case [39]. There are determinant formulas of norm of the Bethe vector in trigonometric $\mathfrak{gl}_3$ [9] and $\mathfrak{gl}_{2|1}$ [51] cases. Also norm of Bethe vectors for higher rank symmetries has been considered before, e.g. in [63]. **Chapter 5** contains analogous determinant representation in quantum affine $U_q(\hat{\mathfrak{gl}}_n)$ case.

1.7 Symmetry of Bethe vector

Using the RTT relation one can prove that the inverse monodromy matrix $\hat{T}$

$$\hat{T}_{ij}(u) = (T(u))^{-1}_{N+1-j,N+1-i} \tag{1.48}$$

satisfies the same RTT relation

$$R_{12}(u-v)\,\hat{T}_1(u)\hat{T}_2(v)=\hat{T}_2(v)\hat{T}_1(u)\,R_{12}(u-v) \tag{1.49}$$

does the monodromy matrix T.

Thus, there are two quantum R-matrix structures for each system with higher rank symmetry. They are associated with two monodromy matrices T and $\hat{T}$, and both have the same R-matrix.

Let us define hatted Bethe vectors $\hat{\mathbb{B}}(\bar{t})$ associated to the monodromy matrix $\hat{T}$ in the same way as for usual Bethe vector $\mathbb{B}(\bar{t})$ with replacement $T_{ij}(t_k)\to\hat{T}_{ij}(t_k)$.

The main point of the **Chapter 6** is a correspondence between $\hat{\mathbb{B}}(\bar{t})$ and $\mathbb{B}(\bar{t})$. One can formulate this result in the following theorem.

Theorem 1.7.1. *The Bethe vectors $\mathbb{B}$ and $\hat{\mathbb{B}}$ of integrable models with $\mathfrak{gl}(N)$-invariant R-matrix are related by*

$$\hat{\mathbb{B}}(\bar{t})=(-1)^{\#\bar{t}}\left(\prod_{s=1}^{N-2}f(\bar{t}^{s+1},\bar{t}^{s})\right)^{-1}\mathbb{B}\big(\mu(\bar{t})\big). \tag{1.50}$$

Here $\#\bar{t}$ is total cardinality of all the sets $\bar{t}^i$, and

$$\mu(\bar{t})\equiv\mu(\{\bar{t}^1,\bar{t}^2,\dots,\bar{t}^{N-1}\})=\{\bar{t}^{N-1}-c,\bar{t}^{N-2}-2c,\dots,\bar{t}^{1}-(N-1)c\}. \tag{1.51}$$

This theorem in the case of $GL(3)$ was proved in [27].

Applying equation (6.4.2) to scalar product of Bethe vectors (1.29) and taking into account that T and $\hat{T}$ satisfy the same RTT-algebra one can get relation for the highest coefficient (1.42) in the scalar product

$$Z\big(\mu(\bar{x})|\mu(\bar{t})\big)=Z(\bar{x}|\bar{t})\prod_{k=1}^{N-2}f(\bar{x}^{k+1},\bar{x}^{k})f(\bar{t}^{k+1},\bar{t}^{k}). \tag{1.52}$$

There is a generalization of the statement of the theorem (1.7.1) and its corollary to super-Yangian and quantum affine cases. We mention it in the end of the **Chapter 6**. The trigonometric analogue of the equation (6.5.12) in the case of $GL(3)$ was proved in [48].

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Chapter 2

Current presentation for the double super-Yangian $DY(\mathfrak{gl}(m|n))$ and Bethe vectors

Introduction:

In this Chapter we considered how to express Bethe vectors in two different ways using two Gauss decompositions. We proved that these two representations give the same Bethe vectors considering actions of monodromy matrix entries onto them. The formula describing co-product property of Bethe vectors was obtained. Also it was proven that if parameters of Bethe vectors satisfy some system of equations (Bethe equations), then Bethe vectors become eigenvectors of the transfer matrix.

Contribution:

I calculated the action of the monodromy matrix entries onto Bethe vectors (4.66) and (4.68). Using these formulas I calculated the action of the transfer matrix onto Bethe vector (4.70) and showed that if parameters of Bethe vector satisfy Bethe equations (4.75), then Bethe vector becomes eigenvector of the transfer matrix. In addition, I used the action formulas in the next Chapters to calculate scalar products of Bethe vectors.

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Current presentation for the super-Yangian double $DY(\mathfrak{gl}(m|n))$ and Bethe vectors

A. A. Hutsalyuk, A. Liashyk, S. Z. Pakuliak, E. Ragoucy, and N. A. Slavnov

Abstract. Bethe vectors are found for quantum integrable models associated with the supersymmetric Yangians $Y(\mathfrak{gl}(m|n)$ in terms of the current generators of the Yangian double $DY(\mathfrak{gl}(m|n))$. The method of projections onto intersections of different types of Borel subalgebras of this infinite-dimensional algebra is used to construct the Bethe vectors. Calculation of these projections makes it possible to express the supersymmetric Bethe vectors in terms of the matrix elements of the universal monodromy matrix. Two different presentations for the Bethe vectors are obtained by using two different but isomorphic current realizations of the Yangian double $DY(\mathfrak{gl}(m|n))$. These Bethe vectors are also shown to obey certain recursion relations which prove their equivalence.

Bibliography: 30 titles.

Keywords: Bethe vector, current algebra, monodromy matrix, Gauss decomposition, projection.

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1. Introduction

The calculation of form factors and correlation functions in quantum integrable models is one of the most important problems in the area of exactly solvable models in statistical physics and low-dimensional quantum mechanics. A lot of results were obtained in this direction starting from the earliest years in the development of the Quantum Inverse Scattering Method (QISM) [1], [2]. For models connected with various deformations of the affine algebra $\widehat{\mathfrak{gl}}(2)$ one of the most important results is a determinant presentation for the particular case of the scalar product in which one of vectors is an eigenvector of the transfer matrix [3]. This result lets us go directly to the problem of calculating the correlation functions [4] of the local operators in integrable models (see the survey [5] and the references there).

One of the most important notions of the QISM is a *Bethe vector*. In $\widehat{\mathfrak{gl}}(2)$-based models the Bethe vector is a monomial in the upper-right element of the monodromy matrix (the creation operator) applied to the pseudo-vacuum vector. It depends on a set of complex variables called Bethe parameters. The distinguishing feature of these vectors is that they become eigenvectors of the transfer matrix if the Bethe parameters satisfy a special system of equations (the Bethe equations). In this case we call them *on-shell Bethe vectors*. Otherwise, if the Bethe parameters are generic complex numbers, then the corresponding vectors are called *off-shell Bethe vectors*, or simply Bethe vectors. In this paper we deal with the universal monodromy matrix. This means that it depends only on the underlying algebra generators. We refer to the corresponding Bethe vectors as universal Bethe vectors.

The main purpose of this paper is to study Bethe vectors in the Yangian double $DY(\mathfrak{gl}(m|n))$. Our first goal is to obtain explicit formulae for them. The second goal is to derive formulae for the action of the monodromy matrix entries on the off-shell Bethe vectors. Achieving these two goals enables us to pose the problem of calculating the scalar products of Bethe vectors, which in turn is necessary for studying the form factors and correlation functions in integrable models with underlying $\mathfrak{gl}(m|n)$ supersymmetry.

For models connected with higher-rank symmetries, the QISM is based on the so-called nested Bethe ansatz, which was elaborated in the pioneering papers [6]–[8]. There a recursive procedure was developed for constructing Bethe vectors corresponding to the algebra $\widehat{\mathfrak{gl}}(N)$ from the known Bethe vectors of the algebra $\widehat{\mathfrak{gl}}(N-1)$. Formally, this method enables us to obtain explicit formulae for Bethe vectors in terms of certain polynomials in the creation operators (upper triangular entries of the monodromy matrix) acting on the pseudo-vacuum vector. However, the procedure is quite involved, and therefore no explicit representations were obtained in the early works mentioned above, with the exception of graphical representations found by Reshetikhin in [9] for models with the algebra $\widehat{\mathfrak{gl}}(3)$. The use of this diagram technique yielded a formula for the scalar products of off-shell Bethe vectors in terms of sums over partitions of the sets of Bethe parameters (a *sum formula*).

In [10] and [11] the Bethe vectors for the integrable models associated with deformed algebras $\widehat{\mathfrak{gl}}(N)$ were obtained as the traces of products of the monodromy matrices, R-matrices, and certain projections. These results were generalized to supersymmetric algebras in [12]. This approach makes it possible in some cases to calculate the norms of the nested Bethe vectors, but not their scalar products.

An alternative approach to the construction of Bethe vectors was proposed in [13]. This method explores the relation between two different realizations of the quantized Hopf algebra $U_q(\widehat{\mathfrak{gl}}(N))$ associated with the affine algebra $\widehat{\mathfrak{gl}}(N)$, the first in terms of the universal monodromy matrix $\mathrm{T}(z)$ and the RTT commutation relations, and the second in terms of the total currents, which are defined by the Gauss decomposition of the monodromy matrix $\mathrm{T}(z)$ [24]. Further, it was shown in [14] that the two different types of formulae for the universal off-shell Bethe vectors (constructed from the monodromy matrix) are related to the two different current realizations of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}(N))$ and their associated projections.

Moreover, the approach using the current generators of the deformed current algebras makes it possible to calculate the action of the monodromy matrix elements on the universal Bethe vectors. These action formulae turned out to be very useful for calculating form factors in the different quantum integrable models connected with rational and trigonometric deformations of the affine algebra $\widehat{\mathfrak{gl}}(3)$ [15]–[17]. Recently, similar results were obtained for the models with the superalgebras $\widehat{\mathfrak{gl}}(1|2)$ and $\widehat{\mathfrak{gl}}(2|1)$ in [18] and [19]. In these works the explicit formulae for the Bethe vectors and the action formulae in [20] and [21] were used in an essential way.

In the present paper we use the approach of [13]. In this framework the universal off-shell Bethe vector is defined as a projection of a product of total currents applied to the pseudo-vacuum vector. We defer the detailed definition to § 3, because it requires the introduction of many new concepts and new notation. For the same reason, we postpone a description of our main results to § 3.5. Here we would like to mention only that we construct explicit formulae for the universal Bethe vectors in terms of the current generators of the Yangian double $DY(\mathfrak{gl}(m|n))$ for two different Gauss decompositions of the universal monodromy matrix and two different current realizations of this algebra. These different Gauss decompositions correspond to the embeddings of $DY(\mathfrak{gl}(m-1|n))$ or $DY(\mathfrak{gl}(m|n-1))$ in $DY(\mathfrak{gl}(m|n))$. On the level of the RTT realization these embeddings are either in the lower-right corner or in the upper-left corner of the universal monodromy matrix. Using the first or the second type of these embeddings, we obtain two different representations for the Bethe vectors, which we denote by $\mathbb{B}(\bar{t})$ and $\widehat{\mathbb{B}}(\bar{t})$, respectively, where $\bar{t}$ is a set of Bethe parameters (3.11). We prove that these two representations are equivalent, that is, $\mathbb{B}(\bar{t}) = \widehat{\mathbb{B}}(\bar{t})$.

The paper is organized as follows. In § 2 we introduce the necessary notation used for calculations in graded vector spaces, as well as the RTT and current realizations of the algebra $DY(\mathfrak{gl}(m|n))$. In § 3 we define universal Bethe vectors using the notion of projections onto intersections of different types of Borel subalgebras. As already mentioned, § 3.5 contains the main results obtained in this paper. § 4 contains calculations of the action of the monodromy matrix elements on Bethe vectors in the generic case of $DY(\mathfrak{gl}(m|n))$. It is proved there, using these action formulae, that the vectors we have constructed become on-shell Bethe vectors if the supersymmetric Bethe equations for the Bethe parameters are satisfied. In § 5 we calculate the projections of a product of currents and present explicit formulae for the off-shell Bethe vectors as sums over partitions of the Bethe parameters. In Appendix A we introduce the notion of composed currents and study the relation between them and the Gauss coordinates of the universal monodromy matrix. Appendix B describes important properties of the projections. Appendix C shows how the Izergin and Cauchy determinants arise in the course of resolving the hierarchical relations in the determination of explicit formulae for the off-shell Bethe vectors.

2. Universal monodromy matrix

In this paper we adopt the following approach. We do not consider any specific supersymmetric exactly solvable models defined by a particular monodromy matrix $\mathrm{T}(z)$ satisfying the standard RTT relation. Instead, we treat a T-operator (2.3) as the *universal* monodromy matrix whose matrix elements are the generating series of the full set of generators of the Yangian double $DY(\mathfrak{gl}(m|n))$ acting in a generic representation space of this algebra, which is a rational deformation of the affine algebra $\widehat{\mathfrak{gl}}(m|n)$. These representations are not specified, except for the requirement that left and right pseudo-vacuum vectors exist, which ensures the applicability of the algebraic Bethe ansatz. To construct Bethe vectors we will use only the one T-operator $\mathrm{T}^{+}(z)$ from the dual pair $\{\mathrm{T}^{+}(z), \mathrm{T}^{-}(z)\}$ which generates the whole algebra $DY(\mathfrak{gl}(m|n))$. The eigenvalues $\lambda_i(z)$ of the diagonal matrix elements on the pseudo-vacuum vectors (see (2.12) and (2.13)) are free functional parameters which can be set equal to zero if necessary.

We first give a definition of $\mathbb{Z}_2$-graded linear spaces and their multiplication rules, and we describe matrices acting in these spaces.

2.1. $\mathbb{Z}_2$-graded linear spaces and notation. Let $\mathbb{C}^{m|n}$ be a $\mathbb{Z}_2$-graded linear space with a basis e_i, $i = 1, \dots, m+n$, where the vectors $\{\mathrm{e}_1, \mathrm{e}_2, \dots, \mathrm{e}_m\}$ are even and the vectors $\{\mathrm{e}_{m+1}, \mathrm{e}_{m+2}, \dots, \mathrm{e}_{m+n}\}$ are odd. The $\mathbb{Z}_2$-grading of the indices is as follows:

$$[i] = 0 \quad \text{for } i = 1, 2, \dots, m \quad \text{and} \quad [i] = 1 \quad \text{for } i = m+1, m+2, \dots, m+n. \tag{2.1}$$

Let $\mathrm{E}_{ij} \in \mathrm{End}(\mathbb{C}^{m|n})$ be the matrix with the only non-zero entry equal to 1 at the intersection of the ith row and jth column.

The basis vectors e_i and the matrices E_{ij} have the following grading:

$$[\mathrm{e}_i] = [i] \quad \text{and} \quad [\mathrm{E}_{ij}] = [i] + [j] \mod 2.$$

The tensor product is also graded according to the rule

$$(\mathrm{E}_{ij} \otimes \mathrm{E}_{kl}) \cdot (\mathrm{E}_{pq} \otimes \mathrm{E}_{rs}) = (-)^{([k]+[l])([p]+[q])} \mathrm{E}_{ij}\mathrm{E}_{pq} \otimes \mathrm{E}_{kl}\mathrm{E}_{rs}.$$

Let P be the graded permutation operator acting in the tensor product $\mathbb{C}^{m|n} \otimes \mathbb{C}^{m|n}$ as follows:

$$\mathrm{P} = \sum_{a,b=1}^{m+n} (-)^{[b]} \mathrm{E}_{ab} \otimes \mathrm{E}_{ba}.$$

Let

$$g(u, v) = \frac{c}{u - v}$$

be a rational function of the spectral parameters u and v and let c be a deformation parameter. By rescaling the spectral parameters it is always possible to set $c = 1$, but we will keep it for later convenience.

We define $\mathrm{R}(u, v) \in \mathrm{End}(\mathbb{C}^{m|n} \otimes \mathbb{C}^{m|n})$ as a rational supersymmetric R-matrix associated with the vector representation of $\mathfrak{gl}(m|n)$,

$$\mathrm{R}(u, v) = \mathbb{I} \otimes \mathbb{I} + g(u, v)\mathrm{P}, \tag{2.2}$$

where we have introduced the identity matrix in $\mathbb{C}^{m|n}$ by

$$\mathbb{I} = \sum_{i=1}^{m+n} \mathrm{E}_{ii}.$$

2.2. Commutation relations for the universal monodromy matrix. The superalgebra $DY(\mathfrak{gl}(m|n))$ is a graded associative algebra with unit $\mathbf{1}$ and is generated by the modes $\mathrm{T}_{i,j}^{(\ell)}$, $\ell \in \mathbb{Z}$, $1 \leqslant i, j \leqslant N+1$, of the T-operators

$$\mathrm{T}^{\pm}(u) = \mathbb{I} \otimes \mathbf{1} + \sum_{\substack{\ell \geqslant 0 \\ \ell < 0}} \sum_{i,j=1}^{N+1} \mathrm{E}_{ij} \otimes \mathrm{T}_{i,j}^{(\ell)} u^{-\ell-1}, \tag{2.3}$$

where $\ell \geqslant 0$ (respectively, $\ell < 0$) refers to the + index (respectively, the − index) in $\mathrm{T}^{\pm}(u)$ and $N = m + n - 1$ is the number of simple roots of the superalgebra $\mathfrak{gl}(m|n)$. The monodromy matrix elements $\mathrm{T}_{i,j}^{\pm}(u)$ are subject to the relations

$$\mathrm{R}(u, v) \cdot (\mathrm{T}^{\mu}(u) \otimes \mathbb{I}) \cdot (\mathbb{I} \otimes \mathrm{T}^{\nu}(v)) = (\mathbb{I} \otimes \mathrm{T}^{\nu}(v)) \cdot (\mathrm{T}^{\mu}(u) \otimes \mathbb{I}) \cdot \mathrm{R}(u, v), \tag{2.4}$$

where $\mu,\nu=\pm$. For the monodromy matrix[1] $\mathrm{T}(u)$ to be globally even, we fix the grading of the monodromy matrix elements as follows:

$$[\mathrm{T}_{i,j}(u)]=[i]+[j] \mod 2.$$

The tensor product of matrices and algebra generators is also graded, that is,

$$(\mathrm{E}_{ij}\otimes \mathrm{T}_{i,j}(u))\cdot(\mathrm{E}_{kl}\otimes \mathrm{T}_{k,l}(v))=(-)^{([i]+[j])([k]+[l])}\mathrm{E}_{ij}\mathrm{E}_{kl}\otimes \mathrm{T}_{i,j}(u)\mathrm{T}_{k,l}(v).$$

The subalgebras formed by the modes $\mathrm{T}^{(\ell)}_{i,j}$ (for $\ell\geqslant 0$ and for $\ell<0$) of the T-operators $\mathrm{T}^{\pm}(u)$ are the standard Borel subalgebras $U(\mathfrak{b}^{\pm})\subset DY(\mathfrak{gl}(m|n))$. These Borel subalgebras are Hopf subalgebras of $DY(\mathfrak{gl}(m|n))$. Their coalgebraic structure is given by the graded coproduct

$$\Delta\big(\mathrm{T}^{\pm}_{i,j}(u)\big)=\sum_{k=1}^{n+m}(-)^{([i]+[k])([k]+[j])}\mathrm{T}^{\pm}_{k,j}(u)\otimes \mathrm{T}^{\pm}_{i,k}(u). \tag{2.5}$$

By the commutation relations (2.4) the universal transfer matrix $\mathfrak{t}(u)$, defined as the supertrace

$$\mathfrak{t}(u)=\mathrm{str}(\mathrm{T}^{+}(u))\equiv\sum_{i=1}^{n+m}(-)^{[i]}\mathrm{T}^{+}_{i,i}(u) \tag{2.6}$$

of the universal monodromy matrix $\mathrm{T}^{+}(u)$, commutes for arbitrary values of the spectral parameters:

$$[\mathfrak{t}(u),\mathfrak{t}(v)]=0.$$

Thus, it can be regarded as a generating function for the commuting integrals of motion in the corresponding supersymmetric quantum integrable model.

All the commutation relations (2.4) can be rewritten in the form

$$\begin{aligned}[\mathrm{T}^{\mu}_{i,j}(u),\mathrm{T}^{\nu}_{k,l}(v)\}&\equiv \mathrm{T}^{\mu}_{i,j}(u)\mathrm{T}^{\nu}_{k,l}(v)-(-)^{([i]+[j])([k]+[l])}\mathrm{T}^{\nu}_{k,l}(v)\mathrm{T}^{\mu}_{i,j}(u)\\ &=(-)^{[i]([k]+[l])+[k][l]}g(u,v)\big(\mathrm{T}^{\nu}_{k,j}(v)\mathrm{T}^{\mu}_{i,l}(u)-\mathrm{T}^{\mu}_{k,j}(u)\mathrm{T}^{\nu}_{i,l}(v)\big),\end{aligned} \tag{2.7}$$

where $\mu,\nu=\pm$. Renaming in (2.7) the indices and the spectral parameters by $i\leftrightarrow k$, $j\leftrightarrow l$, and $u\leftrightarrow v$, we obtain the equivalent relation

$$\begin{aligned}[\mathrm{T}^{\mu}_{i,j}(u),\mathrm{T}^{\nu}_{k,l}(v)\}&= \mathrm{T}^{\mu}_{i,j}(u)\mathrm{T}^{\nu}_{k,l}(v)-(-)^{([i]+[j])([k]+[l])}\mathrm{T}^{\nu}_{k,l}(v)\mathrm{T}^{\mu}_{i,j}(u)\\ &=(-)^{[l]([i]+[j])+[i][j]}g(u,v)\big(\mathrm{T}^{\mu}_{i,l}(u)\mathrm{T}^{\nu}_{k,j}(v)-\mathrm{T}^{\nu}_{i,l}(v)\mathrm{T}^{\mu}_{k,j}(u)\big).\end{aligned} \tag{2.8}$$

Note that, according to the commutation relations (2.7) and (2.8), the odd matrix elements of the monodromy matrix do not commute, in contrast to the even ones:

$$\mathrm{T}^{\mu}_{i,j}(u)\mathrm{T}^{\nu}_{i,j}(v)=\frac{h_{[i]}(v,u)}{h_{[j]}(v,u)}\mathrm{T}^{\nu}_{i,j}(v)\mathrm{T}^{\mu}_{i,j}(u). \tag{2.9}$$

[1]We use the notation $\mathrm{T}(u)$ to denote either $\mathrm{T}^{+}(u)$ or $\mathrm{T}^{-}(u)$ when both matrices share the same properties.

Here and below we use the graded rational functions[2]

$$f_{[i]}(u,v) = 1 + g_{[i]}(u,v) = 1 + \frac{c_{[i]}}{u-v} = \frac{u-v+c_{[i]}}{u-v}\,, \qquad h_{[i]}(u,v) = \frac{f_{[i]}(u,v)}{g_{[i]}(u,v)}$$

and[3]

$$c_{[i]} = (-)^{[i]}c.$$

Below we also use the notation

$$\epsilon_{i,j} = 1 - \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker symbol.

2.3. Morphism of $DY(\mathfrak{gl}(m|n))$, singular vectors, and Gauss decompositions. Since the R-matrix (2.2) and the universal monodromy matrix (2.3) are globally even, one can easily check that the map[4]

$$\Psi\colon \mathrm{T}^{\pm}_{ij}(u) \to (-)^{[i]([j]+1)}\mathrm{T}^{\mp}_{ji}(u) \tag{2.10}$$

is an antimorphism of $DY(\mathfrak{gl}(m|n))$ which is a super- (or equivalently, graded) transposition compatible with the notion of super-trace. This map satisfies

$$\Psi(A\cdot B) = (-)^{[A][B]}\Psi(B)\cdot\Psi(A) \tag{2.11}$$

for arbitrary elements $A, B \in DY(\mathfrak{gl}(m|n))$ and will be used to relate right and left states, or equivalently, Bethe vectors and the dual ones.

Let $|0\rangle$ and $\langle 0|$ be vectors satisfying the conditions

$$\mathrm{T}^{\pm}_{i,j}(u)|0\rangle = 0, \quad i>j, \qquad \mathrm{T}^{\pm}_{i,i}(u)|0\rangle = \lambda^{\pm}_i(u)|0\rangle, \quad i = 1,\ldots,N+1, \tag{2.12}$$

$$\langle 0|\mathrm{T}^{\pm}_{i,j}(u) = 0, \quad i<j, \qquad \langle 0|\mathrm{T}^{\pm}_{i,i}(u) = \lambda^{\pm}_i(u)\langle 0|, \quad i = 1,\ldots,N+1, \tag{2.13}$$

where in (2.12) the monodromy matrix elements are acting to the right, while in (2.13) they are acting to the left. Such vectors, if they exist, are called *singular vectors*. If the pseudo-vacuum vectors $|0\rangle$ and $\langle 0|$ belong to the finite-dimensional representations of the Yangian double $DY(\mathfrak{gl}(m|n))$, then the functions $\lambda^{\pm}_i(u)$ are coinciding rational functions of the spectral parameter [22] expanded in the different domains: the function $\lambda^{+}_i(u)$ is a series with respect to u^{-1} and the same function $\lambda^{-}_i(u)$ is a series with respect to u. In what follows we will use the same notation $\lambda_i(u)$ for the functions $\lambda^{\pm}_i(u)$.

For the T-operators fixed by the relations (2.4) we have two possibilities for introducing the Gauss coordinates. The first possibility is to introduce $\mathrm{F}^{\pm}_{j,i}(u)$,

---

[2] We will keep the usual notation $f(u,v) = \dfrac{u-v+c}{u-v}$ and $h(u,v) = \dfrac{u-v+c}{c}$ and use it occasionally.

[3] Introduction of this graded deformation parameter lets us write many relations systematically, and this is why we do not scale the deformation parameter c to be equal to 1.

[4] We keep the superscripts $\pm$ in order to make the antimorphism compatible with the inclusion of a central charge in the Yangian double.

$\mathrm{E}^{\pm}_{i,j}(u)$, $1 \leqslant i < j \leqslant N+1$, and $k^{\pm}_{\ell}(u)$, $\ell = 1,\dots,N+1$, such that

$$\mathrm{T}^{\pm}_{i,j}(u) = \mathrm{F}^{\pm}_{j,i}(u)k^{\pm}_{i}(u) + \sum_{1\leqslant \ell < i} \mathrm{F}^{\pm}_{j,\ell}(u)k^{\pm}_{\ell}(u)\mathrm{E}^{\pm}_{\ell,i}(u), \tag{2.14}$$

$$\mathrm{T}^{\pm}_{i,i}(u) = k^{\pm}_{i}(u) + \sum_{1\leqslant \ell < i} \mathrm{F}^{\pm}_{i,\ell}(u)k^{\pm}_{\ell}(u)\mathrm{E}^{\pm}_{\ell,i}(u), \tag{2.15}$$

$$\mathrm{T}^{\pm}_{j,i}(u) = k^{\pm}_{i}(u)\mathrm{E}^{\pm}_{i,j}(u) + \sum_{1\leqslant \ell < i} \mathrm{F}^{\pm}_{i,\ell}(u)k^{\pm}_{\ell}(u)\mathrm{E}^{\pm}_{\ell,j}(u). \tag{2.16}$$

In the second case we introduce $\widehat{\mathrm{F}}^{\pm}_{j,i}(u)$, $\widehat{\mathrm{E}}^{\pm}_{i,j}(u)$, $1 \leqslant i < j \leqslant N+1$, and $\widehat{k}^{\pm}_{\ell}(u)$, $\ell = 1,\dots,N+1$, such that

$$\mathrm{T}^{\pm}_{i,j}(u) = \widehat{\mathrm{F}}^{\pm}_{j,i}(u)\widehat{k}^{+}_{j}(u) + \sum_{j<\ell\leqslant N+1} (-)^{([\ell]+[i])([\ell]+[j])}\widehat{\mathrm{F}}^{\pm}_{\ell,i}(u)\widehat{k}^{\pm}_{\ell}(u)\widehat{\mathrm{E}}^{\pm}_{j,\ell}(u), \tag{2.17}$$

$$\mathrm{T}^{\pm}_{j,j}(u) = \widehat{k}^{\pm}_{j}(u) + \sum_{j<\ell\leqslant N+1} (-)^{([\ell]+[j])}\widehat{\mathrm{F}}^{\pm}_{\ell,j}(u)\widehat{k}^{\pm}_{\ell}(u)\widehat{\mathrm{E}}^{\pm}_{j,\ell}(u), \tag{2.18}$$

$$\mathrm{T}^{\pm}_{j,i}(u) = \widehat{k}^{\pm}_{j}(u)\widehat{\mathrm{E}}^{\pm}_{i,j}(u) + \sum_{j<\ell\leqslant N+1} (-)^{([\ell]+[i])([\ell]+[j])}\widehat{\mathrm{F}}^{\pm}_{\ell,j}(u)\widehat{k}^{\pm}_{\ell}(u)\widehat{\mathrm{E}}^{\pm}_{i,\ell}(u). \tag{2.19}$$

One can verify that the antimorphism (2.10) and the Gauss decomposition (2.14)–(2.16) imply the following formulae for the Gauss coordinates:

$$\begin{gathered}\Psi\big(\mathrm{F}^{\pm}_{j,i}(u)\big) = (-)^{[i]([j]+1)}\mathrm{E}^{\mp}_{i,j}(u), \quad \Psi\big(\mathrm{E}^{\pm}_{i,j}(u)\big) = (-)^{[j]([i]+1)}\mathrm{F}^{\mp}_{j,i}(u),\\ \Psi\big(k^{\pm}_{\ell}(u)\big) = k^{\mp}_{\ell}(u).\end{gathered} \tag{2.20}$$

Similarly,

$$\begin{gathered}\Psi\big(\widehat{\mathrm{F}}^{\pm}_{j,i}(u)\big) = (-)^{[i]([j]+1)}\widehat{\mathrm{E}}^{\mp}_{i,j}(u), \quad \Psi\big(\widehat{\mathrm{E}}^{\pm}_{i,j}(u)\big) = (-)^{[j]([i]+1)}\widehat{\mathrm{F}}^{\mp}_{j,i}(u),\\ \Psi\big(\widehat{k}^{\pm}_{\ell}(u)\big) = \widehat{k}^{\mp}_{\ell}(u).\end{gathered}$$

The Gauss decomposition formulae also imply that

$$\begin{aligned}&\mathrm{E}^{\pm}_{i,j}(u)|0\rangle = \widehat{\mathrm{E}}^{\pm}_{i,j}(u)|0\rangle = 0, \quad i<j, \qquad k^{\pm}_{\ell}(u)|0\rangle = \widehat{k}^{\pm}_{\ell}(u)|0\rangle = \lambda^{\pm}_{\ell}(u)|0\rangle;\\ &\langle 0|\mathrm{F}^{\mp}_{j,i}(u) = \langle 0|\widehat{\mathrm{F}}^{\mp}_{j,i}(u) = 0, \quad i<j, \qquad \langle 0|k^{\mp}_{\ell}(u) = \langle 0|\widehat{k}^{\mp}_{\ell}(u) = \lambda^{\mp}_{\ell}(u)\langle 0|.\end{aligned}$$

2.4. Current realizations of $DY(\mathfrak{gl}(m|n))$. Let

$$F_i(u) = \mathrm{F}^{+}_{i+1,i}(u) - \mathrm{F}^{-}_{i+1,i}(u) \quad \text{and} \quad E_i(u) = \mathrm{E}^{+}_{i,i+1}(u) - \mathrm{E}^{-}_{i,i+1}(u)$$

be total currents [23]. Note that according to (2.20) we have

$$\begin{aligned}\Psi(F_i(u)) &= -(-)^{[i]([i+1]+1)}E_i(u) = -E_i(u),\\ \Psi(E_i(u)) &= -(-)^{[i+1]([i]+1)}F_i(u) = -(-)^{\delta_{i,m}}F_i(u),\end{aligned} \qquad i = 1,\dots,N. \tag{2.21}$$

This proves that the graded transposition is an idempotent of order 4 and its square counts the number of odd elements modulo 2.

Using straightforward calculations [24], [25] and the Gauss decomposition (2.14)–(2.16), we can obtain the following non-trivial commutation relations in terms of the total currents $F_i(t)$ and $E_i(t)$ and the Cartan currents $k_i^{\pm}(t)$:

$$\begin{aligned} k_i^{\pm}(u)F_i(v)k_i^{\pm}(u)^{-1} &= f_{[i]}(v,u)F_i(v), \\ k_{i+1}^{\pm}(u)F_i(v)k_{i+1}^{\pm}(u)^{-1} &= f_{[i+1]}(u,v)F_i(v), \end{aligned} \tag{2.22}$$

$$\begin{aligned} k_i^{\pm}(u)^{-1}E_i(v)k_i^{\pm}(u) &= f_{[i]}(v,u)E_i(v), \\ k_{i+1}^{\pm}(u)^{-1}E_i(v)k_{i+1}^{\pm}(u) &= f_{[i+1]}(u,v)E_i(v), \end{aligned} \tag{2.23}$$

$$((u-v)\epsilon_{i,m} - c_{[i]})F_i(u)F_i(v) = ((u-v)\epsilon_{i,m} + c_{[i]})F_i(v)F_i(u), \tag{2.24}$$

$$((u-v)\epsilon_{i,m} + c_{[i]})E_i(u)E_i(v) = ((u-v)\epsilon_{i,m} - c_{[i]})E_i(v)E_i(u), \tag{2.25}$$

$$(u-v)F_i(u)F_{i+1}(v) = (u-v-c_{[i+1]})F_{i+1}(v)F_i(u), \tag{2.26}$$

$$(u-v-c_{[i+1]})E_i(u)E_{i+1}(v) = (u-v)E_{i+1}(v)E_i(u), \tag{2.27}$$

$$\begin{aligned} [E_i(u), F_j(v)\} &= E_i(u)F_j(v) - (-)^{([i]+[i+1])([j]+[j+1])}F_j(v)E_i(u) \\ &= \delta_{i,j}c_{[i+1]}\delta(u,v)\big(k_{i+1}^{-}(u)\cdot k_i^{-}(u)^{-1} - k_{i+1}^{+}(v)\cdot k_i^{+}(v)^{-1}\big), \end{aligned} \tag{2.28}$$

where $\delta(u,v)$ is the rational δ-function given by (2.32). These calculations also lead to the Serre relations. For the simple root currents $F_i(u)$, $i = 1,\dots,N$, they have the form

$$\begin{aligned} &\mathrm{Sym}_{u_1,u_2}\big(((u_2-u_1)\delta_{i,m} - c_{[i+1]})\big(F_i(u_1)F_i(u_2)F_{i+1}(v) \\ &\qquad - 2F_i(u_1)F_{i+1}(v)F_i(u_2) + F_{i+1}(v)F_i(u_1)F_i(u_2)\big)\big) = 0, \end{aligned} \tag{2.29}$$

$$\begin{aligned} &\mathrm{Sym}_{u_1,u_2}\big(((u_1-u_2)\delta_{i,m} + c_{[i]})\big(F_i(u_1)F_i(u_2)F_{i-1}(v) \\ &\qquad - 2F_i(u_1)F_{i-1}(v)F_i(u_2) + F_{i-1}(v)F_i(u_1)F_i(u_2)\big)\big) = 0, \end{aligned} \tag{2.30}$$

$$\begin{aligned} &\mathrm{Sym}_{u_1,u_2}\big((u_1-u_2+c)\big[F_m(u_1)F_m(u_2)F_{m-1}(v_1)F_{m+1}(v_2) \\ &\qquad - 2F_m(u_1)F_{m-1}(v_1)F_m(u_2)F_{m+1}(v_2)\big] \\ &\qquad + 2c\; F_{m-1}(v_1)F_m(u_1)F_m(u_2)F_{m+1}(v_2) \\ &\qquad + (u_2-u_1+c)\big[F_{m-1}(v_1)F_{m+1}(v_2)F_m(u_1)F_m(u_2) \\ &\qquad - 2F_{m-1}(v_1)F_m(u_1)F_{m+1}(v_2)F_m(u_2)\big]\big) = 0. \end{aligned} \tag{2.31}$$

Analogous formulae for the currents $E_i(u)$, $i = 1,\dots,N$, can be obtained by applying the antimorphism Ψ to these relations. This amounts to replacing $F_i(u)$ by $E_i(u)$ and c by $-c$ in (2.29)–(2.31).

The rational, or equivalently, additive δ-function used in (2.28) can be represented as a difference of two series:

$$\delta(u,v) = \delta(v,u) = \frac{1}{(u-v)_>} - \frac{1}{(u-v)_<} = \sum_{n\in\mathbb{Z}} \frac{v^n}{u^{n+1}}, \tag{2.32}$$

where

$$\frac{1}{(u-v)_>} = \frac{1}{u}\sum_{k\geqslant 0}\Big(\frac{v}{u}\Big)^k \quad \text{and} \quad \frac{1}{(u-v)_<} = -\frac{1}{v}\sum_{k\geqslant 0}\Big(\frac{u}{v}\Big)^k. \tag{2.33}$$

Here the symbol > in the rational function $\dfrac{1}{(u-v)_>}$ means that $|u|>|v|$ and this rational function should be represented as the first series in (2.33). In turn, the symbol < in the rational function $\dfrac{1}{(u-v)_<}$ means that $|u|<|v|$ and this rational function should be represented as the second series in (2.33). Below we will also use the notation $\dfrac{1}{(u-v)_{\lessgtr}}$ to stress that one can use either of the two series expansions in (2.33) for the rational function $\dfrac{1}{u-v}$.

It is known [14] that another current realization of the Yangian double $DY(\mathfrak{gl}(m|n))$ can be obtained using a different Gauss decomposition of the monodromy matrix, as in (2.17)–(2.19). The commutation relations between the Cartan currents $\widehat{k}_i^{\pm}(u)$ and the simple root total currents $\widehat{F}_i(u)$ and $\widehat{E}_i(u)$ given by

$$\widehat{F}_i(u)=\widehat{\mathrm{F}}^{+}_{i+1,i}(u)-\widehat{\mathrm{F}}^{-}_{i+1,i}(u),\qquad \widehat{E}_i(u)=\widehat{\mathrm{E}}^{+}_{i,i+1}(u)-\widehat{\mathrm{E}}^{-}_{i,i+1}(u) \tag{2.34}$$

are gathered below:

$$\begin{aligned}\widehat{k}_i^{\pm}(u)\widehat{F}_i(v)\widehat{k}_i^{\pm}(u)^{-1}&=f_{[i]}(v,u)\widehat{F}_i(v),\\ \widehat{k}_{i+1}^{\pm}(u)\widehat{F}_i(v)\widehat{k}_{i+1}^{\pm}(u)^{-1}&=f_{[i+1]}(u,v)\widehat{F}_i(v),\end{aligned} \tag{2.35}$$

$$\begin{aligned}\widehat{k}_i^{\pm}(u)^{-1}\widehat{E}_i(v)\widehat{k}_i^{\pm}(u)&=f_{[i]}(v,u)\widehat{E}_i(v),\\ \widehat{k}_{i+1}^{\pm}(u)^{-1}\widehat{E}_i(v)\widehat{k}_{i+1}^{\pm}(u)&=f_{[i+1]}(u,v)\widehat{E}_i(v),\end{aligned} \tag{2.36}$$

$$((u-v)\epsilon_{i,m}+c_{[i]})\widehat{F}_i(u)\widehat{F}_i(v)=((u-v)\epsilon_{i,m}-c_{[i]})\widehat{F}_i(v)\widehat{F}_i(u), \tag{2.37}$$
$$((u-v)\epsilon_{i,m}-c_{[i]})\widehat{E}_i(u)\widehat{E}_i(v)=((u-v)\epsilon_{i,m}+c_{[i]})\widehat{E}_i(v)\widehat{E}_i(u), \tag{2.38}$$
$$(u-v-c_{[i+1]})\widehat{F}_i(u)\widehat{F}_{i+1}(v)=(u-v)\widehat{F}_{i+1}(v)\widehat{F}_i(u), \tag{2.39}$$
$$(u-v)\widehat{E}_i(u)\widehat{E}_{i+1}(v)=(u-v-c_{[i+1]})\widehat{E}_{i+1}(v)\widehat{E}_i(u), \tag{2.40}$$

$$\begin{aligned}[\widehat{E}_i(u),\widehat{F}_j(v)\}&=\widehat{E}_i(u)\widehat{F}_j(v)-(-)^{([i]+[i+1])([j]+[j+1])}\widehat{F}_j(v)\widehat{E}_i(u)\\ &=\delta_{i,j}c_{[i+1]}\delta(u,v)\big(\widehat{k}_i^{+}(u)\cdot\widehat{k}_{i+1}^{+}(u)^{-1}-\widehat{k}_i^{-}(v)\cdot\widehat{k}_{i+1}^{-}(v)^{-1}\big).\end{aligned} \tag{2.41}$$

The Serre relations for the simple root currents $\widehat{E}_i(u)$, $i=1,\dots,N$, now have the form

$$\begin{aligned}&\mathrm{Sym}_{u_1,u_2}\Big(((u_2-u_1)\delta_{i,m}-c_{[i+1]})\big(\widehat{E}_i(u_1)\widehat{E}_i(u_2)\widehat{E}_{i+1}(v)\\ &\qquad-2\widehat{E}_i(u_1)\widehat{E}_{i+1}(v)\widehat{E}_i(u_2)+\widehat{E}_{i+1}(v)\widehat{E}_i(u_1)\widehat{E}_i(u_2)\big)\Big)=0,\end{aligned} \tag{2.42}$$
$$\begin{aligned}&\mathrm{Sym}_{u_1,u_2}\Big(((u_1-u_2)\delta_{i,m}+c_{[i]})(\widehat{E}_i(u_1)\widehat{E}_i(u_2)\widehat{E}_{i-1}(v)\\ &\qquad-2\widehat{E}_i(u_1)\widehat{E}_{i-1}(v)\widehat{E}_i(u_2)+\widehat{E}_{i-1}(v)\widehat{E}_i(u_1)\widehat{E}_i(u_2))\Big)=0,\end{aligned} \tag{2.43}$$

$$\begin{aligned}
&\mathrm{Sym}_{u_1,u_2}\big((u_1-u_2+c)\big[\widehat{E}_m(u_1)\widehat{E}_m(u_2)\widehat{E}_{m-1}(v_1)\widehat{E}_{m+1}(v_2)\\
&\quad-2\widehat{E}_m(u_1)\widehat{E}_{m-1}(v_1)\widehat{E}_m(u_2)\widehat{E}_{m+1}(v_2)\big]\\
&\quad+2c\widehat{E}_{m-1}(v_1)\widehat{E}_m(u_1)\widehat{E}_m(u_2)\widehat{E}_{m+1}(v_2)\\
&\quad+(u_2-u_1+c)\big[\widehat{E}_{m-1}(v_1)\widehat{E}_{m+1}(v_2)\widehat{E}_m(u_1)\widehat{E}_m(u_2)\\
&\quad-2\widehat{E}_{m-1}(v_1)\widehat{E}_m(u_1)\widehat{E}_{m+1}(v_2)\widehat{E}_m(u_2)\big]\big)=0. \qquad (2.44)
\end{aligned}$$

Thanks to the antimorphism Ψ, there are analogous relations for the currents $\widehat{F}_i(u)$, $i=1,\dots,N$, with the replacements $\widehat{E}_i(u)\to\widehat{F}_i(u)$ and $c\to -c$ in the formulae (2.42)–(2.44). The action of the antimorphism (2.10) on the currents $\widehat{F}_i(u)$, $\widehat{E}_i(u)$, and $\widehat{k}_\ell(u)$ is given by the same formulae as in (2.21).

Note that in the commutation relations (2.24), (2.25), (2.37), and (2.38) one can replace $c_{[i]}$ by $c_{[i+1]}$. Indeed, $c_{[i]}=c_{[i+1]}$ when $i\neq m$, while for $i=m$ the factor $(u-v)\epsilon_{i,m}$ vanishes, and thus it does not matter whether we use $c_{[i]}$ or $c_{[i+1]}$.

3. Universal Bethe vectors

It follows from the commutation relations (2.4) that the subalgebras $U^{\pm}$ generated by the modes of the T-operators $\mathrm{T}_{ij}^{(n)}$ form two Borel subalgebras of $DY(\mathfrak{gl}(m|n))$. Moreover, by (2.5) they are Hopf subalgebras. We call $U^{\pm}$ the *standard Borel subalgebras* of the Yangian double $DY(\mathfrak{gl}(m|n))$.

As we already mentioned, the universal Bethe vectors are constructed from the matrix elements of one universal monodromy matrix T^{+}_{ij}. These operators belong to the standard 'positive' Borel subalgebra U^{+}. The goal of this section is to express the universal Bethe vectors in terms of the current generators of the Yangian double $DY(\mathfrak{gl}(m|n))$, using the approach developed in [13], [14], and [26].

In this paper we consider formulae for the Bethe vectors compatible with two different ways of embedding an algebra of smaller rank in an algebra of larger rank. Namely, from the explicit formulae for the right Bethe vectors $\mathbb{B}(\bar t)$ (see (5.17)) one can conclude that the Bethe vector $\mathbb{B}(\bar t)$ is obtained by resolving the hierarchical relations based on the embedding of the Yangian double $DY(\mathfrak{gl}(m-1|n))$ in the larger algebra $DY(\mathfrak{gl}(m|n))$. Similarly, it follows from (5.25) that the Bethe vector $\widehat{\mathbb{B}}(\bar t)$ is obtained by resolving the hierarchical relations based on the embedding of the Yangian double $DY(\mathfrak{gl}(m|n-1))$ in the larger algebra $DY(\mathfrak{gl}(m|n))$. To express the Bethe vectors $\mathbb{B}(\bar t)$ and $\widehat{\mathbb{B}}(\bar t)$ in terms of the current generators we will use two different types of Gauss decompositions of the monodromy matrix elements and the corresponding current generators [14].

The general theory of the relation between Bethe vectors and currents was developed in the paper [26] and then applied in [13] and [14] to the construction of the hierarchical Bethe vectors for quantum integrable models associated with the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}(N))$. The main tool used in those papers was the language of projections onto intersections of Borel subalgebras of different type.

To describe the Bethe vectors $\mathbb{B}(\bar{t})$ and $\mathbb{C}(\bar{t})$ we will use the current Borel subalgebras associated with the Gauss decomposition (2.14)–(2.16) and the antimorphism (2.10). For the Bethe vectors $\widehat{\mathbb{B}}(\bar{t})$ and $\widehat{\mathbb{C}}(\bar{t})$ we will use the same antimorphism and the current Borel subalgebras associated with the second Gauss decomposition (2.17)–(2.19).

3.1. Notation and conventions. We will denote sets of variables by bars over letters: $\bar{u}$, $\bar{v}$, and so on. To simplify formulae below, we use a shortened notation for products of functions depending on one or two variables. Namely, whenever we indicate that a function λ_j depends on a set of variables, the notation $\lambda_j(\bar{u})$ stands for the product of the functions $\lambda_j(u_\ell)$ over the set $\bar{u}$. Similarly, the notation $f_{[i]}(\bar{u},\bar{v})$ (or $g_{[i]}(\bar{u},\bar{v})$, or $h_{[i]}(\bar{u},\bar{v})$) denotes the double product of these functions over the corresponding sets. For example,

$$\lambda_j(\bar{u}) = \prod_{u_\ell\in\bar{u}} \lambda_j(u_\ell) \quad \text{and} \quad f_{[i]}(\bar{u},\bar{v}) = \prod_{u_\ell\in\bar{u},\ v_{\ell'}\in\bar{v}} f_{[i]}(u_\ell, v_{\ell'}).$$

Moreover, we use the same convention when considering products of commuting operators. For example,

$$\mathrm{T}_{i,j}(\bar{u}) = \prod_{\ell} \mathrm{T}_{i,j}(u_\ell) \qquad \text{for } [i]+[j]=0 \mod 2.$$

We also introduce several rational functions which will appear in the text below. First, for any function $x(u_1,u_2)$ we set

$$\Delta_x(\bar{u}) = \prod_{1\leqslant \ell<\ell'\leqslant a} x(u_{\ell'},u_\ell) \quad \text{and} \quad \Delta'_x(\bar{u}) = \prod_{1\leqslant \ell<\ell'\leqslant a} x(u_\ell,u_{\ell'}),$$

where $a=\#\bar{u}$.

Second, for arbitrary sets of parameters $\bar{u}$ and $\bar{v}$ we define

$$\gamma_i(\bar{u}) = \frac{\Delta_{f_{[i]}}(\bar{u})}{\Delta_h(\bar{u})^{\delta_{i,m}}} \quad \text{and} \quad \gamma_i(\bar{u},\bar{v}) = \frac{f_{[i]}(\bar{u},\bar{v})}{h(\bar{u},\bar{v})^{\delta_{i,m}}}\,. \tag{3.1}$$

The first function coincides with $\Delta_{f_{[i]}}(\bar{u})$ for $i\neq m$ and with $\Delta_g(\bar{u})$ for $i=m$. The second function coincides with $f_{[i]}(\bar{u},\bar{v})$ for $i\neq m$ and with $g(\bar{u},\bar{v})$ for $i=m$. Similarly, we define

$$\widehat{\gamma}_i(\bar{u}) = \frac{\Delta_{f_{[i+1]}}(\bar{u})}{\Delta'_h(\bar{u})^{\delta_{i,m}}} \quad \text{and} \quad \widehat{\gamma}_i(\bar{u},\bar{v}) = \frac{f_{[i+1]}(\bar{u},\bar{v})}{h(\bar{v},\bar{u})^{\delta_{i,m}}}\,.$$

For $i\neq m$,

$$\widehat{\gamma}_i(\bar{u}) = \Delta_{f_{[i+1]}}(\bar{u}) \quad \text{and} \quad \widehat{\gamma}_i(\bar{u},\bar{v}) = f_{[i+1]}(\bar{u},\bar{v}),$$

while for $i=m$,

$$\widehat{\gamma}_m(\bar{u}) = \Delta'_g(\bar{u}) \quad \text{and} \quad \widehat{\gamma}_m(\bar{u},\bar{v}) = g(\bar{v},\bar{u}).$$

Note that the function $\gamma_m(\bar{u})$ differs from $\widehat{\gamma}_m(\bar{u})$ by the factor $(-)^{\#\bar{u}(\#\bar{u}-1)/2}$. Similarly,

$$\gamma_m(\bar{u},\bar{v}) = (-)^{\#\bar{u}\#\bar{v}}\widehat{\gamma}_m(\bar{u},\bar{v}). \tag{3.2}$$

Also, note that $\gamma_i(\bar{u}) = \widehat{\gamma}_i(\bar{u})$ and $\gamma_i(\bar{u},\bar{v}) = \widehat{\gamma}_i(\bar{u},\bar{v})$ for $i\neq m$.

3.2. Deformed symmetrization. For any formal series $G(\bar{t})$ depending on the set of variables $\bar{t}$ (see (3.11) below) we define the *deformed symmetrization* (or c-symmetrization) to be the sum[5]

$$\overline{\mathrm{Sym}}_{\bar{t}}\, G(\bar{t}) = \sum_{\sigma\in S_{\bar{r}}} \prod_{s=1}^{N} \prod_{\substack{\ell<\ell' \\ \sigma^s(\ell)>\sigma^s(\ell')}} \frac{(t^s_{\sigma^s(\ell')} - t^s_{\sigma^s(\ell)})\epsilon_{s,m} + c_{[s]}}{(t^s_{\sigma^s(\ell')} - t^s_{\sigma^s(\ell)})\epsilon_{s,m} - c_{[s]}} G({}^{\sigma}\bar{t}), \tag{3.3}$$

where $S_{\bar{r}} = S_{r_1} \times \cdots \times S_{r_N}$ is the direct product of the groups S_{r_s} of permutations of the integers $1,\dots,r_s$, $s = 1,\dots,N$, and ${}^{\sigma}\bar{t}$ is the corresponding permuted set of Bethe parameters (3.11). By the arguments at the end of § 2.4, the formula for the deformed symmetrization can easily be written as

$$\overline{\mathrm{Sym}}_{\bar{t}}\, G(\bar{t}) = \sum_{\sigma\in S_{\bar{r}}} \prod_{s=1}^{N} \prod_{\substack{\ell<\ell' \\ \sigma^s(\ell)>\sigma^s(\ell')}} \frac{(t^s_{\sigma^s(\ell')} - t^s_{\sigma^s(\ell)})\epsilon_{s,m} + c_{[s+1]}}{(t^s_{\sigma^s(\ell')} - t^s_{\sigma^s(\ell)})\epsilon_{s,m} - c_{[s+1]}} G({}^{\sigma}\bar{t}). \tag{3.4}$$

In what follows we will use either (3.3) or (3.4), depending on the situation.

We say that a series $Q(\bar{t})$ is *c-symmetric* if

$$\overline{\mathrm{Sym}}_{\bar{t}}\, Q(\bar{t}) = \biggl(\prod_{s=1}^{N} r_s!\biggr) Q(\bar{t}).$$

Note that for $s = m$ the product over ℓ and ℓ' is equal to $(-)^{P(\sigma^m)}$, where $P(\sigma^m)$ is the parity of the permutation σ^m, and the sum over all permutations σ^m is nothing else but the antisymmetrization over the set $\bar{t}^m$.

3.3. The Bethe vector $\mathbb{B}(\bar{t})$ and the dual Bethe vector $\mathbb{C}(\bar{t})$. We first explain the relation between the Bethe vector $\mathbb{B}(\bar{t})$ and the current presentation (2.22)–(2.28).

Let $U_F \subset DY(\mathfrak{gl}(m|n))$ be the $DY(\mathfrak{gl}(m|n))$ subalgebra generated by the modes of the simple root currents $F_i^{(\ell)}$, $i = 1,\dots,N$, $\ell \in \mathbb{Z}$, and by the modes of the 'positive' Cartan currents $k_j^{(\ell')}$, $j = 1,\dots,N+1$, $\ell' \geqslant 0$. In the framework of the quantum double construction, the subalgebra $U_E \subset DY(\mathfrak{gl}(m|n))$ dual to U_F is generated by the modes of the simple root currents $E_i^{(\ell)}$, $i = 1,\dots,N$, $\ell \in \mathbb{Z}$, and by the modes of the 'negative' Cartan currents $k_j^{(\ell')}$, $j = 1,\dots,N+1$, $\ell' < 0$.

We call the subalgebras U_F and U_E *current Borel subalgebras*. They are Hopf subalgebras of $DY(\mathfrak{gl}(m|n))$ with respect to the so-called Drinfeld coproduct

$$\begin{aligned}
\Delta^{(D)}(F_i(z)) &= F_i(z) \otimes \mathbf{1} + k^+_{i+1}(z)(k^+_i(z))^{-1} \otimes F_i(z), \\
\Delta^{(D)}(k^\pm_j(z)) &= k^\pm_j(z) \otimes k^\pm_j(z), \\
\Delta^{(D)}(E_i(z)) &= \mathbf{1} \otimes E_i(z) + E_i(z) \otimes k^-_{i+1}(z)(k^-_i(z))^{-1},
\end{aligned} \tag{3.5}$$

which obviously differs from the coproduct given by (2.5).

[5]Recall that $N = m+n-1$ is the number of simple roots of the superalgebra $\mathfrak{gl}(m|n)$.

In order to express the Bethe vectors $\mathbb{B}(\bar{t})$ in terms of the current generators, we need only the one current Borel subalgebra U_F and its coalgebraic properties given by the first two equalities in (3.5). Consider the following intersections of this current Borel subalgebra with the standard Borel subalgebras $U^{\pm}$:

$$U_F^- = U_F \cap U^- \quad \text{and} \quad U_F^+ = U_F \cap U^+. \tag{3.6}$$

Each of these intersections is a subalgebra of $DY(\mathfrak{gl}(m|n))$ [26], and they are coideals with respect to the coproduct (3.5):

$$\Delta^{(D)}(U_F^+) = U_F^+ \otimes U_F \quad \text{and} \quad \Delta^{(D)}(U_F^-) = U_F \otimes U_F^-. \tag{3.7}$$

To see this we introduce the expansion of the following combination of Cartan currents:

$$k_{i+1}^+(z)(k_i^+(z))^{-1} = \mathbf{1} + \sum_{\ell \geqslant 0} \kappa_i^{(\ell)} z^{-\ell-1}.$$

Then the coproduct (3.5) maps the modes $F_i^{(\ell)}$ of the currents $F_i(z)$ to

$$\Delta^{(D)}(F_i^{(\ell)}) = F_i^{(\ell)} \otimes \mathbf{1} + \mathbf{1} \otimes F_i^{(\ell)} + \sum_{\ell' \geqslant 0} \kappa_i^{(\ell')} \otimes F_i^{(\ell-\ell'-1)}. \tag{3.8}$$

The properties (3.7) become obvious in view of (3.8).

According to the Cartan–Weyl construction of the Yangian double we have to find a global ordering on the generators of this algebra. There are two different choices for this ordering. We choose the ordering such that elements in the subalgebra U_F^- precede elements of the subalgebra U_F^+ [26], [27]. We say that an arbitrary element $\mathscr{F} \in U_F$ is ordered if it is represented in the form

$$\mathscr{F} = \mathscr{F}_- \cdot \mathscr{F}_+,$$

where $\mathscr{F}_\pm \in U_F^\pm$.

According to the general theory [26] one can define the projections of any ordered elements of the subalgebra U_F on the subalgebras (3.6) using the formulae

$$P_f^+(\mathscr{F}_- \cdot \mathscr{F}_+) = \varepsilon(\mathscr{F}_-)\mathscr{F}_+, \quad P_f^-(\mathscr{F}_- \cdot \mathscr{F}_+) = \mathscr{F}_-\varepsilon(\mathscr{F}_+), \quad \mathscr{F}_\pm \in U_F^\pm, \tag{3.9}$$

where the counit map $\varepsilon\colon U_F \to \mathbb{C}$ is defined by the rules

$$\varepsilon(F_i^{(\ell)}) = 0, \quad \varepsilon(\mathbf{1}) = 1, \quad \varepsilon(k_j^{(\ell)}) = 0.$$

Let $\overline{U}_F$ be the completion of U_F, which is formed by infinite sums of monomials that are ordered products of the form

$$\mathscr{A}_{i_1}^{(\ell_1)} \cdots \mathscr{A}_{i_a}^{(\ell_a)}, \qquad \ell_1 \leqslant \cdots \leqslant \ell_a,$$

where $\mathscr{A}_{i_l}^{(\ell_l)}$ is either $F_{i_l}^{(\ell_l)}$ or $k_{i_l}^{(\ell_l)}$. It can be proved [26] that

1) the action of the projections (3.9) extends to the algebra $\overline{U}_F$;
2) for any $\mathscr{F} \in \overline{U}_F$ with $\Delta^{(D)}(\mathscr{F}) = \mathscr{F}' \otimes \mathscr{F}''$ we have

$$\mathscr{F} = P_f^-(\mathscr{F}') \cdot P_f^+(\mathscr{F}''). \tag{3.10}$$

The formula (3.10) is an important tool for calculating the universal Bethe vectors. It allows us to present an arbitrary product of currents in the ordered form using simple formulae for the Drinfeld current coproducts.

Now we can define the universal Bethe vector. Let

$$\bar{t} = \{t_1^1, \dots, t_{r_1}^1; t_1^2, \dots, t_{r_2}^2; \dots; t_1^N, \dots, t_{r_N}^N\} \tag{3.11}$$

be a set of parameters. The superscript labels the different types of Bethe parameters and refers to the simple root numbering, and the subscript counts the number of parameters of a given type. There are r_ℓ Bethe parameters of type $\ell = 1, \dots, N$.

Let $\overleftarrow{\prod\limits_a} A_a$ $\Big($respectively, $\overrightarrow{\prod\limits_a} A_a\Big)$ denote the ordered product of non-commuting operators A_a such that A_ℓ is on the right (respectively, on the left) of $A_{\ell'}$ for $\ell' \geqslant \ell$:

$$\overleftarrow{\prod_{j\geqslant a\geqslant i}} A_a = A_j\ A_{j-1}\ \cdots\ A_{i+1}\ A_i \quad \text{and} \quad \overrightarrow{\prod_{i\leqslant a\leqslant j}} A_a = A_i\ A_{i+1}\ \cdots\ A_{j-1}\ A_j.$$

We define an ordered product of total currents,

$$\mathscr{F}(\bar{t}\,) = \overrightarrow{\prod_{1\leqslant a\leqslant N}} \Bigg(\overrightarrow{\prod_{1\leqslant \ell\leqslant r_a}} F_a(t_\ell^a) \Bigg), \tag{3.12}$$

which is a formal series with respect to the ratios t_k^b/t_l^c $(b > c)$ and t_i^a/t_j^a $(i > j)$ and takes values in the completion $\overline{U}_F$ (see [26]). The product (3.12) has poles for some values of the ratios t_k^b/t_l^c and t_i^a/t_j^a. The operator-valued coefficients at these poles take values in the completion $\overline{U}_F$ and can be identified with composed root currents (see Appendix A). Note also that in view of the commutation relations between currents, the product (3.12) as well as its projections are c-symmetric.

Let us introduce the normalized product of currents

$$\mathsf{F}(\bar{t}\,) = \frac{\prod_{\ell=1}^N \gamma_\ell(\bar{t}^\ell)}{\prod_{\ell=1}^{N-1} f_{[\ell+1]}(\bar{t}^{\ell+1}, \bar{t}^\ell)} \mathscr{F}(\bar{t}\,), \tag{3.13}$$

where γ_ℓ is given by (3.1). Then the universal off-shell Bethe vector $\mathbb{B}(\bar{t}\,)$ is defined as the action of the projection on this normalized product, applied to the singular vector $|0\rangle$:

$$\mathbb{B}(\bar{t}\,) = P_f^+\big(\mathsf{F}(\bar{t}\,)\big) \prod_{s=1}^N \lambda_s(\bar{t}^s)|0\rangle. \tag{3.14}$$

Note that in view of the commutation relations (2.24) and (2.26) between currents the normalized product of currents (3.13) is symmetric with respect to permutations of Bethe parameters of the same type.

The normalization of the universal off-shell Bethe vector is chosen so that it removes all zeros and poles originating from products of currents. For example, according to the commutation relations (2.24), the products of currents $\mathscr{F}_\ell(\bar{t}^\ell)$ have poles when $t_j^\ell - t_i^\ell + c_{[\ell]} = 0$ for $j > i$ and $\ell \neq m$, and zeros for all ℓ when $t_j^\ell - t_i^\ell = 0$. The potential singularities are compensated by the rational functions in the numerator of the prefactor in (3.13). On the other hand, the products of currents $\mathscr{F}_\ell(\bar t^\ell)\mathscr{F}_{\ell+1}(\bar t^{\ell+1})$ have poles when $t_j^{\ell+1}-t_i^\ell=0$ and zeros when $t_j^{\ell+1}-t_i^\ell+c_{[\ell+1]}=0$ for all i, j. These possible singularities are compensated by the product of the rational functions $f_{[\ell+1]}(\bar t^{\ell+1},\bar t^\ell)^{-1}$ in the denominator of the prefactor in (3.13).

Our strategy is to calculate first the projection in (3.14) and then to rewrite the result of this calculation as some polynomial in the monodromy matrix elements. This will be done in § 5. Then we define the dual Bethe vector $\mathbb{C}(\bar t\,)$ by the formula

$$\mathbb{C}(\bar t\,)=\Psi\big(\mathbb{B}(\bar t\,)\big), \tag{3.15}$$

where the antimorphism (2.10) is extended from the algebra to vectors of the representation of this algebra using the relations $\Psi(|0\rangle)=\langle 0|$ and $\Psi(\langle 0|)=|0\rangle$.

Alternatively, the formula for the dual Bethe vector can be found via the projection method and another choice of the current Borel subalgebra, the Drinfeld coproduct, and the associated projections from the ordered product of currents

$$\mathscr{E}(\bar t\,)=\overleftarrow{\prod_{N\geqslant a\geqslant 1}}\bigg(\overleftarrow{\prod_{r_a\geqslant \ell\geqslant 1}}E_a(t^a_\ell)\bigg).$$

We do not perform these calculations in this paper.

3.4. The Bethe vector $\widehat{\mathbb{B}}(\bar t\,)$ and the dual Bethe vector $\widehat{\mathbb{C}}(\bar t\,)$. For the Bethe vector $\widehat{\mathbb{B}}(\bar t\,)$ and the dual Bethe vector $\widehat{\mathbb{C}}(\bar t\,)$ one has to explore the second current realization (2.35)–(2.41) of the Yangian double $DY(\mathfrak{gl}(m|n))$ given by the currents $\widehat F_i(z)$, $\widehat E_i(z)$, and $\widehat k_j^\pm(z)$, which are related to the monodromy matrix elements through the Gauss decomposition (2.17)–(2.19) and the Frenkel–Ding formulae (2.34).

As in the previous subsections, to describe the Bethe vector $\widehat{\mathbb{B}}(\bar t\,)$ we define a Borel subalgebra $\widehat U_F$ such that the 'positive' Cartan currents $\widehat k_j^+(z)$ are in $\widehat U_F$ and have the coalgebraic properties

$$\begin{aligned}\widehat\Delta^{(D)}(\widehat F_i(z))&=\mathbf{1}\otimes\widehat F_i(z)+\widehat F_i(z)\otimes\widehat k_i^+(z)(\widehat k_{i+1}^+(z))^{-1},\\ \widehat\Delta^{(D)}(\widehat k_j^+(z))&=\widehat k_j^+(z)\otimes\widehat k_j^+(z).\end{aligned} \tag{3.16}$$

We again consider the intersections of this current Borel subalgebra with the standard Borel subalgebras $\widehat U^\pm$,

$$\widehat U_F^-=\widehat U_F\cap\widehat U^-\quad\text{and}\quad \widehat U_F^+=\widehat U_F\cap\widehat U^+, \tag{3.17}$$

and check the coideal properties of these intersections,

$$\widehat\Delta^{(D)}(\widehat U_F^+)=\widehat U_F\otimes\widehat U_F^+\quad\text{and}\quad\widehat\Delta^{(D)}(\widehat U_F^-)=\widehat U_F^-\otimes\widehat U_F$$

with respect to the coproduct (3.16).

Using the same cycling ordering for the Cartan–Weyl generators of $\widehat U_F$ as we used for ordering elements in U_F, we say that an arbitrary element $\widehat{\mathscr{F}}\in\widehat U_F$ is ordered if

$$\widehat{\mathscr{F}}=\widehat{\mathscr{F}}_-\cdot\widehat{\mathscr{F}}_+,$$

where $\widehat{\mathscr{F}}_\pm\in\widehat U_F^\pm$.

Again, according to the general theory formulated in [26] one can define the projections of any ordered elements of the subalgebras $\widehat{U}_F$ and $\widehat{U}_E$ on the subalgebras (3.17) by using the formulae

$$\widehat{P}^+_f(\widehat{\mathscr{F}}_- \cdot \widehat{\mathscr{F}}_+) = \widehat{\varepsilon}(\widehat{\mathscr{F}}_-)\widehat{\mathscr{F}}_+, \quad \widehat{P}^-_f(\widehat{\mathscr{F}}_- \cdot \widehat{\mathscr{F}}_+) = \widehat{\mathscr{F}}_-\widehat{\varepsilon}(\widehat{\mathscr{F}}_+), \qquad \widehat{\mathscr{F}}_\pm \in \widehat{U}^\pm_F, \tag{3.18}$$

where the counit map $\widehat{\varepsilon}\colon DY(\mathfrak{gl}(m|n)) \to \mathbb{C}$ is defined by the rules

$$\widehat{\varepsilon}(\widehat{F}^{(\ell)}_i) = 0 \quad \text{and} \quad \widehat{\varepsilon}(\widehat{k}^{(\ell)}_j) = 0,$$

and $\widehat{F}^{(\ell)}_i$ and $\widehat{k}^{(\ell)}_j$ are modes of the currents $\widehat{F}_i(z)$ and $\widehat{k}^+_i(z)$ in the second current realization of the Yangian double $DY(\mathfrak{gl}(m|n))$.

Defining the completion $\overline{\widehat{U}}_F$, we can verify [26] that:

1) the action of the projections (3.18) extends to the algebras $\overline{\widehat{U}}_F$;

2) for any $\widehat{\mathscr{F}} \in \overline{\widehat{U}}_F$ with $\widehat{\Delta}^{(D)}(\widehat{\mathscr{F}}) = \widehat{\mathscr{F}}' \otimes \widehat{\mathscr{F}}''$ we have

$$\widehat{\mathscr{F}} = \widehat{P}^-_f(\widehat{\mathscr{F}}'') \cdot \widehat{P}^+_f(\widehat{\mathscr{F}}'). \tag{3.19}$$

For the set (3.11) of Bethe parameters we consider the normalized ordered product of currents

$$\widehat{\mathsf{F}}(\bar t) = \frac{\prod_{\ell=1}^{N} \hat\gamma_\ell(\bar t^\ell)}{\prod_{\ell=1}^{N-1} f_{[\ell+1]}(\bar t^{\ell+1}, \bar t^\ell)} \widehat{\mathscr{F}}(\bar t), \tag{3.20}$$

where

$$\widehat{\mathscr{F}}(\bar t) = \prod_{N\geqslant a\geqslant 1}^{\longleftarrow} \biggl(\prod_{r_a\geqslant \ell\geqslant 1}^{\longleftarrow} \widehat{F}_a(t^a_\ell) \biggr). \tag{3.21}$$

The universal off-shell Bethe vectors associated with the second current realization of the Yangian double $DY(\mathfrak{gl}(m|n))$ are defined in terms of the action of the above projections on the singular vector $|0\rangle$ as follows:

$$\widehat{\mathbb{B}}(\bar t) = \widehat{P}^+_f\bigl(\widehat{\mathsf{F}}(\bar t)\bigr) \prod_{s=1}^{N} \lambda_{s+1}(\bar t^s)|0\rangle. \tag{3.22}$$

The normalization of this universal off-shell Bethe vector is again chosen in such a way as to remove all zeros and poles arising from products of currents.

The dual Bethe vector $\widehat{\mathbb{C}}(\bar t)$ is defined using the antimorphism (2.10):

$$\widehat{\mathbb{C}}(\bar t) = \Psi\bigl(\widehat{\mathbb{B}}(\bar t)\bigr). \tag{3.23}$$

3.5. Main results. In this paper we verify the following.

• The two different ways of constructing the Bethe vectors lead in the end to the same result, that is,

$$\mathbb{B}(\bar t) = \widehat{\mathbb{B}}(\bar t) \quad \text{and} \quad \mathbb{C}(\bar t) = \widehat{\mathbb{C}}(\bar t). \tag{3.24}$$

In § 4 we will prove this statement for the Bethe vectors $\mathbb{B}(\bar t)$ and $\widehat{\mathbb{B}}(\bar t)$ only. The proof for the dual vectors $\mathbb{C}(\bar t)$ and $\widehat{\mathbb{C}}(\bar t)$ follows from application of the antimorphism Ψ to the first equality in (3.24).

• Bethe vectors become on-shell, or equivalently, become eigenvectors of the supersymmetric transfer matrix $\mathfrak{t}(z)$ (2.6) with the eigenvalue (4.78), if the Bethe equations (4.75) for the parameters (3.11) are satisfied.

• Explicit formulae for the Bethe vectors in terms of the monodromy matrix elements are given by (5.17) and (5.25). Explicit formulae for the dual vectors can be obtained using the antimorphism (2.10).

• The coproduct properties for the Bethe vectors are given in the relations (4.8) and (4.9). They express the coproduct of a Bethe vector in term of Bethe vectors belonging to the two copies of $DY(\mathfrak{gl}(m|n))$ arising under application of the coproduct.

4. Formulae for the action of the monodromy matrix elements

The goal of the present section is to prove that the Bethe vectors $\mathbb{B}(\bar t\,)$ and $\widehat{\mathbb{B}}(\bar t\,)$ coincide. After obtaining formulae for the universal off-shell Bethe vectors in terms of elements of the monodromy matrix (see § 5), we will see that a direct proof of the equality (3.24) is a rather complicated combinatorial problem. Instead, we will prove it by checking that both of these vectors satisfy the same recurrence relations with respect to the action of the upper triangular and diagonal monodromy matrix elements on these vectors. To check this statement it is not necessary to get explicit formulae for the universal off-shell Bethe vectors in terms of the monodromy matrix elements. Before starting this analysis, we show that the Bethe vectors $\mathbb{B}(\bar t\,)$ and $\widehat{\mathbb{B}}(\bar t\,)$ have the same coproduct properties that follow from the coproduct (2.5) for the monodromy matrix.

4.1. Coproduct properties of the Bethe vectors. Calculating the coproduct of the product of the currents $F_i(t)$ using the first formula in (3.5), we get that the Drinfeld coproduct of the ordered product of simple root currents $\mathscr{F}(\bar t\,)$ is

$$\Delta^{(D)}\big(\mathscr{F}(\bar t\,)\big) = \sum_{0\leqslant s_1\leqslant r_1}\cdots\sum_{0\leqslant s_N\leqslant r_N}\prod_{\ell=1}^{N}\frac{1}{s_\ell!\,(r_\ell-s_\ell)!} \times \overline{\mathrm{Sym}}_{\bar t}\left(Z_{\bar s}(\bar t\,)\mathscr{F}(\bar t\,')\prod_{s=1}^{N}\prod_{\ell=s_\ell+1}^{r_\ell}k^+_{s+1}(t^s_\ell)k^+_s(t^s_\ell)^{-1}\otimes\mathscr{F}(\bar t\,'')\right), \tag{4.1}$$

where the sets $\bar t\,'$ and $\bar t\,''$ are

$$\bar t\,' = \{t^1_1,\dots,t^1_{s_1};t^2_1,\dots,t^2_{s_2};\dots;t^N_1,\dots,t^N_{s_N}\},$$
$$\bar t\,'' = \{t^1_{s_1+1},\dots,t^1_{r_1};t^2_{s_2+1},\dots,t^2_{r_2};\dots;t^N_{s_N+1},\dots,t^N_{r_N}\},$$

and $Z_{\bar s}(\bar t\,)$ is the rational function

$$Z_{\bar s}(\bar t\,) = \prod_{a=1}^{N-1}\prod_{\substack{s_a<\ell\leqslant r_a\\ 0<\ell'\leqslant s_{a+1}}}\frac{t^a_\ell - t^{a+1}_{\ell'} - c_{[a+1]}}{t^a_\ell - t^{a+1}_{\ell'}} = \prod_{a=1}^{N-1}\prod_{\substack{s_a<\ell\leqslant r_a\\ 0<\ell'\leqslant s_{a+1}}} f_{[a+1]}(t^{a+1}_{\ell'},t^a_\ell).$$

The formula (4.1) enables us to obtain the coalgebraic properties of the normalized product of currents (3.13) with respect to the Drinfeld coproduct. Indeed, the c-symmetrization can be transformed into the usual symmetrization over the set $\{\bar{t}^s\}$ due to the property

$$\gamma_s(\bar{t}^s)\,\overline{\mathrm{Sym}}_{\bar{t}^s}(G(\bar{t}^s)) = \mathrm{Sym}_{\bar{t}^s}(\gamma_s(\bar{t}^s)G(\bar{t}^s)). \tag{4.2}$$

Then the symmetrization can be replaced by the sum over partitions and subsequent symmetrization over each subset:

$$\mathrm{Sym}_{\bar{t}^s}(\,\cdot\,) = \sum_{\bar{t}^s\Rightarrow\{\bar{t}^s_{\mathrm{I}},\bar{t}^s_{\mathrm{II}}\}} \mathrm{Sym}_{\bar{t}^s_{\mathrm{I}}}\ \mathrm{Sym}_{\bar{t}^s_{\mathrm{II}}}(\,\cdot\,). \tag{4.3}$$

Here the summation is over the partitions of the set $\{\bar{t}^s\}$ into two disjoint subsets $\{\bar{t}^s_{\mathrm{I}}\}$ and $\{\bar{t}^s_{\mathrm{II}}\}$ with cardinalities $\#\bar{t}^s_{\mathrm{I}} + \#\bar{t}^s_{\mathrm{II}} = \#\bar{t}^s$, where

$$\bar{t} = \{\bar{t}^1,\dots,\bar{t}^N\} \Rightarrow \bar{t}_{\mathrm{I}} \cup \bar{t}_{\mathrm{II}} \tag{4.4}$$

and

$$\bar{t}_{\mathrm{I}} = \{\bar{t}^1_{\mathrm{I}},\dots,\bar{t}^N_{\mathrm{I}}\}, \qquad \bar{t}_{\mathrm{II}} = \{\bar{t}^1_{\mathrm{II}},\dots,\bar{t}^N_{\mathrm{II}}\}. \tag{4.5}$$

Using (4.2) and (4.3) and the fact that the normalized product of currents $\mathsf{F}(\bar{t})$ is symmetric with respect to permutations in each set of Bethe parameters $\bar{t}^\ell$, $\ell = 1,\dots,N$, we can transform (4.1) into a sum over the partitions given by (4.4) and (4.5):

$$\Delta^{(D)}\big(\mathsf{F}(\bar{t})\big) = \sum_{\mathrm{part}} \frac{\prod_{s=1}^{N}\gamma_s(\bar{t}^s_{\mathrm{II}},\bar{t}^s_{\mathrm{I}})}{\prod_{s=1}^{N-1} f_{[s+1]}(\bar{t}^{s+1}_{\mathrm{II}},\bar{t}^s_{\mathrm{I}})}\mathsf{F}(\bar{t}_{\mathrm{I}})\prod_{s=1}^{N} k^+_{s+1}(\bar{t}^s_{\mathrm{II}})k^+_s(\bar{t}^s_{\mathrm{II}})^{-1}\otimes\mathsf{F}(\bar{t}_{\mathrm{II}}). \tag{4.6}$$

With the help of the Drinfeld coproduct (3.16) for the second current realization of $DY(\mathfrak{gl}(m|n))$ we can show that the coproduct of the normalized product of currents (3.20) is given by

$$\widehat{\Delta}^{(D)}\big(\widehat{\mathsf{F}}(\bar{t})\big) = \sum_{\mathrm{part}} \frac{(-)^{\#\bar{t}^m_{\mathrm{I}}\cdot\#\bar{t}^m_{\mathrm{II}}}\prod_{s=1}^{N}\widehat{\gamma}_s(\bar{t}^s_{\mathrm{II}},\bar{t}^s_{\mathrm{I}})}{\prod_{s=1}^{N-1} f_{[s+1]}(\bar{t}^{s+1}_{\mathrm{II}},\bar{t}^s_{\mathrm{I}})}\widehat{\mathsf{F}}(\bar{t}_{\mathrm{I}})\otimes\widehat{\mathsf{F}}(\bar{t}_{\mathrm{II}})\prod_{s=1}^{N}\widehat{k}^+_s(\bar{t}^s_{\mathrm{I}})\widehat{k}^+_{s+1}(\bar{t}^s_{\mathrm{I}})^{-1}, \tag{4.7}$$

where the summation is over the disjoint subsets defined by (4.4) and (4.5).

We can use the formulae (4.6) and (4.7) to establish the coproduct properties of the universal Bethe vectors (3.14) and (3.22). It was proved in [26] that for any elements $\mathscr{F} \in \overline{U}_F$ and $\widehat{\mathscr{F}} \in \overline{\widehat{U}}_F$ the following equations hold:

$$\Delta\big(P^+_f(\mathscr{F})\big) \equiv (P^+_f\otimes P^+_f)\big(\Delta^{(D)}(\mathscr{F})\big) \mod U^+_F\otimes J,$$
$$\Delta\big(\widehat{P}^+_f(\widehat{\mathscr{F}})\big) \equiv (\widehat{P}^+_f\otimes \widehat{P}^+_f)\big(\widehat{\Delta}^{(D)}(\widehat{\mathscr{F}})\big) \mod \widehat{U}^+_F\otimes \widehat{J},$$

where J and $\widehat{J}$ are ideals in the corresponding subalgebras which annihilate the singular vector $|0\rangle$. A proper definition of these ideals is given in the beginning of the next subsection. Using these equalities and the formulae (4.6) and (4.7), we get that

$$\mathbb{B}(\bar{t}) = \sum_{\mathrm{part}} \frac{\prod_{s=1}^{N}\gamma_s(\bar{t}^s_{\mathrm{II}},\bar{t}^s_{\mathrm{I}})}{\prod_{s=1}^{N-1} f_{[s+1]}(\bar{t}^{s+1}_{\mathrm{II}},\bar{t}^s_{\mathrm{I}})}\mathbb{B}^{(1)}(\bar{t}_{\mathrm{I}})\prod_{s=1}^{N}\lambda^{(1)}_{s+1}(\bar{t}^s_{\mathrm{II}})\otimes\mathbb{B}^{(2)}(\bar{t}_{\mathrm{II}})\prod_{s=1}^{N}\lambda^{(2)}_s(\bar{t}^s_{\mathrm{I}}) \tag{4.8}$$

and

$$\widehat{\mathbb{B}}(\bar t\,) = \sum_{\rm part} \frac{(-)^{\#\bar t^m_{\rm I}\cdot\#\bar t^m_{\rm II}} \prod_{s=1}^N \widehat{\gamma}_s(\bar t^s_{\rm II},\bar t^s_{\rm I})}{\prod_{s=1}^{N-1} f_{[s+1]}(\bar t^{s+1}_{\rm II},\bar t^s_{\rm I})} \widehat{\mathbb{B}}^{(1)}(\bar t_{\rm I}) \prod_{s=1}^N \lambda^{(1)}_{s+1}(\bar t^s_{\rm II}) \otimes \widehat{\mathbb{B}}^{(2)}(\bar t_{\rm II}) \prod_{s=1}^N \lambda^{(2)}_s(\bar t^s_{\rm I}). \tag{4.9}$$

Taking (3.2) into account, we conclude that the universal Bethe vectors $\mathbb{B}(\bar t\,)$ and $\widehat{\mathbb{B}}(\bar t\,)$ have the same coproduct properties, which indicates that they may coincide. Below we will show that they satisfy the same recurrence relations, thereby proving that they do coincide.

The coproduct formulae (4.1) and (4.6) are very powerful tools for calculating the projection of a product of currents. Indeed, using the fundamental property in (3.10) of the projections $P^{\pm}_f$, we get from (4.1) that

$$\mathscr{F}(\bar t\,) = \sum_{0\leqslant s_1\leqslant r_1} \cdots \sum_{0\leqslant s_N\leqslant r_N} \prod_{\ell=1}^N \frac{1}{s_\ell!\,(r_\ell-s_\ell)!}\,\overline{\mathrm{Sym}}_{\bar t}\,\big(Z_{\bar s}(\bar t\,)P^-_f(\mathscr{F}(\bar t\,'))P^+_f(\mathscr{F}(\bar t\,''))\big), \tag{4.10}$$

and from (4.6) that

$$\mathsf{F}(\bar t\,) = \sum_{\bar t^s\Rightarrow\{\bar t^s_{\rm I},\bar t^s_{\rm II}\}} \frac{\prod_{s=1}^N \gamma_s(\bar t^s_{\rm II},\bar t^s_{\rm I})}{\prod_{s=1}^{N-1} f_{[s+1]}(\bar t^{s+1}_{\rm II},\bar t^s_{\rm I})} P^-_f\big(\mathsf{F}(\bar t_{\rm I})\big)\cdot P^+_f\big(\mathsf{F}(\bar t_{\rm II})\big). \tag{4.11}$$

This equality and the analogous equality for the product of currents $\widehat{F}_i(t)$ will be used in the § 5 to solve the hierarchical relations for the nested Bethe vectors and to obtain explicit formulae for them in terms of the monodromy matrix elements. This will be achieved by an explicit calculation of the projection of the corresponding products of currents, which reduces to a calculation presented in Appendix C.

4.2. Ideals of the Yangian double and presentations of the projections. To calculate the action of monodromy matrix elements on Bethe vectors, we have to formulate an important auxiliary statement about the action of monodromy matrix elements $\mathrm{T}^+_{i,j}(z)$ on 'negative' projections of composed currents $P^-_f\big(F_{k,l}(w)\big)$ and $\widehat{P}^-_f\big(\widehat{F}_{k,l}(w)\big)$ modulo certain ideals. This can be proved in the same way as used in [27] for the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}(N))$, and therefore we just sketch it below.

Let $U^\pm_F$ and $U^\pm_E$ be the intersections of the standard Borel subalgebras $U^\pm$ and the current Borel subalgebras U_F and U_E used in § 3.3. Let $I\subset DY(\mathfrak{gl}(m|n))$ be the ideal constructed from the elements of the form $\mathscr{F}_-\cdot\mathscr{F}$ such that $\mathscr{F}_-\in U^-_F$, $\mathscr{F}\in U_F$, and $\varepsilon(\mathscr{F}_-)=0$. Here and below, ε is the counit in the Hopf algebra $DY(\mathfrak{gl}(m|n))$. It is clear from the definition (3.9) of the projection P^+_f that the whole ideal I is annihilated by it: $P^+_f(I)=0$. Let $K\subset DY(\mathfrak{gl}(m|n))$ be the ideal generated by the elements which contain any combination of the 'negative' Cartan currents $k^-_j(u)$. By the commutation relations in $DY(\mathfrak{gl}(m|n))$, K is indeed an ideal because the 'negative' Cartan currents cannot be annihilated by any of the commutation relations in $DY(\mathfrak{gl}(m|n))$. Let $J\subset DY(\mathfrak{gl}(m|n))$ be the ideal generated by the elements of the form $\mathscr{F}\cdot\mathscr{E}_+$ such that $\mathscr{E}_+\in U^+_E$, $\mathscr{F}\in U^+_F$, and $\varepsilon(\mathscr{E}_+)=0$. By the definition of this ideal, any element in J annihilates the right vacuum vector: $J|0\rangle = 0$. Below we will use the symbols $\sim_I$, $\sim_K$, and $\sim_J$ to denote equalities in the Yangian double $DY(\mathfrak{gl}(m|n))$ modulo terms from the corresponding ideals I, K, and J. Similarly, starting from the current Borel subalgebras $\widehat{U}_F$ and $\widehat{U}_E$, we define the ideals $\widehat{I}$, $\widehat{K}$, and $\widehat{J}$ and the equivalence relations $\sim_{\widehat{I}}$, $\sim_{\widehat{K}}$, and $\sim_{\widehat{J}}$.

Since the off-shell Bethe vectors defined in (3.14) and (3.22) obviously do not belong to the ideals I and K nor the ideals $\widehat{I}$ and $\widehat{K}$, we can compute the action of the monodromy matrix elements on the Bethe vectors modulo these ideals. Moreover, since the ideals J and $\widehat{J}$ annihilate the vacuum vector $|0\rangle$, we can also skip the terms from these ideals when calculating the action of the monodromy matrix on the projections of currents.

Using the commutation relations (2.7) and (2.8) between the Gauss coordinates of the 'positive' and 'negative' monodromy matrices, as well as the relations (A.32) and (A.36) between the 'negative' projections of composed currents and the Gauss coordinates, we can prove the following.

Proposition 4.1 (see [27]).

$$\mathrm{T}^+_{i,j}(z)\cdot P^-_f\big(F_{k,l}(w)\big) \sim_{I,K} -\phi_k c_{[l,k]}\delta_{j,l}\, g(z,w)\mathrm{T}^+_{i,k}(z), \tag{4.12}$$

$$\mathrm{T}^+_{i,j}(z)\cdot \widehat{P}^-_f\big(\widehat{F}_{k,l}(w)\big) \sim_{\widehat{I},\widehat{K}} -\widehat{\phi}_l c_{[l,k]}\delta_{i,k}\, g(z,w)\mathrm{T}^+_{l,j}(z), \tag{4.13}$$

where $c_{[l,k]}$ *is given by* (A.30), *and*[6]

$$\begin{aligned} \phi_k &= (-)^{([i]+[j])[k]+[i][j]} && \text{for } k > j,\\ \widehat{\phi}_l &= (-)^{1+[i]} && \text{for } l < i. \end{aligned} \tag{4.14}$$

Remark 4.1. One can extend the values of the indices k and l in (4.14) to the values $k = j$ and $l = i$:

$$\phi_k = 1 \;\text{ for }\; k = j \qquad \text{and} \qquad \widehat{\phi}_l = 1 \;\text{ for }\; l = i$$

(this extension will be justified later; see Proposition 4.6).

Sketch of the proof of Proposition 4.1. The appearance of the Kronecker symbols $\delta_{j,l}$ and $\delta_{i,k}$ in (4.12) and (4.13), respectively, was proved in [27]. Let us give arguments which fix the rest of the terms on the right-hand side of (4.12) and (4.13), including the phases (4.14). To do this we consider the equations (4.12) and (4.13) applied to a right singular vector.

It is clear from the Gauss decompositions (2.14) that

$$\mathrm{F}^-_{k,l}(w)|0\rangle = \mathrm{T}^-_{l,k}(w)\mathrm{T}^-_{l,l}(w)^{-1}|0\rangle.$$

Then the equation (2.8) can be interpreted as

$$\mathrm{T}^+_{i,j}(z)\mathrm{T}^-_{l,k}(w)\mathrm{T}^-_{l,l}(w)^{-1} \sim_{I,K} (-)^{[k]([i]+[j])+[i][j]} g(z,w)\mathrm{T}^+_{i,k}(z)\mathrm{T}^-_{l,j}(w)\mathrm{T}^-_{l,l}(w)^{-1}, \tag{4.15}$$

[6]The asymmetry in the symbols ϕ_k and $\widehat{\phi}_l$ is related to the asymmetry in the different Gauss decompositions.

and due to the Kronecker symbol $\delta_{j,l}$ on the right-hand side of (4.12) the 'negative' monodromy matrix elements on the right-hand side of (4.15) cancel each other. Taking (A.32) into account, we get that $\phi_k = (-)^{[k]([i]+[j])+[i][j]}$.

Similarly, it follows from the Gauss decomposition (2.17) that

$$\widehat{\mathrm{F}}^{-}_{k,l}(w)|0\rangle = \mathrm{T}^{-}_{l,k}(w)\mathrm{T}^{-}_{k,k}(w)^{-1}|0\rangle.$$

Then the equation (2.7) can be interpreted as

$$\mathrm{T}^{+}_{i,j}(z)\mathrm{T}^{-}_{l,k}(w)\mathrm{T}^{-}_{k,k}(w)^{-1} \sim_{\widehat{I},\widehat{K}} (-)^{1+[i]([k]+[l])+[k][l]}g(z,w)\mathrm{T}^{+}_{l,j}(z)\mathrm{T}^{-}_{i,k}(w)\mathrm{T}^{-}_{k,k}(w)^{-1}, \tag{4.16}$$

and due to the Kronecker symbol $\delta_{i,k}$ on the right-hand side of (4.13) the 'negative' monodromy matrix elements on the right-hand side of (4.16) disappear, leading to $\widehat{\phi}_l = (-)^{1+[i]}$. □

We conclude this subsection by formulating the following proposition.

Proposition 4.2. *The off-shell Bethe vectors given by* (3.14) *and* (3.22) *satisfy the same recurrence relations following from the action by the upper triangular monodromy matrix elements* $\mathrm{T}_{i,j}(z)$, $i \leqslant j$, *on these vectors. This implies that the Bethe vectors coincide*:

$$\mathbb{B}(\bar{t}) = \widehat{\mathbb{B}}(\bar{t}).$$

The proof of this proposition will be given in the next two subsections, §§ 4.3 and 4.4.

4.3. Auxiliary presentations for the projections. To calculate the action of the upper triangular and diagonal monodromy matrix elements on the Bethe vectors (3.14) and (3.22), we have to obtain a special presentation for the projections of the products of simple root total currents. A systematic way to get such a presentation is based on techniques elaborated in [28]. Below we use the results contained in that paper, adapting them to the case under consideration.

Proposition 4.3. *The following identities hold for* $i < j$:

$$\begin{aligned} P^{-}_{f}\bigl(F_i(t^i)\cdots F_j(t^j)\bigr) = \sum_{\ell=0}^{j-i} c^{-1}_{[i,i+\ell+1]} \prod_{s=i}^{i+\ell-1} g_{[s+1]}(t^{s+1},t^s) P^{-}_{f}\bigl(F_{i+\ell+1,i}(t^{i+\ell})\bigr) \\ \times P^{-}_{f}\bigl(F_{i+\ell+1}(t^{i+\ell+1})\cdots F_j(t^j)\bigr), \end{aligned} \tag{4.17}$$

$$\begin{aligned} \widehat{P}^{-}_{f}\bigl(\widehat{F}_j(t^j)\cdots \widehat{F}_i(t^i)\bigr) = \sum_{\ell=0}^{j-i} c^{-1}_{[j-\ell,j+1]} \prod_{s=j-\ell}^{j+1} g_{[s+1]}(t^{s+1},t^s) \widehat{P}^{-}_{f}\bigl(\widehat{F}_{j+1,j-\ell}(t^{j-\ell})\bigr) \\ \times \widehat{P}^{-}_{f}\bigl(\widehat{F}_{j-\ell-1}(t^{j-\ell-1})\cdots \widehat{F}_i(t^i)\bigr). \end{aligned} \tag{4.18}$$

Proof. The two equalities can be proved similarly, using the definitions of the projections. Therefore, we give a detailed proof only for (4.17). We start from the definition

$$\begin{aligned} P^{-}_{f}\bigl(F_i(t^i)\cdots F_j(t^j)\bigr) = P^{-}_{f}\bigl(F_i(t^i)\bigr)\cdot P^{-}_{f}\bigl(F_{i+1}(t^{i+1})\cdots F_j(t^j)\bigr) \\ + P^{-}_{f}\bigl(F^{(+)}_i(t^i)F_{i+1}(t^{i+1})F_{i+2}(t^{i+2})\cdots F_j(t^j)\bigr). \end{aligned} \tag{4.19}$$

Using the definition

$$F_i^{(+)}(t^i) = \int dw\, \frac{F_i(w)}{t^i - w}$$

and the commutation relation

$$F_i(u)F_{i+1}(v) = \frac{u - v - c_{[i+1]}}{(u-v)_<} F_{i+1}(v)F_i(u) - \delta(u,v)F_{i+2,i}(v), \tag{4.20}$$

which is a particular case of the definition of the composed current (A.1) or (A.9), we get that

$$\begin{aligned} F_i^{(+)}(t^i)F_{i+1}(t^{i+1}) &= f_{[i+1]}(t^{i+1}, t^i)F_{i+1}(t^{i+1})F_i^{(+)}(t^i; t^{i+1}) \\ &\quad + c_{[i+1]}^{-1} g_{[i+1]}(t^{i+1}, t^i)F_{i+2,i}(t^{i+1}), \end{aligned} \tag{4.21}$$

where $F_i^{(+)}(t^i; t^{i+1}) = F_i^{(+)}(t^i) - \dfrac{c_{[i+1]}}{t^{i+1} - t^i + c_{[i+1]}} F_i^{(+)}(t^{i+1})$. Because of the commutativity of the current $F_i(t)$ with $F_{i+2}(t^{i+2}) \cdots F_j(t^j)$, the first term in (4.21) vanishes under the 'negative' projection in the second term of (4.19). On the other hand, by the second relation in (A.13),

$$F_{i+2,i}(t^{i+1}) = -\mathscr{S}_{F_i^{(0)}}\big(F_{i+1}(t^{i+1})\big) + c_{[i+1]}F_{i+1}(t^{i+1})F_i^{(+)}(t^{i+1}), \tag{4.22}$$

where the operators $\mathscr{S}_{F_i^{(0)}}(\,\cdot\,)$ are called screening operators and are defined by (B.1). The second term on the right-hand side of (4.22) also vanishes under the 'negative' projection in the second line of (4.19). Thus, (4.19) turns into

$$\begin{aligned} P_f^-\big(F_i(t^i)\cdots F_j(t^j)\big) &= P_f^-\big(F_i(t^i)\big)\cdot P_f^-\big(F_{i+1}(t^{i+1})\cdots F_j(t^j)\big) \\ &\quad - c_{[i+1]}^{-1} g_{[i+1]}(t^{i+1}, t^i)\mathscr{S}_{F_i^{(0)}}\Big(P_f^-\big(F_{i+1}(t^{i+1})\cdots F_j(t^j)\big)\Big). \end{aligned} \tag{4.23}$$

In the second line of (4.23) we obtain the 'negative' projection of the product of currents $F_{i+1}(t^{i+1}) \cdots F_j(t^j)$. Therefore, we can use this equality recursively to get in the first step that

$$\begin{aligned} P_f^-\big(F_i(t^i)\cdots F_j(t^j)\big) &= P_f^-\big(F_i(t^i)\big)\cdot P_f^-\big(F_{i+1}(t^{i+1})\cdots F_j(t^j)\big) \\ &\quad + c_{[i,i+2]}^{-1} g_{[i+1]}(t^{i+1}, t^i) P_f^-\big(F_{i+2,i}(t^{i+1})\big) P_f^-\big(F_{i+2}(t^{2+1})\cdots F_j(t^j)\big) \\ &\quad + c_{[i,i+3]}^{-1} g_{[i+1]}(t^{i+1}, t^i) g_{[i+2]}(t^{i+2}, t^{i+1}) \\ &\qquad\qquad \times \mathscr{S}_{F_i^{(0)}}\Big(\mathscr{S}_{F_{i+1}^{(0)}}\Big(P_f^-\big(F_{i+2}(t^{i+2})\cdots F_j(t^j)\big)\Big)\Big), \end{aligned}$$

where we have again used (4.22) and the commutativity of the screening operators and the projections (see Appendix B). Continuing this recursion process, we prove (4.17). The equality (4.18) can be proved similarly starting from the commutation relations

$$\widehat{F}_{i+1}(u)\widehat{F}_i(v) = \frac{u - v + c_{[i+1]}}{(u-v)_<} \widehat{F}_i(v)\widehat{F}_{i+1}(u) + \delta(u,v)\widehat{F}_{i+2,i}(v)$$

and using the first equality in (A.17). □

For each simple root index $i = 1, \dots, N$ we introduce the following notation for ordered products of currents:

$$\mathscr{F}_i(\bar t^i) = F_i(t^i_1)\cdots F_i(t^i_{r_i}) \quad \text{and} \quad \widehat{\mathscr{F}}_i(\bar t^i) = \widehat{F}_i(t^i_{r_i})\cdots \widehat{F}_i(t^i_1).$$

Using the normal ordering relation (3.10) (in the form (4.10)) and (4.17), we can prove the following statement.

Proposition 4.4. *The equality*

$$\begin{aligned} P^+_f\big(\mathscr{F}_1(\bar t^1)\cdots \mathscr{F}_N(\bar t^N)\big) &= P^+_f\big(\mathscr{F}_1(\bar t^1)\cdots \mathscr{F}_{N-1}(\bar t^{N-1})\big)\cdot \mathscr{F}_N(\bar t^N) \\ &- \sum_{\ell=1}^{N} \mathbb{C}_\ell\, \overline{\mathrm{Sym}}_{\bar t^\ell,\dots,\bar t^N}\big[\mathbb{G}_\ell(\bar t^{\ell-1},\dots,\bar t^N) c^{-1}_{[\ell,N+1]} P^-_f\big(F_{N+1,\ell}(t^N_1)\big) \\ &\times P^+_f\big(\mathscr{F}_1(\bar t^1)\cdots \mathscr{F}_{\ell-1}(\bar t^{\ell-1})\mathscr{F}_\ell(\bar t^\ell_1)\cdots \mathscr{F}_{N-1}(\bar t^{N-1}_1)\big)\cdot \mathscr{F}_N(\bar t^N_1)\big] + \mathbb{W} \end{aligned} \tag{4.24}$$

holds, where for $1 \leqslant \ell \leqslant N$ *the rational functions*

$$\mathbb{G}_\ell(\bar t^{\ell-1},\dots,\bar t^N) = f_{[\ell]}(t^\ell_1, \bar t^{\ell-1}) \prod_{s=\ell}^{N-1} g_{[s+1]}(t^{s+1}_1, t^s_1) f_{[s+1]}(t^{s+1}_1, \bar t^s_1) \tag{4.25}$$

appear along with the combinatorial factors

$$\mathbb{C}_\ell = \prod_{s=\ell}^{N} \frac{1}{(r_s-1)!}\,. \tag{4.26}$$

In (4.24), $\mathbb{W}$ *denotes terms having the structure* $P^-_f\big(F_{j_1,i_1}(w_1)\big)P^-_f\big(F_{j_2,i_2}(w_2)\big)\mathscr{F}$ *with* $j_1 \geqslant j_2$ *for some element* $\mathscr{F} \in \overline{U}_F$.

In (4.24) we used the shortened notation

$$\bar t^\ell_i = \{t^\ell_1,\dots,t^\ell_{i-1}, t^\ell_{i+1},\dots,t^\ell_{r_\ell}\}, \qquad i = 1,\dots,r_\ell,$$

where the Bethe parameter t^ℓ_i is omitted from the set $\bar t^\ell$, $\ell = 1,\dots,N$.

By (4.12), the action of any monodromy matrix element $\mathrm{T}^+_{i,j}(z)$ on the terms $\mathbb{W}$ belongs to the ideal I, except for the terms proportional to $\delta_{j,i_1}\delta_{j_1,i_2}$. These terms are irrelevant in view of the condition $j_1 \geqslant j_2 > i_2$.

Proof. It was proved in [28] that the projection

$$P^+_f\big(\mathscr{F}_1(\bar t^1)\cdots \mathscr{F}_N(\bar t^N)\big)$$

can be represented in the form[7]

$$P^+_f(\mathscr{F}_1\cdots \mathscr{F}_N) = \sum \mathscr{P}\big(P^-_f(F_{N+1,\ell})\big)\cdot P^+_f(\mathscr{F}_1\cdots \mathscr{F}_{N-1})\cdot \mathscr{F}_N, \tag{4.27}$$

where $\mathscr{P}\big(P^-_f(F_{N+1,\ell})\big)$ is a certain polynomial with rational coefficients in the 'negative' projections of the composed currents $F_{N+1,\ell}$, $\ell = 1,\dots,N$, and the $\mathscr{F}_\ell$

[7] In fact, this was proved in [28] for the case of the currents $\widehat{F}_\ell$, but it can easily be repeated for the currents F_ℓ, leading to (4.27).

are the products of currents corresponding to the simple roots ℓ. For brevity we did not write the arguments of the currents in (4.27).

It was shown in [28] that only 'negative' projections of currents $P_f^-\big(F_{N+1,\ell}(t)\big)$ appear on the right-hand side of (4.27). The other 'negative' projections of currents $P_f^-\big(F_{\ell',\ell}(t)\big)$ with $N \geqslant \ell' > \ell$ do not appear. The main reason for such a phenomenon is the factorization of projections of products of currents. We will demonstrate this phenomenon below in the simplest non-trivial case of $N = 2$, using the normal ordering relation (4.10).

Moreover, by (4.12) it is enough to keep in (4.27) only the first-order polynomials in the 'negative' projections of composed currents. Indeed, after the action of the monodromy matrix element $\mathrm{T}^+_{i,j}(z)$ on a product of two 'negative' projections of composed currents $P_f^-\big(F_{N+1,\ell_1}(t)\big)\cdot P_f^-\big(F_{N+1,\ell_2}(t)\big)$, the terms which are not in the ideals I and K are proportional to $\delta_{j,\ell_1}\delta_{N+1,\ell_2}$, and they vanish because $\ell_2 < N+1$.

Let us show how relations of the type (4.27) arise in the simple case of $m = 2$ and $n = 1$. We rename the sets of parameters as $\bar{t}^1 \equiv \bar{u}$ and $\bar{t}^2 \equiv \bar{v}$ with cardinalities $\#\bar{u} = a$ and $\#\bar{v} = b$ to simplify the formulae below. In this case the formula (4.10) can be rewritten as

$$
\begin{aligned}
P_f^+\big(F_1(u_1)\cdots F_1(u_a)\cdot F_2(v_1)\cdots F_2(v_b)\big) &= F_1(u_1)\cdots F_1(u_a)\cdot F_2(v_1)\cdots F_2(v_b)\\
&- \overline{\mathrm{Sym}}_{\bar{u}}\,\frac{1}{(a-1)!}P_f^-\big(F_1(u_1)\big)P_f^+\big(F_1(u_2)\cdots F_1(u_a)\cdot F_2(v_1)\cdots F_2(v_b)\big)\\
&- \overline{\mathrm{Sym}}_{\bar{v}}\,\frac{f(v_1,\bar{u})}{(b-1)!}P_f^-\big(F_2(v_1)\big)P_f^+\big(F_1(u_1)\cdots F_1(u_a)\cdot F_2(v_2)\cdots F_2(v_b)\big)\\
&- \overline{\mathrm{Sym}}_{\bar{u},\bar{v}}\,\frac{f(v_1,\bar{u}_1)}{(a-1)!\,(b-1)!}P_f^-\big(F_1(u_1)F_2(v_1)\big)\\
&\qquad\times P_f^+\big(F_1(u_2)\cdots F_1(u_a)\cdot F_2(v_2)\cdots F_2(v_b)\big) + \mathbb{W}.
\end{aligned}
\tag{4.28}
$$

We keep the double symmetrized term in (4.28) because it is the source of the 'negative' projection of composed currents $P_f^-\big(F_{3,1}(v)\big)$ (see (4.29) below), while the quadratic terms from $P_f^-\big(F_1(u_1)F_2(v_1)\big)$ disappear in the next step of the recursion.

Applying (4.28) recursively, we can replace the 'positive' projections by the corresponding products of total currents. Using the equality

$$
P_f^-\big(F_1(u)F_2(v)\big) = P_f^-\big(F_1(u)\big)P_f^-\big(F_2(v)\big) + c^{-1}g(v,u)P_f^-\big(F_{3,1}(v)\big),
\tag{4.29}
$$

which is a direct consequence of (4.20), we obtain instead of (4.28) the equality of formal series (recall that $F_{i+1,i}(t) \equiv F_i(t)$)

$$
\begin{aligned}
P_f^+\big(F_1(u_1)\cdots F_1(u_a)\cdot F_2(v_1)\cdots F_2(v_b)\big) &= F_1(u_1)\cdots F_1(u_a)\cdot F_2(v_1)\cdots F_2(v_b)\\
&- \overline{\mathrm{Sym}}_{\bar{u}}\,\frac{1}{(a-1)!}P_f^-\big(F_{2,1}(u_1)\big)F_1(u_2)\cdots F_1(u_a)\cdot F_2(v_1)\cdots F_2(v_b)\\
&- \overline{\mathrm{Sym}}_{\bar{v}}\,\frac{f(v_1,\bar{u})}{(b-1)!}P_f^-\big(F_{3,2}(v_1)\big)F_1(u_1)\cdots F_1(u_a)\cdot F_2(v_2)\cdots F_2(v_b)\\
&- \overline{\mathrm{Sym}}_{\bar{u},\bar{v}}\,\frac{c^{-1}g(v_1,u_1)f(v_1,\bar{u}_1)}{(a-1)!\,(b-1)!}\\
&\qquad\times P_f^-\big(F_{3,1}(v_1)\big)F_1(u_2)\cdots F_1(u_a)\cdot F_2(v_2)\cdots F_2(v_b) + \mathbb{W},
\end{aligned}
$$

where the terms denoted by $\mathbb{W}$ again belong to the ideal I after the action of any monodromy matrix element. Finally, using the normal ordering rule (4.10) for the product of currents $\mathscr{F}_1$, we can replace these products by their 'positive' projections to obtain

$$
\begin{aligned}
&P_f^+\big(F_1(u_1)\cdots F_1(u_a)\cdot F_2(v_1)\cdots F_2(v_b)\big)\\
&\quad= P_f^+\big(F_1(u_1)\cdots F_1(u_a)\big)\cdot F_2(v_1)\cdots F_2(v_b)\\
&\qquad - \overline{\mathrm{Sym}}_{\bar v}\,\frac{f(v_1,\bar u)}{(b-1)!}P_f^-\big(F_{3,2}(v_1)\big)P_f^+\big(F_1(u_1)\cdots F_1(u_a)\big)\cdot F_2(v_2)\cdots F_2(v_b)\\
&\qquad - \overline{\mathrm{Sym}}_{\bar u,\bar v}\,\frac{c^{-1}g(v_1,u_1)f(v_1,\bar u_1)}{(a-1)!\,(b-1)!}P_f^-\big(F_{3,1}(v_1)\big)\\
&\qquad\qquad\times P_f^+\big(F_1(u_2)\cdots F_1(u_a)\big)\cdot F_2(v_2)\cdots F_2(v_b)+\mathbb{W}.
\end{aligned}
\tag{4.30}
$$

We see that the terms containing the 'negative' projection of a current $P_f^-\big(F_{2,1}(u_1)\big)$ disappear from the final formula (4.30).

Now we prove the statement of Proposition 4.4 in the general case, using the normal ordering relation (4.10). Taking into account the arguments above, we write

$$
\begin{aligned}
&P_f^+\big(\mathscr{F}_1(\bar t^1)\cdots\mathscr{F}_{N-1}(\bar t^{N-1})\mathscr{F}_N(\bar t^N)\big)=\mathscr{F}_1(\bar t^1)\cdots\mathscr{F}_{N-1}(\bar t^{N-1})\mathscr{F}_N(\bar t^N)\\
&\quad-\sum_{\ell=1}^{N}\mathbb{C}_\ell\,\overline{\mathrm{Sym}}_{\bar t^\ell,\dots,\bar t^N}\Bigg[f_{[\ell]}(t_1^\ell,\bar t^{\ell-1})\prod_{s=\ell}^{N-1}f_{[s+1]}(t_1^{s+1},\bar t_1^s)P_f^-\big(F_\ell(t_1^\ell)\cdots F_N(t_1^N)\big)\\
&\qquad\qquad\times P_f^+\big(\mathscr{F}_1(\bar t^1)\cdots\mathscr{F}_{\ell-1}(\bar t^{\ell-1})\cdot\mathscr{F}_\ell(\bar t_1^\ell)\cdots\mathscr{F}_N(\bar t_1^N)\big)\Bigg]+\mathbb{W},
\end{aligned}
\tag{4.31}
$$

where we keep only the terms containing $P_f^-\big(F_\ell(t_1^\ell)\cdots F_N(t_1^N)\big)$ as the source of the 'negative' projection of a composed current $P_f^-\big(F_{N+1,\ell}(t_1^N)\big)$, and $\mathbb{W}$ denotes terms which give elements of the ideal I after the action of any monodromy matrix element $\mathrm{T}^+_{i,j}(z)$. Using (4.17), we can replace (4.31) by

$$
\begin{aligned}
&P_f^+\big(\mathscr{F}_1(\bar t^1)\cdots\mathscr{F}_{N-1}(\bar t^{N-1})\mathscr{F}_N(\bar t^N)\big)=\mathscr{F}_1(\bar t^1)\cdots\mathscr{F}_{N-1}(\bar t^{N-1})\mathscr{F}_N(\bar t^N)\\
&\quad-\sum_{\ell=1}^{N}\mathbb{C}_\ell\,\overline{\mathrm{Sym}}_{\bar t^\ell,\dots,\bar t^N}\big[\mathbb{G}_\ell(\bar t^{\ell-1},\dots,\bar t^N)c^{-1}_{[\ell,N+1]}P_f^-\big(F_{N+1,\ell}(t_1^N)\big)\\
&\qquad\qquad\times P_f^+\big(\mathscr{F}_1(\bar t^1)\cdots\mathscr{F}_{\ell-1}(\bar t^{\ell-1})\cdot\mathscr{F}_\ell(\bar t_1^\ell)\cdots\mathscr{F}_N(\bar t_1^N)\big)\big]+\mathbb{W}.
\end{aligned}
\tag{4.32}
$$

Now we can use a result from [28] asserting that only 'negative' projections of composed currents $P_f^-\big(F_{N+1,\ell}(t_1^N)\big)$, $\ell=1,\dots,N$, appear on the right-hand side of (4.27). This allows us to replace the first term $\mathscr{F}_1(\bar t^1)\cdots\mathscr{F}_{N-1}(\bar t^{N-1})\mathscr{F}_N(\bar t^N)$ on the right-hand side of (4.32) by

$$
P_f^+\big(\mathscr{F}_1(\bar t^1)\cdots\mathscr{F}_{N-1}(\bar t^{N-1})\big)\mathscr{F}_N(\bar t^N).
$$

Similarly, the 'positive' projections of products of composed currents

$$
P_f^+\big(\mathscr{F}_1(\bar t^1)\cdots\mathscr{F}_{\ell-1}(\bar t^{\ell-1})\cdot\mathscr{F}_\ell(\bar t_1^\ell)\cdots\mathscr{F}_N(\bar t_1^N)\big)
$$

under the sum sign in (4.32) can be replaced by

$$P_f^+\big(\mathscr{F}_1(\bar t^1)\cdots\mathscr{F}_{\ell-1}(\bar t^{\ell-1})\cdot\mathscr{F}_\ell(\bar t^\ell_1)\cdots\mathscr{F}_{N-1}(\bar t_1^{N-1})\big)\cdot\mathscr{F}_N(\bar t_1^N),$$

and this replacement changes only the structure of the elements in $\mathbb{W}$. This finishes the proof of Proposition 4.4. □

Similarly, using (3.19) and (4.18), we can prove the following statement.

Proposition 4.5. *The equality*

$$\begin{aligned}\widehat P_f^+\big(\widehat{\mathscr{F}}_N(\bar t^N)\cdots\widehat{\mathscr{F}}_1(\bar t^1)\big) &= P_f^+\big(\widehat{\mathscr{F}}_N(\bar t^N)\cdots\widehat{\mathscr{F}}_2(\bar t^2)\big)\cdot\widehat{\mathscr{F}}_1(\bar t^1)\\ &-\sum_{\ell=1}^{N}\overline{\mathrm{Sym}}_{\bar t^1,\dots,\bar t^\ell}\big[\widehat{\mathbb{C}}_\ell\widehat{\mathbb{G}}_\ell(\bar t^1,\dots,\bar t^{\ell+1})c^{-1}_{[1,\ell+1]}\widehat P_f^-\big(\widehat F_{\ell+1,1}(t^1_{r_1})\big)\\ &\times\widehat P_f^+\big(\widehat{\mathscr{F}}_N(\bar t^N)\cdots\widehat{\mathscr{F}}_{\ell+1}(\bar t^{\ell+1})\widehat{\mathscr{F}}_\ell(\bar t^\ell_{r_\ell})\cdots\widehat{\mathscr{F}}_2(\bar t^2_{r_2})\big)\cdot\widehat{\mathscr{F}}_1(\bar t^1_{r_1})\big]+\widehat{\mathbb{W}}\end{aligned}\tag{4.33}$$

holds, where for $1\leqslant\ell\leqslant N$ *the rational functions*

$$\widehat{\mathbb{G}}_\ell(\bar t^1,\dots,\bar t^{\ell+1})=f_{[\ell+1]}(\bar t^{\ell+1},t^\ell_{r_\ell})\prod_{s=1}^{\ell-1}g_{[s+1]}(t^{s+1}_{r_{s+1}},t^s_{r_s})f_{[s+1]}(\bar t^{s+1}_{r_{s+1}},t^s_{r_s})\tag{4.34}$$

appear along with the combinatorial factors

$$\widehat{\mathbb{C}}_\ell=\prod_{s=1}^{\ell}\frac{1}{(r_s-1)!}\,.\tag{4.35}$$

The symbol $\widehat{\mathbb{W}}$ *denotes terms with the structure* $P_f^-\big(\widehat F_{j_1,1}(w_1)\big)P_f^-\big(\widehat F_{j_2,1}(w_2)\big)$.

Again, the action of any monodromy matrix element $\mathrm{T}^+_{i,j}(z)$ on $\widehat{\mathbb{W}}$ belongs to the ideal $\widehat I$ in view of (4.13). The terms not belonging to this ideal are proportional to $\delta_{i,j_1}\delta_{1,j_2}$, and they vanish due to the condition $1<j_2$.

4.4. Action of the monodromy matrix element $\mathrm{T}^+_{i,j}(z)$. Let us apply the monodromy matrix element $\mathrm{T}^+_{i,j}(z)$ from the left to (4.24) and (4.33). As one can easily verify, the structure of the action formulae differs significantly in the cases $i\leqslant j$ and $i>j$.

The action of the monodromy matrix elements $\mathrm{T}_{i,j}(z)$ for $i<j$ leads to recursion relations which relate Bethe vectors depending on fewer Bethe parameters to Bethe vectors depending on more of these parameters. If we prove that the action formulae for $i<j$ are the same for $\mathbb{B}(\bar t)$ and $\widehat{\mathbb{B}}(\bar t)$, then this will mean that these vectors satisfy the same recurrence relations, and thus $\mathbb{B}(\bar t)$ and $\widehat{\mathbb{B}}(\bar t)$ coincide.

The action formulae for the diagonal monodromy matrix elements $\mathrm{T}_{i,i}(z)$ lead to the Bethe equations. They prove that the Bethe vectors become eigenvectors of the transfer matrix if the Bethe equations are satisfied.

Finally, the action formulae for the monodromy matrix elements $\mathrm{T}_{i,j}(z)$ with $i>j$ are necessary for calculating the scalar products of Bethe vectors. This last problem is beyond the scope of the present paper, and we will consider the general action formulae in this case in a separate publication. From now on, we restrict ourselves to the action of the monodromy matrix elements $\mathrm{T}_{i,j}(z)$ with $i \leqslant j$.

We introduce the shortened notation

$$\mathscr{F}_\ell \equiv \mathscr{F}(\bar t^\ell), \quad \mathscr{F}'_\ell \equiv \mathscr{F}(\bar t^\ell_1), \quad \text{and} \quad \mathscr{F}''_\ell \equiv \mathscr{F}(\bar t^\ell_{r_\ell})$$

and the analogous notation

$$\widehat{\mathscr{F}}_\ell \equiv \widehat{\mathscr{F}}(\bar t^\ell), \quad \widehat{\mathscr{F}}'_\ell \equiv \widehat{\mathscr{F}}(\bar t^\ell_1) \quad \text{and} \quad \widehat{\mathscr{F}}''_\ell \equiv \widehat{\mathscr{F}}(\bar t^\ell_{r_\ell}),$$

where $\bar t^\ell_1 = \{\bar t^\ell\} \setminus \{t^\ell_1\}$ and $\bar t^\ell_{r_\ell} = \{\bar t^\ell\} \setminus \{t^\ell_{r_\ell}\}$ are the sets of Bethe parameters of the same type with either the first or the last element omitted.

For $1 \leqslant \ell \leqslant N$ we introduce the two sets of rational functions

$$\begin{aligned} \mathbb{G}^q_\ell(\bar t^{\ell-1}, \dots, \bar t^q) &= f_{[\ell]}(t^\ell_1, \bar t^{\ell-1}) \\ &\quad \times \prod_{s=\ell}^{q-1} g_{[s+1]}(t^{s+1}_1, t^s_1) f_{[s+1]}(t^{s+1}_1, \bar t^s_1), \qquad \ell \leqslant q \leqslant N, \\ \widehat{\mathbb{G}}^p_\ell(\bar t^p, \dots, \bar t^{\ell+1}) &= f_{[\ell+1]}(\bar t^{\ell+1}, t^\ell_{r_\ell}) \\ &\quad \times \prod_{s=p}^{\ell-1} g_{[s+1]}(t^{s+1}_{r_{s+1}}, t^s_{r_s}) f_{[s+1]}(\bar t^{s+1}_{r_{s+1}}, t^s_{r_s}), \qquad 1 \leqslant p \leqslant \ell. \end{aligned} \tag{4.36}$$

The rational functions in (4.25) and (4.34) are particular cases of the functions in (4.36):

$$\mathbb{G}_\ell(\bar t) \equiv \mathbb{G}^N_\ell(\bar t) \quad \text{and} \quad \widehat{\mathbb{G}}(\bar t) \equiv \widehat{\mathbb{G}}^1_\ell(\bar t).$$

For $q = j+1, \dots, N+1$ and $p = 1, \dots, i-1$ we also define the rational functions

$$\mathbb{Z}^q_j(z; \bar t) = g(z, t^{q-1}_1) \mathbb{G}^{q-1}_j(\bar t) \quad \text{and} \quad \widehat{\mathbb{Z}}^p_i(z; \bar t) = g(z, t^p_{r_p}) \widehat{\mathbb{G}}^p_{i-1}(\bar t). \tag{4.37}$$

We extend these definitions to $q = j$ and $p = i$ by setting $\mathbb{Z}^j_j(z; \bar t) = \widehat{\mathbb{Z}}^i_i(z; \bar t) \equiv 1$. Finally, let

$$\mathbb{C}^{\ell'}_\ell = \prod_{s=\ell}^{\ell'} \frac{1}{(r_s - 1)!}.$$

Then the combinatorial factors given by (4.26) and (4.35) are

$$\mathbb{C}_\ell \equiv \mathbb{C}^N_\ell \quad \text{and} \quad \widehat{\mathbb{C}}_\ell \equiv \mathbb{C}^\ell_1.$$

Proposition 4.6. *The following equivalence relations hold*:

$$\begin{aligned} \mathrm{T}^+_{i,j}(z) \cdot P^+_f\big(\mathscr{F}_1 \cdots \mathscr{F}_N\big) &\sim_{I,K} \sum_{q=j}^{N+1} \overline{\mathrm{Sym}}_{\bar t^j, \dots, \bar t^{q-1}} \big[\phi_q \mathbb{C}^{q-1}_j \mathbb{Z}^q_j(z; \bar t) \\ &\quad \times \mathrm{T}^+_{i,q}(z) \cdot \mathscr{F}_1 \cdots \mathscr{F}_{j-1} \mathscr{F}'_j \cdots \mathscr{F}'_{q-1}\big] \cdot \mathscr{F}_q \cdots \mathscr{F}_N, \end{aligned} \tag{4.38}$$

$$\begin{aligned} \mathrm{T}^+_{i,j}(z) \cdot \widehat{P}^+_f\big(\widehat{\mathscr{F}}_N \cdots \widehat{\mathscr{F}}_1\big) &\sim_{\widehat{I},\widehat{K}} \sum_{p=1}^{i} \overline{\mathrm{Sym}}_{\bar t^p, \dots, \bar t^{i-1}} \big[\widehat{\phi}_p \mathbb{C}^p_{i-1} \widehat{\mathbb{Z}}^p_i(z; \bar t) \\ &\quad \times \mathrm{T}^+_{p,j}(z) \cdot \widehat{\mathscr{F}}_N \cdots \widehat{\mathscr{F}}_i \widehat{\mathscr{F}}''_{i-1} \cdots \widehat{\mathscr{F}}''_p\big] \cdot \widehat{\mathscr{F}}_{p-1} \cdots \widehat{\mathscr{F}}_1, \end{aligned} \tag{4.39}$$

where the sign factors ϕ_q *for* $q = j+1, \ldots, N+1$ *and* $\widehat{\phi}_p$ *for* $p = 1, \ldots, i-1$ *are given by* (4.14), *and* $\phi_j = \widehat{\phi}_i \equiv 1$.

Proof. We begin the proof with the relation (4.38). Assume that $j = N+1$. Then by (4.12), under the action of $\mathrm{T}^+_{i,N+1}(z)$ the sum over ℓ on the right-hand side of (4.24) and also the terms $\mathbb{W}$ give elements of the ideal I. As a result,

$$\begin{aligned}\mathrm{T}^+_{i,N+1}(z) \cdot P^+_f\big(\mathscr{F}_1(\bar t^1) \cdots \mathscr{F}_N(\bar t^N)\big) \\ \sim_{I,K} \mathrm{T}^+_{i,N+1}(z) \cdot P^+_f\big(\mathscr{F}_1(\bar t^1) \cdots \mathscr{F}_{N-1}(\bar t^{N-1})\big) \cdot \mathscr{F}_N(\bar t^N).\end{aligned} \tag{4.40}$$

Again using (4.24) for the projection $P^+_f\big(\mathscr{F}_1(\bar t^1) \cdots \mathscr{F}_{N-1}(\bar t^{N-1})\big)$, we can continue this process and get that

$$\mathrm{T}^+_{i,N+1}(z) \cdot P^+_f\big(\mathscr{F}_1(\bar t^1) \cdots \mathscr{F}_N(\bar t^N)\big) \sim_{I,K} \mathrm{T}^+_{i,N+1}(z) \cdot \mathscr{F}_1(\bar t^1) \cdots \mathscr{F}_N(\bar t^N). \tag{4.41}$$

Assume now that $j \leqslant N$. Then by (4.12), besides the first term as in (4.40) there will be a contribution of the term corresponding to $\ell = j$ in the sum on the right-hand side of (4.24), so that

$$\begin{aligned}\mathrm{T}^+_{i,j}(z) \cdot P^+_f\big(\mathscr{F}_1 \cdots \mathscr{F}_N\big) &\sim_{I,K} \mathrm{T}^+_{i,j}(z) \cdot P^+_f\big(\mathscr{F}_1 \cdots \mathscr{F}_{N-1}\big) \cdot \mathscr{F}_N \\ &+ \overline{\mathrm{Sym}}_{\bar t^j,\ldots,\bar t^N}\big[\phi_{N+1} g(z, t^N_1) \mathbb{C}^N_j \mathbb{G}^N_j(\bar t) \\ &\qquad \times \mathrm{T}^+_{i,N+1}(z) \cdot \mathscr{F}_1 \cdots \mathscr{F}_{j-1} \mathscr{F}'_j \cdots \mathscr{F}'_{N-1} \cdot \mathscr{F}'_N\big].\end{aligned} \tag{4.42}$$

In view of (4.41) and (4.12) we can omit the projection operator P^+_f applied to the product of currents $\mathscr{F}_1 \cdots \mathscr{F}_{j-1} \mathscr{F}'_j \cdots \mathscr{F}'_{N-1}$.

We leave the second term on the right-hand side of (4.42) as it is and consider the first term. In this term we have the projection $P^+_f\big(\mathscr{F}_1 \cdots \mathscr{F}_{N-1}\big)$ and we can again use the presentation (4.24) for a product of currents in the smaller-rank algebra $\mathfrak{gl}(m|n-1)$. As before, the only contribution comes from the regular term and the one term with $\ell = j$ in the sum over ℓ. We obtain

$$\begin{aligned}\mathrm{T}^+_{i,j}(z) \cdot P^+_f(\mathscr{F}_1 \cdots \mathscr{F}_N) &\sim_{I,K} \mathrm{T}^+_{i,j}(z) \cdot P^+_f(\mathscr{F}_1 \cdots \mathscr{F}_{N-2}) \cdot \mathscr{F}_{N-1}\mathscr{F}_N \\ &+ \overline{\mathrm{Sym}}_{\bar t^j,\ldots,\bar t^{N-1}}[\phi_N g(z, t^{N-1}_1) \mathbb{C}^{N-1}_j \mathbb{G}^{N-1}_j(\bar t) \\ &\qquad \times \mathrm{T}^+_{i,N}(z) \cdot \mathscr{F}_1 \cdots \mathscr{F}_{j-1} \mathscr{F}'_j \cdots \mathscr{F}'_{N-1}] \cdot \mathscr{F}_N \\ &+ \overline{\mathrm{Sym}}_{\bar t^j,\ldots,\bar t^N}[\phi_{N+1} g(z, t^N_1) \mathbb{C}^N_j \mathbb{G}^N_j(\bar t) \mathrm{T}^+_{i,N+1}(z) \cdot \mathscr{F}_1 \cdots \mathscr{F}_{j-1} \mathscr{F}'_j \cdots \mathscr{F}'_N].\end{aligned}$$

Continuing this process, we conclude that the action of the monodromy matrix element $\mathrm{T}^+_{i,j}(z)$ on the projection $P^+_f(\mathscr{F}_1 \cdots \mathscr{F}_N)$ modulo elements in the ideals I and K is given by

$$\begin{aligned}\mathrm{T}^+_{i,j}(z) \cdot P^+_f(\mathscr{F}_1 \cdots \mathscr{F}_N) &\sim_{I,K} \mathrm{T}^+_{i,j}(z) \cdot \mathscr{F}_1 \cdots \mathscr{F}_N \\ &+ \sum_{q=j+1}^{N+1} \overline{\mathrm{Sym}}_{\bar t^j,\ldots,\bar t^{q-1}}[\phi_q g(z, t^{q-1}_1) \mathbb{C}^{q-1}_j \mathbb{G}^{q-1}_j(\bar t) \\ &\qquad \times \mathrm{T}^+_{i,q}(z) \cdot \mathscr{F}_1 \cdots \mathscr{F}_{j-1} \mathscr{F}'_j \cdots \mathscr{F}'_{q-1}] \cdot \mathscr{F}_q \cdots \mathscr{F}_N.\end{aligned} \tag{4.43}$$

Using the relation (4.33) and arguments similar to those above, we get that

$$\mathrm{T}^{+}_{i,j}(z)\cdot \widehat{P}^{+}_{f}(\widehat{\mathscr{F}}_N\cdots\widehat{\mathscr{F}}_1)\sim_{\widehat{I},\widehat{K}}\mathrm{T}^{+}_{i,j}(z)\cdot\widehat{\mathscr{F}}_N\cdots\widehat{\mathscr{F}}_1 \\ +\sum_{p=1}^{i-1}\overline{\mathrm{Sym}}_{\bar t^{p},\dots,\bar t^{i-1}}[\widehat{\phi}_p g(z,t^p_{r_p})\widehat{\mathbb{C}}^p_{i-1}\widehat{\mathbb{G}}^p_{i-1}(\bar t) \\ \times \mathrm{T}^{+}_{p,j}(z)\cdot\widehat{\mathscr{F}}_N\cdots\widehat{\mathscr{F}}_i\widehat{\mathscr{F}}''_{i-1}\cdots\widehat{\mathscr{F}}''_p]\cdot\widehat{\mathscr{F}}_{p-1}\cdots\widehat{\mathscr{F}}_1. \tag{4.44}$$

With the notation (4.37) the formulae (4.43) and (4.44) are equivalent to the assertion of Proposition 4.6. □

The next step is to use the explicit representations for the monodromy matrix element $\mathrm{T}^{+}_{i,j}(z)$ in terms of the Gauss coordinates (2.14),

$$\mathrm{T}^{+}_{i,q}(z)=\sum_{1\leqslant p\leqslant i}\mathrm{F}^{+}_{q,p}(z)k^{+}_p(z)\mathrm{E}^{+}_{p,i}(z), \tag{4.45}$$

and in terms of the 'hatted' Gauss coordinates (2.17),

$$\mathrm{T}^{+}_{p,j}(z)=\sum_{j\leqslant q\leqslant N+1}(-)^{([q]+[p])([q]+[j])}\widehat{\mathrm{F}}^{+}_{q,p}(z)\widehat{k}^{+}_q(z)\widehat{\mathrm{E}}^{+}_{j,q}(z), \tag{4.46}$$

where we have formally set $\mathrm{F}^{+}_{i,i}(z)=\widehat{\mathrm{F}}^{+}_{j,j}(z)=\mathrm{E}^{+}_{i,i}(z)=\widehat{\mathrm{E}}^{+}_{j,j}(z)\equiv 1$. These representations allow us to move the Gauss coordinates $\mathrm{E}^{+}_{p,i}(z)$ and $\widehat{\mathrm{E}}^{+}_{j,q}(z)$ through the corresponding products of currents.

As we will demonstrate below, for $i\leqslant j$ these permutations transform the product of currents in (4.38) into the product

$$\mathscr{F}_1\cdots\mathscr{F}_{p-1}\cdot\mathscr{F}''_p\cdots\mathscr{F}''_{i-1}\cdot\mathscr{F}_i\cdots\mathscr{F}_{j-1}\cdot\mathscr{F}'_j\cdots\mathscr{F}'_{q-1}\cdot\mathscr{F}_q\cdots\mathscr{F}_N, \tag{4.47}$$

and the product of currents in (4.39) into the product

$$\widehat{\mathscr{F}}_N\cdots\widehat{\mathscr{F}}_q\cdot\widehat{\mathscr{F}}'_{q-1}\cdots\widehat{\mathscr{F}}'_j\cdot\widehat{\mathscr{F}}_{j-1}\cdots\widehat{\mathscr{F}}_i\cdot\widehat{\mathscr{F}}''_{i-1}\cdots\widehat{\mathscr{F}}''_p\cdot\widehat{\mathscr{F}}_{p-1}\cdots\widehat{\mathscr{F}}_1 \tag{4.48}$$

for $p=1,\dots,i$ and $q=j,\dots,N+1$.

According to (A.29) and (A.35), the Gauss coordinates $\mathrm{F}^{+}_{q,p}(z)$ and $\widehat{\mathrm{F}}^{+}_{q,p}(z)$ can be replaced by the total composed currents $F_{q,p}(z)$ and $\widehat{F}_{q,p}(z)$ modulo terms in the ideal I. Then by (A.5) and (A.7) the products of currents

$$\mathscr{F}''_p\cdots\mathscr{F}''_{i-1}\cdot\mathscr{F}_i\cdots\mathscr{F}_{j-1}\cdot\mathscr{F}'_j\cdots\mathscr{F}'_{q-1}$$

and

$$\widehat{\mathscr{F}}'_{q-1}\cdots\widehat{\mathscr{F}}'_j\cdot\widehat{\mathscr{F}}_{j-1}\cdots\widehat{\mathscr{F}}_i\cdot\widehat{\mathscr{F}}''_{i-1}\cdots\widehat{\mathscr{F}}''_p \tag{4.49}$$

in (4.47) and (4.48) will be extended by the corresponding simple root currents depending on the auxiliary parameter z.

This observation shows that the action of the monodromy matrix element $\mathrm{T}^{+}_{i,j}(z)$ on the projections of currents $P^{+}_{f}(\mathscr{F}_1\cdots\mathscr{F}_N)$ and $\widehat{P}^{+}_{f}(\widehat{\mathscr{F}}_N\cdots\widehat{\mathscr{F}}_1)$ have a similar structure. This is the first sign that the recursion relations for the Bethe vectors (3.14) and (3.22) coincide.

Let us be more precise. In view of (A.37) the Gauss coordinate $\mathrm{E}^+_{p,i}(z)$ commutes with all the products of currents $\mathscr{F}_q \cdots \mathscr{F}_{j-1}$ except $\mathscr{F}_p \cdots \mathscr{F}_{i-1}$. This is because by (A.37) the Gauss coordinate $\mathrm{E}^+_{p,i}(z)$ is constructed from modes of the currents $E_p(z), E_{p+1}(z), \ldots, E_{i-1}(z)$. From the commutation relations (2.28) for the simple root total currents we obtain the commutation relations of the simple root Gauss coordinates,

$$[\mathrm{E}^+_{i,i+1}(v), \mathrm{F}^+_{i+1,i}(u)\} = \frac{c_{[i+1]}}{(v-u)_{\lessgtr}}\big(k^+_{i+1}(v)k^+_i(v)^{-1} - k^+_{i+1}(u)k^+_i(u)^{-1}\big),$$
$$[\mathrm{E}^+_{i,i+1}(v), \mathrm{F}^-_{i+1,i}(u)\} = \frac{c_{[i+1]}}{(v-u)_{>}}\big(k^+_{i+1}(v)k^+_i(v)^{-1} - k^-_{i+1}(u)k^-_i(u)^{-1}\big),$$

which also follow from (2.8). From this we conclude that

$$[\mathrm{E}^+_{p,p+1}(z), F_p(t)\} \sim_K g_{[p+1]}(t,z)\psi^+_p(t),$$

where

$$\psi^+_p(t) = k^+_{p+1}(t)k^+_p(t)^{-1}.$$

We recall that $[\,\cdot\,,\,\cdot\,\}$ is the graded commutator defined in (2.28). Using this commutation relation, the commutation relations of the Cartan currents with the total currents $F_p(t)$, and the definition of deformed symmetrization (3.3), we have

$$[\mathrm{E}^+_{p,p+1}(z), \mathscr{F}_p(\bar t^p)\} \sim_K \frac{(-)^{(r_p-1)\delta_{p,m}}}{(r_p-1)!}\,\overline{\mathrm{Sym}}_{\bar t^p}\big[g_{[p+1]}(t^p_{r_p},z)\mathscr{F}_p(\bar t^p_{r_p})\psi^+_p(t^p_{r_p})\big]. \quad (4.50)$$

Let us explain the appearance of the phase factor $(-)^{(r_m-1)}$ in this formula for $p=m$. Using the definition of the graded commutator in (2.28), the commutativity $\psi^+_m(t)F_m(t') = F_m(t')\psi^+_m(t)$, and the anticommutativity of the currents $F_m(t)$, we conclude that

$$\begin{aligned}
&[\mathrm{E}^+_{m,m+1}(z), \mathscr{F}_m(\bar t^m)\}\\
&\quad\sim_K \sum_{\ell=1}^{r_m}(-)^{\ell-1}g(z,t^m_\ell)F_m(t^m_1)\cdots F_m(t^m_{\ell-1})\psi^+_m(t^m_\ell)F_m(t^m_{\ell+1})\cdots F_m(t^m_{r_m})\\
&\quad\sim_K \frac{(-)^{(r_m-1)}}{(r_m-1)!}\,\mathrm{ASym}_{\bar t^m}\big(g(z,t^m_{r_m})F_m(t^m_1)\cdots F_m(t^m_{r_m-1})\psi^+_m(t^m_{r_m})\big),
\end{aligned}$$

where the symbol $\mathrm{ASym}_{\bar t^m}(\,\cdot\,)$ stands for antisymmetrization over the set of variables $\bar t^m$. It coincides with the deformed symmetrization $\overline{\mathrm{Sym}}_{\bar t^m}(\,\cdot\,)$ (see (3.3)) over the same set.

Within the product of screening operators $\mathscr{S}_{E^{(0)}_{i-1}} \cdots \mathscr{S}_{E^{(0)}_{p+1}}$ in the formula (A.37) for the Gauss coordinate $\mathrm{E}^+_{p,i}(z)$, only the screening operator $\mathscr{S}_{E^{(0)}_{p+1}}$ does not commute with the Cartan current $k^+_{p+1}(t^p_{r_p})$:

$$\mathscr{S}_{E^{(0)}_{p+1}}\big(k^+_{p+1}(t^p_{r_p})\big) = -c_{[p+1]}k^+_{p+1}(t^p_{r_p})\mathrm{E}^+_{p+1,p+2}(t^p_{r_p}),$$

which can be obtained from the commutation relation (2.23). Again using (A.37), we find that

$$[\mathrm{E}^+_{p,i}(z),\mathscr{F}_p(\bar t^p)\}\sim_K \frac{(-)^{(r_p-1)\delta_{p,m}}}{(r_p-1)!}\,\overline{\mathrm{Sym}}_{\bar t^p}[g_{[p+1]}(t^p_{r_p},z)\mathscr{F}_p(\bar t^p_{r_p})\psi^+_p(t^p_{r_p})\mathrm{E}^+_{p+1,i}(t^p_{r_p})]. \tag{4.51}$$

In view of the result

$$\mathrm{E}^+_{p,i}(z)\cdot\mathscr{F}_{p+1}\cdots\mathscr{F}_{i-1}\sim_J 0$$

we can represent (4.51) as an action of the Gauss coordinate $\mathrm{E}^+_{p,i}(z)$ on the product of currents $\mathscr{F}_p\cdots\mathscr{F}_{i-1}$ modulo elements in the ideals K and J

$$\begin{aligned}\mathrm{E}^+_{p,i}(z)\cdot\mathscr{F}_p(\bar t^p)\cdots\mathscr{F}_i(\bar t^i)\sim_{K,J}&\frac{(-)^{(r_p-1)\delta_{p,m}}}{(r_p-1)!}\,\overline{\mathrm{Sym}}_{\bar t^p}[g_{[p+1]}(t^p_{r_p},z)\mathscr{F}_p(\bar t^p_{r_p})\psi^+_p(t^p_{r_p})\\&\times\mathrm{E}^+_{p+1,i}(t^p_{r_p})\cdot\mathscr{F}_{p+1}(\bar t^{p+1})\cdots\mathscr{F}_i(\bar t^i)].\end{aligned}\tag{4.52}$$

In the last line of (4.52) we can use (4.51) again, and by repeating the calculations finally get that

$$\begin{aligned}\mathrm{E}^+_{p,i}(z)\cdot\mathscr{F}_p(\bar t^p)\cdots\mathscr{F}_{i-1}(\bar t^{i-1})\mathscr{F}_i(\bar t^i)\sim_{K,J}\epsilon_p\prod_{s=p}^{i-1}(-)^{(r_s-1)\delta_{s,m}}\\\times\overline{\mathrm{Sym}}_{\bar t^p,\dots,\bar t^{i-1}}\Big[\mathbb{C}^{i-1}_p\ \widehat{\mathbb{Z}}^p_i(z;\bar t)\mathscr{F}_p(\bar t^p_{r_p})\cdots\mathscr{F}_{i-1}(\bar t^{i-1}_{r_{i-1}})\mathscr{F}_i(\bar t^i)\\\times\prod_{s=p}^{i-1}k^+_{s+1}(t^s_{r_s})k^+_s(t^s_{r_s})^{-1}\Big],\end{aligned}\tag{4.53}$$

where ϵ_p is the sign factor

$$\epsilon_i=1\quad\text{and}\quad\epsilon_p=(-)^{1+[i]}\quad\text{for } p=1,2,\dots,i-1. \tag{4.54}$$

We recall that the rational function $\widehat{\mathbb{Z}}^p_i(z;\bar t)$ is defined by (4.36) and (4.37).

Similarly, taking into account that the Gauss coordinate $\widehat{\mathrm{E}}^+_{j,q}(z)$ does not commute only with the product of currents $\widehat{\mathscr{F}}_{q-1}(\bar t^{q-1})\cdots\widehat{\mathscr{F}}_j(\bar t^j)$ in the product (4.49), we find that

$$\begin{aligned}&(-)^{([q]+[p])([q]+[j])}\widehat{\mathrm{E}}^+_{j,q}(z)\cdot\widehat{\mathscr{F}}_{q-1}(\bar t^{q-1})\cdots\widehat{\mathscr{F}}_j(\bar t^j)\widehat{\mathscr{F}}_{j-1}(\bar t^{j-1})\\&\sim_{\widehat K,\widehat J}\widehat\epsilon_q\prod_{s=j}^{q-1}(-)^{(r_s-1)\delta_{s,m}}\,\overline{\mathrm{Sym}}_{\bar t^j,\dots,\bar t^{q-1}}\Big[\mathbb{C}^{q-1}_j\mathbb{Z}^q_j(z;\bar t)\widehat{\mathscr{F}}_{q-1}(\bar t^{q-1}_1)\times\cdots\\&\qquad\times\widehat{\mathscr{F}}_j(\bar t^j_1)\widehat{\mathscr{F}}_{j-1}(\bar t^{j-1})\prod_{s=j}^{q-1}\widehat k^+_s(t^s_1)\widehat k^+_{s+1}(t^s_1)^{-1}\Big],\end{aligned}\tag{4.55}$$

where $\widehat\epsilon_q$ is the sign factor

$$\widehat\epsilon_j=1\quad\text{and}\quad\widehat\epsilon_q=(-)^{([j]+[p])[q]+[j][p]}\quad\text{for } q=j+1,j+2,\dots,N, \tag{4.56}$$

and the rational function $\mathbb{Z}^q_j(z;\bar t)$ is defined by (4.36) and (4.37).

The Gauss coordinates $\mathrm{F}^+_{q,p}(z)$ and $\widehat{\mathrm{F}}^+_{q,p}(z)$ in (4.45) and (4.46) can be replaced by the products of the corresponding currents (see the formulae (A.5), (A.29) and (A.7), (A.34), respectively):

$$\mathrm{F}^+_{q,p}(z) \sim_I \prod_{s=p}^{q-2} f_{[s+1]}(z_{s+1}, z_s)^{-1} F_p(z_p) \cdots F_{q-1}(z_{q-1})\big|_{z_p=\cdots=z_{q-1}=z}, \tag{4.57}$$

$$\widehat{\mathrm{F}}^+_{q,p}(z) \sim_{\widehat{I}} \prod_{s=p}^{q-2} f_{[s+1]}(z_{s+1}, z_s)^{-1} \widehat{F}_{q-1}(z_{q-1}) \cdots \widehat{F}_p(z_p)\big|_{z_p=\cdots=z_{q-1}=z}, \tag{4.58}$$

where we have changed the order in the products of currents and have introduced an auxiliary set of variables $\bar{z} = \{z_p, \ldots, z_{q-1}\}$, which in the end should all be set equal to the parameter z.

Combining (4.38), the Gauss decomposition (4.45), the action (4.53) of the Gauss coordinates $\mathrm{E}^+_{p,i}(z)$, and the formula (4.57), we can obtain the action formulae of the monodromy matrix elements $\mathrm{T}^+_{i,j}(z)$ on the unnormalized Bethe vector

$$\mathscr{B}(\bar{t}\,) = P^+_f\big(\mathscr{F}(\bar{t}\,)\big) \prod_{\ell=1}^{N} \lambda_\ell(\bar{t}^\ell)|0\rangle,$$

where the ordered product of simple root currents $\mathscr{F}(\bar{t}\,)$ is given by (3.12). We have

$$\begin{aligned}
\mathrm{T}^+_{i,j}(z) \cdot \mathscr{B}(\bar{t}\,) &= \sum_{p=1}^{i} \sum_{q=j}^{N+1} \phi_q \epsilon_p \mathbb{C}_p^{i-1} \mathbb{C}_j^{q-1} \prod_{s=p}^{i-1} (-)^{(r_s-1)\delta_{s,m}} \\
&\times \overline{\mathrm{Sym}}_{\bar{t}^p,\ldots,\bar{t}^{i-1},\bar{t}^j,\ldots,\bar{t}^{q-1}} \Bigg[\frac{\widehat{\mathbb{Z}}_i^p(z; \bar{t}^p, \ldots, \bar{t}^i) \mathbb{Z}_j^q(z; \bar{t}^{j-1}, \ldots, \bar{t}^{q-1})}{\mathbb{X}(\bar{z}; \bar{t}^p, \ldots, \bar{t}^{q-1})} \\
&\times \mathscr{B}\big(\bar{t}^1, \ldots, \bar{t}^{p-1}, \{z_p, \bar{t}^p_{r_p}\}, \ldots, \{z_{i-1}, \bar{t}^{i-1}_{r_{i-1}}\}, \{z_i, \bar{t}^i\}, \ldots, \{z_{j-1}, \bar{t}^{j-1}\}, \\
&\qquad \{z_j, \bar{t}^j_1\}, \ldots, \{z_{q-1}, \bar{t}^{q-1}_1\}, \bar{t}^q, \ldots, \bar{t}^N\big) \\
&\times \frac{\lambda_{p+1}(t^p_{r_p}) \cdots \lambda_i(t^{i-1}_{r_{i-1}}) \lambda_j(t^j_1) \cdots \lambda_{q-1}(t^{q-1}_1)}{\lambda_p(z_p) \cdots \lambda_{q-1}(z_{q-1})} \lambda_p(z) \Bigg]\Bigg|_{z_p=\cdots=z_{q-1}=z},
\end{aligned} \tag{4.59}$$

where we have introduced yet another rational function $\mathbb{X}(\bar{z}, \bar{t}^p, \ldots, \bar{t}^{q-1})$ depending on the auxiliary set $\bar{z}$ and the Bethe parameters:

$$\begin{aligned}
\mathbb{X}(\bar{z}; \bar{t}^p, \ldots, \bar{t}^{q-1}) &= \prod_{s=p}^{i-1} f_{[s+1]}(z_{s+1}, \{z_s, \bar{t}^s_{r_s}\}) \\
&\times \prod_{s=i}^{j-1} f_{[s+1]}(z_{s+1}, \{z_s, \bar{t}^s\}) \prod_{s=j}^{q-2} f_{[s+1]}(z_{s+1}, \{z_s, \bar{t}^s_1\}) f_{[p]}(\bar{t}^p_{r_p}, z_p)^{-1}.
\end{aligned} \tag{4.60}$$

Similarly, using (4.39), the Gauss decomposition (4.46), the action (4.55) of the Gauss coordinate $\widehat{\mathrm{E}}^+_{j,q}(z)$, and the formula (4.58), we can calculate the action formula of the monodromy matrix element $\mathrm{T}^+_{i,j}(z)$ on the unnormalized Bethe vector

$$\widehat{\mathscr{B}}(\bar t) = \widehat{P}^+_f\big(\widehat{\mathscr{F}}(\bar t)\big)\prod_{\ell=1}^{N}\widehat{k}^+_{\ell+1}(\bar t^\ell)|0\rangle,$$

where the ordered product of currents $\widehat{\mathscr{F}}(\bar t)$ is given by (3.21). We have

$$\begin{aligned}\mathrm{T}^+_{i,j}(z)\cdot\widehat{\mathscr{B}}(\bar t) &= \sum_{p=1}^{i}\sum_{q=j}^{N+1}\widehat{\phi}_p\widehat{\epsilon}_q\mathbb{C}_p^{i-1}\mathbb{C}_j^{q-1}\prod_{s=j}^{q-1}(-)^{(r_s-1)\delta_{s,m}}\\ &\times\overline{\mathrm{Sym}}_{\bar t^p,\dots,\bar t^{i-1},\bar t^j,\dots,\bar t^{q-1}}\Bigg[\frac{\widehat{\mathbb{Z}}^p_i(z;\bar t^p,\dots,\bar t^i)\mathbb{Z}^q_j(z;\bar t^{j-1},\dots,\bar t^{q-1})}{\widehat{\mathbb{X}}(\bar z;\bar t^p,\dots,\bar t^{q-1})}\\ &\times\widehat{\mathscr{B}}\big(\bar t^1,\dots,\bar t^{p-1},\{z_p,\bar t^p_{r_p}\},\dots,\{z_{i-1},\bar t^{i-1}_{r_{i-1}}\},\{z_i,\bar t^i\},\dots,\{z_{j-1},\bar t^{j-1}\},\\ &\qquad\{z_j,\bar t^j_1\},\dots,\{z_{q-1},\bar t^{q-1}_1\},\bar t^q,\dots,\bar t^N\big)\\ &\times\frac{\lambda_{p+1}(t^p_{r_p})\cdots\lambda_i(t^{i-1}_{r_{i-1}})\lambda_j(t^j_1)\cdots\lambda_{q-1}(t^{q-1}_1)}{\lambda_{p+1}(z_p)\cdots\lambda_q(z_{q-1})}\lambda_q(z)\Bigg]\Bigg|_{z_p=\dots=z_{q-1}=z},\end{aligned}\tag{4.61}$$

where we have introduced another rational function $\widehat{\mathbb{X}}(\bar z,\bar t^p,\dots,\bar t^{q-1})$ depending on the auxiliary set $\bar z$ and the Bethe parameters:

$$\begin{aligned}\widehat{\mathbb{X}}\big(\bar z,\bar t^p,\dots,\bar t^{q-1}\big) &= \prod_{s=p}^{i-2}f_{[s+1]}\big(\{z_{s+1},\bar t^{s+1}_{r_{s+1}}\},z_s\big)\prod_{s=i-1}^{j-2}f_{[s+1]}\big(\{z_{s+1},\bar t^{s+1}\},z_s\big)\\ &\times\prod_{s=j-1}^{q-2}f_{[s+1]}\big(\{z_{s+1},\bar t^{s+1}_1\},z_s\big)f_{[q]}\big(z_{q-1},\bar t^{q-1}_1\big)^{-1}.\end{aligned}\tag{4.62}$$

Let us compare the phase factors in the first rows of (4.59) and (4.61). Using the definitions of these factors in (4.14), (4.54), and (4.56), we observe that $\widehat{\phi}_p = \epsilon_p$ for $p = 1,\dots,i$. On the other hand, at first glance

$$\phi_q = (-)^{[q][j]+([q]+[j])[i]}\tag{4.63}$$

seems to differ from

$$\widehat{\epsilon}_q = (-)^{[q][j]+([q]+[j])[p]}.\tag{4.64}$$

However, this is not true, because of the restrictions on p, i, j, and q. If the parities of the indices $[p]$ and $[i]$ coincide, then the factors (4.63) and (4.64) also coincide. Now consider the case where the parities of $[p]$ and $[i]$ are different. Recall that $p \leqslant i$. By the definition of the grading (see (2.1)), this means that $[p] = 0$ and $[i] = 1$. But in this subsection we consider the action of diagonal and upper triangular monodromy matrix elements $\mathrm{T}^+_{i,j}(z)$ on Bethe vectors. This means that there is the restriction $p \leqslant i \leqslant j \leqslant q$, so that if $[p] \ne [i]$, then $[j] = [q] = 1$ and both factors in (4.63) and (4.64) are equal to -1. Below we will denote these phase factors as

$$\phi_q\epsilon_p = \widehat{\phi}_p\widehat{\epsilon}_q = \varphi_{p,q}.\tag{4.65}$$

We can now restore the normalizations of the Bethe vectors (3.14) and (3.22) and observe that the actions of the diagonal and upper triangular monodromy matrix elements on these Bethe vectors lead to the same recurrence relations. This means that the Bethe vectors given by (3.14) and (3.22) coincide.

We start our restoration of the normalization with the Bethe vectors (3.14) using (4.59). Note that the deformed symmetrization in the action formula (4.61) turns into the usual symmetrization in (4.66) in view of the property (4.2). Using the explicit expressions for the rational functions (4.25), (4.34), and (4.60), we get that

$$\mathrm{T}^{+}_{i,j}(z)\cdot\mathbb{B}(\bar t)=\sum_{p=1}^{i}\sum_{q=j}^{N+1}\varphi_{p,q}\mathbb{C}_p^{i-1}\mathbb{C}_j^{q-1}\\ \times \mathrm{Sym}_{\bar t^p,\dots,\bar t^{i-1},\bar t^j,\dots,\bar t^{q-1}}\big[\mathbb{D}(\bar t)\mathbb{Y}(z,\bar t)\Lambda(z;\bar t)\mathbb{B}(\{z,\bar t\}')\big], \quad (4.66)$$

where the sign factor $\varphi_{p,q}$ is given by (4.65), and the Bethe vector $\mathbb{B}(\{z,\bar t\}')$ on the right-hand side of this equality depends on the following set of parameters:

$$\{z,\bar t\}'=\{\bar t^1,\dots,\bar t^{p-1},\{z,\bar t^p_{r_p}\},\dots,\{z,\bar t^{i-1}_{r_{i-1}}\},\{z,\bar t^i\},\dots,\{z,\bar t^{j-1}\},\\ \{z,\bar t^j_1\},\dots,\{z,\bar t^{q-1}_1\},\bar t^q,\dots,\bar t^N\}.$$

The rational function $\mathbb{D}(\bar t)$ is given by the product

$$\mathbb{D}(\bar t)=\prod_{s=p}^{i-1}\frac{f_{[s]}(t^s_{r_s},\bar t^s_{r_s})}{[(-)^{r_s-1}h(t^s_{r_s},\bar t^s_{r_s})]^{\delta_{s,m}}}\prod_{s=j}^{q-1}\frac{f_{[s]}(\bar t^s_1,t^s_1)}{h(\bar t^s_1,t^s_1)^{\delta_{s,m}}}. \quad (4.67)$$

The form of the other two rational functions $\mathbb{Y}(z,\bar t)$ and $\Lambda(z;\bar t)$ strongly depends on the values of p and q. For $p<i$ and $q>j$

$$\mathbb{Y}(z,\bar t)=f_{[p]}(z,\bar t^{p-1})f_{[q]}(\bar t^q,z)\prod_{s=p}^{i-1}h(\bar t^s_{r_s},z)^{\delta_{s,m}}\prod_{s=i}^{j-1}h(\bar t^s,z)^{\delta_{s,m}}\prod_{s=j}^{q-1}h(\bar t^s_1,z)^{\delta_{s,m}}\\ \times\frac{g(z,t^p_{r_p})\prod_{s=p}^{i-2}g_{[s+1]}(t^{s+1}_{r_{s+1}},t^s_{r_s})}{\prod_{s=p-1}^{i-2}f_{[s+1]}(t^{s+1}_{r_{s+1}},\bar t^s)}\,\frac{g(z,t^{q-1}_1)\prod_{s=j}^{q-2}g_{[s+1]}(t^{s+1}_1,t^s_1)}{\prod_{s=j}^{q-1}f_{[s+1]}(\bar t^{s+1},t^s_1)},$$

$$\Lambda(z;\bar t)=\frac{\lambda_{p+1}(t^p_{r_p})\cdots\lambda_i(t^{i-1}_{r_{i-1}})\lambda_j(t^j_1)\cdots\lambda_{q-1}(t^{q-1}_1)}{\lambda_{p+1}(z)\cdots\lambda_{q-1}(z)},$$

for $p=i$ and $q>j$

$$\mathbb{Y}(z,\bar t)=f_{[i]}(z,\bar t^{i-1})f_{[q]}(\bar t^q,z)\prod_{s=i}^{j-1}h(\bar t^s,z)^{\delta_{s,m}}\\ \times\prod_{s=j}^{q-1}h(\bar t^s_1,z)^{\delta_{s,m}}\frac{g(z,t^{q-1}_1)\prod_{s=j}^{q-2}g_{[s+1]}(t^{s+1}_1,t^s_1)}{\prod_{s=j}^{q-1}f_{[s+1]}(\bar t^{s+1},t^s_1)},$$

$$\Lambda(z;\bar t)=\frac{\lambda_j(t^j_1)\cdots\lambda_{q-1}(t^{q-1}_1)}{(\lambda_{i+1}(z)\cdots\lambda_{q-1}(z))^{\theta_{i+1,q-1}}},$$

for $p<i$ and $q=j$

$$\mathbb{Y}(z,\bar t\,) = f_{[p]}(z,\bar t^{p-1})f_{[j]}(\bar t^{j},z)\prod_{s=p}^{i-1} h(\bar t^{s}_{r_s},z)^{\delta_{s,m}}$$

$$\times\prod_{s=i}^{j-1} h(\bar t^{s},z)^{\delta_{s,m}}\,\frac{g(z,t^{p}_{r_p})\prod_{s=p}^{i-2} g_{[s+1]}(t^{s+1}_{r_{s+1}},t^{s}_{r_s})}{\prod_{s=p-1}^{i-2} f_{[s+1]}(t^{s+1}_{r_{s+1}},\bar t^{s})}\,,$$

$$\Lambda(z;\bar t\,) = \frac{\lambda_{p+1}(t^{p}_{r_p})\cdots\lambda_i(t^{i-1}_{r_{i-1}})}{(\lambda_{p+1}(z)\cdots\lambda_{j-1}(z))^{\theta_{p+1,j-1}}}\,,$$

and finally, for $p=i$ and $q=j$

$$\mathbb{Y}(z,\bar t\,) = f_{[i]}(z,\bar t^{i-1})f_{[j]}(\bar t^{j},z)\prod_{s=i}^{j-1} h(\bar t^{s},z)^{\delta_{s,m}},$$

$$\Lambda(z;\bar t\,) = \frac{\lambda_i(z)^{\delta_{ij}}}{(\lambda_{i+1}(z)\cdots\lambda_{j-1}(z))^{\theta_{i+1,j-1}}}\,,$$

where $\theta_{i,j}$ is the Heaviside step function

$$\theta_{i,j} = \begin{cases} 1, & i\leqslant j,\\ 0, & i>j.\end{cases}$$

Now we restore the normalization of the Bethe vectors (3.22) using (4.61). Again using the explicit expressions for the rational functions (4.25), (4.34), and (4.62), we obtain the action formula

$$\mathrm{T}^{+}_{i,j}(z)\cdot\widehat{\mathbb{B}}(\bar t\,) = \sum_{p=1}^{i}\sum_{q=j}^{N+1}\varphi_{p,q}\mathbb{C}_p^{i-1}\mathbb{C}_j^{q-1}$$

$$\times\,\mathrm{Sym}_{\bar t^{p},\dots,\bar t^{i-1},\bar t^{j},\dots,\bar t^{q-1}}\big[\widehat{\mathbb{D}}(\bar t\,)\mathbb{Y}(z,\bar t\,)\Lambda(z;\bar t\,)\widehat{\mathbb{B}}(\{z,\bar t\}')\big], \qquad (4.68)$$

where the only difference from the action formula (4.66) is that the function $\mathbb{D}(\bar t\,)$ is replaced by

$$\widehat{\mathbb{D}}(\bar t\,) = \prod_{s=p}^{i-1}\frac{f_{[s+1]}(t^{s}_{r_s},\bar t^{s}_{r_s})}{h(\bar t^{s}_{r_s},t^{s}_{r_s})^{\delta_{s,m}}}\prod_{s=j}^{q-1}\frac{f_{[s+1]}(\bar t^{s}_{1},t^{s}_{1})}{[(-)^{r_s-1}h(t^{s}_{1},\bar t^{s}_{1})]^{\delta_{s,m}}}\,. \qquad (4.69)$$

Comparing the action formulae (4.66) and (4.68), we can prove Proposition 4.2 if we prove that the functions $\mathbb{D}(\bar t\,)$ and $\widehat{\mathbb{D}}(\bar t\,)$ actually coincide. First of all, we recall that for $s\neq m$ the rational functions $f_{[s]}(u,v)$ and $f_{[s+1]}(u,v)$ in the definitions of the functions (4.67) and (4.69) coincide. A difference is possible only in the case when $s=m$, since by definition

$$f_{[m]}(u,v) = \frac{u-v+c}{u-v} \quad\text{and}\quad f_{[m+1]}(u,v) = \frac{u-v-c}{u-v}\,.$$

Assume first that $m\notin\{p,\dots,i-1\}$ and $m\notin\{j,\dots,q-1\}$. Then the functions (4.67) and (4.69) coincide. If $m\in\{p,\dots,i-1\}$, then both the factors in the functions $\mathbb{D}(\bar t)$ and $\widehat{\mathbb{D}}(\bar t)$ that depend on the Bethe parameters $\bar t^m$ are equal to $g(t^m_{r_m}, \bar t^m_{r_m})$. Similarly, if $m \in \{j, \dots, q-1\}$, then these factors are equal to $g(\bar t^m_1, t^m_1)$. This means that in the Yangian double $DY(\mathfrak{gl}(m|n))$ the Bethe vectors constructed using the first current realization (3.14) coincide with the Bethe vectors constructed using the second current realization (3.22).

This concludes the proof of the main statement formulated in Proposition 4.2.

4.5. Actions of the diagonal elements and the Bethe equations. In this subsection we consider the action of the universal transfer matrix $\mathfrak{t}(z)$ in (2.6) on Bethe vectors. For this we must find the action of the diagonal monodromy matrix elements. Hence we should set $i = j$ on the right-hand side of the action formula (4.66). Since the action formulae (4.66) and (4.68) are equivalent, we use the first of them. We have

$$\mathfrak{t}(z) \cdot \mathbb{B}(\bar t) = \sum_{i=1}^{N+1} (-)^{[i]} \sum_{p=1}^{i} \sum_{q=i}^{N+1} \varphi_{p,q} \mathbb{C}_p^{i-1} \mathbb{C}_i^{q-1} \\ \times \mathrm{Sym}_{\bar t^p, \dots, \bar t^{q-1}} \big[\mathbb{D}(\bar t)\mathbb{Y}(z, \bar t)\Lambda(z; \bar t)\mathbb{B}(\{z, \bar t\}')\big], \tag{4.70}$$

where

$$\{z, \bar t\}' = \{\bar t^1, \dots, \bar t^{p-1}, \{z, \bar t^p_{r_p}\}, \dots, \{z, \bar t^{i-1}_{r_{i-1}}\}, \{z, \bar t^i_1\}, \dots, \{z, \bar t^{q-1}_1\}, \bar t^q, \dots, \bar t^N\}$$

and, we recall, $N = m + n - 1$.

Among all the terms on the right-hand side of (4.70) there are the so-called 'wanted' terms corresponding to $p = q = i$. One can easily see that their sum is equal to

$$\sum_{i=1}^{N+1} (-)^{[i]} \lambda_i(z) f_{[i]}(z, \bar t^{i-1}) f_{[i]}(\bar t^i, z) \mathbb{B}(\bar t).$$

Let us compare the terms in (4.70) coming from the actions of the monodromy matrix elements $\mathrm{T}_{i,i}(z)$ and $\mathrm{T}_{i+1,i+1}(z)$. In both cases they correspond to the terms in the sums over p and q on the right-hand side of (4.66) for $p = i$ and $q = i+1$. For the action of the matrix element $(-)^{[i]}\mathrm{T}_{i,i}(z)$ these terms are

$$\frac{1}{(r_i - 1)!} \mathrm{Sym}_{\bar t^i} \bigg[\frac{\lambda_i(t^i_1)}{f_{[i+1]}(\bar t^{i+1}, t^i_1)} \frac{f_{[i]}(\bar t^i_1, t^i_1)}{h(\bar t^i_1, t^i_1)^{\delta_{i,m}}} \mathbb{B}\big(\bar t^1, \dots, \bar t^{i-1}, \{z, \bar t^i_1\}, \bar t^{i+1}, \dots, \bar t^N\big) \\ \times g(z, t^i_1) f_{[i]}(z, \bar t^{i-1}) f_{[i+1]}(\bar t^{i+1}, z) h(\bar t^i_1, z)^{\delta_{i,m}} \bigg]. \tag{4.71}$$

For the action of the matrix element $(-)^{[i+1]}\mathrm{T}_{i+1,i+1}(z)$ the analogous terms are

$$\frac{(-)^{1+(r_i-1)\delta_{i,m}}}{(r_i - 1)!} \mathrm{Sym}_{\bar t^i} \bigg[\frac{\lambda_{i+1}(t^i_{r_i})}{f_{[i]}(t^i_{r_i}, \bar t^{i-1})} \frac{f_{[i]}(t^i_{r_i}, \bar t^i_{r_i})}{h(t^i_{r_i}, \bar t^i_{r_i})^{\delta_{i,m}}} \\ \times \mathbb{B}\big(\bar t^1, \dots, \bar t^{i-1}, \{z, \bar t^i_{r_i}\}, \bar t^{i+1}, \dots, \bar t^N\big) \\ \times g(z, t^i_{r_i}) f_{[i]}(z, \bar t^{i-1}) f_{[i+1]}(\bar t^{i+1}, z) h(\bar t^i_{r_i}, z)^{\delta_{i,m}} \bigg]. \tag{4.72}$$

The symmetrizations in (4.71) and (4.72) can be replaced by summations over $\ell = 1, \dots, r_i$:

$$\sum_{\ell=1}^{r_i}\Big[\frac{\lambda_i(t^i_\ell)}{f_{[i+1]}(\bar t^{i+1}, t^i_\ell)}\,\frac{f_{[i]}(\bar t^i_\ell, t^i_\ell)}{h(\bar t^i_\ell, t^i_\ell)^{\delta_{i,m}}}\mathbb{B}\big(\bar t^1, \dots, \bar t^{i-1}, \{z, \bar t^i_\ell\}, \bar t^{i+1}, \dots, \bar t^N\big)$$
$$\times\, g(z, t^i_\ell) f_{[i]}(z, \bar t^{i-1}) f_{[i+1]}(\bar t^{i+1}, z) h(\bar t^i_\ell, z)^{\delta_{i,m}}\Big] \tag{4.73}$$

and

$$-\,(-)^{(r_i-1)\delta_{i,m}}\sum_{\ell=1}^{r_i}\Big[\frac{\lambda_{i+1}(t^i_\ell)}{f_{[i]}(t^i_\ell, \bar t^{i-1})}\,\frac{f_{[i]}(t^i_\ell, \bar t^i_\ell)}{h(t^i_\ell, \bar t^i_\ell)^{\delta_{i,m}}}\mathbb{B}\big(\bar t^1, \dots, \bar t^{i-1}, \{z, \bar t^i_\ell\}, \bar t^{i+1}, \dots, \bar t^N\big)$$
$$\times\, g(z, t^i_\ell) f_{[i]}(z, \bar t^{i-1}) f_{[i+1]}(\bar t^{i+1}, z) h(\bar t^i_\ell, z)^{\delta_{i,m}}\Big]. \tag{4.74}$$

If the set of Bethe parameters $\bar t$ satisfies the system of equations

$$\frac{\lambda_{i+1}(t^i_\ell)}{\lambda_i(t^i_\ell)} = (-)^{(r_i-1)\delta_{i,m}}\,\frac{f_{[i]}(\bar t^i_\ell, t^i_\ell)}{h(\bar t^i_\ell, t^i_\ell)^{\delta_{i,m}}}\,\frac{h(t^i_\ell, \bar t^i_\ell)^{\delta_{i,m}}}{f_{[i]}(t^i_\ell, \bar t^i_\ell)}\,\frac{f_{[i]}(t^i_\ell, \bar t^{i-1})}{f_{[i+1]}(\bar t^{i+1}, t^i_\ell)}\,, \tag{4.75}$$

then the terms in (4.73) and (4.74) cancel each other. If $i \neq m$, then the equations (4.75) become the standard Bethe equations analogous to those arising in the algebra $\mathfrak{gl}(N+1)$:

$$\frac{\lambda_{i+1}(t^i_\ell)}{\lambda_i(t^i_\ell)} = \frac{f_{[i]}(\bar t^i_\ell, t^i_\ell)}{f_{[i]}(t^i_\ell, \bar t^i_\ell)}\,\frac{f_{[i]}(t^i_\ell, \bar t^{i-1})}{f_{[i+1]}(\bar t^{i+1}, t^i_\ell)}\,. \tag{4.76}$$

For $i = m$ the Bethe equations (4.75) simplify to

$$\frac{\lambda_{m+1}(t^m_\ell)}{\lambda_m(t^m_\ell)} = \frac{f(t^m_\ell, \bar t^{m-1})}{f(t^m_\ell, \bar t^{m+1})}\,. \tag{4.77}$$

This simplified form of the Bethe equations is typical for the models of free fermions, but one should remember that in the case under consideration the parameters t^m_ℓ are coupled through the equations (4.76) with $i = m \pm 1$.

If the Bethe equations are satisfied, then the Bethe vector becomes an eigenvector of the transfer matrix (2.6):

$$\mathfrak{t}(z) \cdot \mathbb{B}(\bar t\,) = \tau(z; \bar t\,)\mathbb{B}(\bar t\,),$$

with the eigenvalue

$$\tau(z; \bar t\,) = \sum_{i=1}^{N+1} (-)^{[i]} \lambda_i(z) f_{[i]}(z, \bar t^{i-1}) f_{[i]}(\bar t^i, z). \tag{4.78}$$

In this case we call $\mathbb{B}(\bar t\,)$ an on-shell Bethe vector. Note that the Bethe equations (4.76) and (4.77) can be regarded as the condition of absence of poles of the eigenvalue (4.78) at the points $z = t^i_\ell$.

Let us verify that all the remaining 'unwanted' terms in the action of the transfer matrix (2.6) on the on-shell Bethe vector vanish. To do this we calculate the general coefficient of the Bethe vector

$$\mathbb{B}(\bar t^1,\dots,\bar t^{i-1},\{z,\bar t^i_{\ell_i}\},\dots,\{z,\bar t^{i+a}_{\ell_{i+a}}\},\bar t^{i+a+1},\dots,\bar t^N) \tag{4.79}$$

for fixed i and[8] $a>0$ in the sum over the index ℓ_b of the Bethe parameters $t^b_{\ell_b}$ for $b=i,\dots,i+a$. These sums arise from the symmetrizations in (4.70). One can see that a vector with Bethe parameters as in (4.79) can arise only from the actions of the diagonal monodromy matrix elements $\mathrm{T}^+_{b,b}(z)$ with $b=i,\dots,i+a+1$. To get such a vector one must take the term with $p=i$ and $q=i+a+1$ in the sums over p and q in (4.66). Recalling the definition of the phase factor $\varphi_{i,i+a+1}$ in (4.65) for each $b=i,\dots,i+a+1$ and denoting it by $\varphi_{i,i+a+1}(b)$, we find that

$$(-)^{[b]}\varphi_{i,i+a+1}(b)=\begin{cases}1 & \text{for } b=i,\\ (-)^{1+[b]} & \text{for } b=i+1,\dots,i+a,\\ -1 & \text{for } b=i+a+1.\end{cases}$$

Substituting the explicit Bethe equation in the function $\Lambda(z;\bar t)$, we get that the coefficient of the Bethe vector (4.79) on the right-hand side of the action formula (4.70) is proportional to the expression

$$g(z,t^i_{\ell_i})^{-1}-\sum_{b=i+1}^{i+a} g(t^b_{\ell_b},t^{b-1}_{\ell_{b-1}})^{-1}-g(z,t^{i+a}_{\ell_{i+a}})^{-1},$$

which obviously vanishes. We note that the same trivial identity was used in [27] (see the unnumbered formula on p. 29 of that paper) to prove that a universal off-shell Bethe vector becomes on-shell if the Bethe equations are satisfied.

5. Explicit formulae for the universal Bethe vectors

5.1. Hierarchical relations for the Bethe vectors $\mathbb{B}(\bar t)$. By calculating the 'positive' projection in the formula (3.14) for the Bethe vector $\mathbb{B}(\bar t)$, we can obtain a hierarchical recurrence relation which connects the Bethe vectors constructed for the Yangian double $DY(\mathfrak{gl}(m|n))$ with the Bethe vectors for $DY(\mathfrak{gl}(m-1|n))$. Let us separate the product of currents $\mathscr{F}_1(\bar t^1)=F_1(t^1_1)\cdots F_1(t^1_{r_1})$ from the product of the other currents $\mathscr{F}_\ell(\bar t^\ell)$, $\ell=2,\dots,N$, and apply the normal ordering rule (4.11) to the latter product. It is obvious from this rule that in order to obtain the desired hierarchical relations for the Bethe vectors (see (5.3) below) it is sufficient to calculate the projection

$$P^+_f\big(\mathscr{F}_1(\bar t^1)\cdot P^-_f\big(\mathscr{F}_2(\bar t^2_{\mathrm{I}})\mathscr{F}_3(\bar t^3_{\mathrm{I}})\cdots\mathscr{F}_N(\bar t^N_{\mathrm{I}})\big)\big). \tag{5.1}$$

Using the property $P^-_f(\mathscr{F}\cdot P^+_f(\mathscr{F}'))=0$ for arbitrary elements $\mathscr{F},\mathscr{F}'\in\overline{U}_F$ such that $\varepsilon(\mathscr{F}')=0$, we reduce the problem to the calculation of the projections

$$P^+_f\big(\mathscr{F}_1(\bar t^1)\cdot P^-_f\big(\mathscr{F}_2(\bar t^2_{\mathrm{I}})\cdot P^-_f(\mathscr{F}_3(\bar t^3_{\mathrm{I}})\cdots P^-_f(\mathscr{F}_N(\bar t^N_{\mathrm{I}}))\cdots)\big)\big). \tag{5.2}$$

[8] The case $a=0$ was considered above to obtain the Bethe equations.

This calculation is given in Appendix C, where it is shown to provide an answer in the form of a sum over partitions of the sets $\bar t^1$ and $\bar t^\ell_{\rm I}$, $\ell=2,\dots,N$, of the Bethe parameters in the expression (5.2).

To obtain the hierarchical relations for the Bethe vectors in the framework of this approach, we use the formula (4.11) to rewrite the Bethe vector (3.14) as a sum over partitions of the sets of Bethe parameters

$$\bar t\,' = \{\bar t^2,\dots,\bar t^N\} \Rightarrow \bar t\,'_{\rm I}\cup \bar t\,'_{\rm II},$$

where

$$\bar t\,'_{\rm I} = \{\bar t^2_{\rm I},\dots,\bar t^N_{\rm I}\} \quad\text{and}\quad \bar t\,'_{\rm II} = \{\bar t^2_{\rm II},\dots,\bar t^N_{\rm II}\}.$$

The primed set of Bethe parameters $\bar t\,'$ differs from the full set $\bar t$ of these parameters (3.11) by excluding the Bethe parameters of the first type $\bar t^1$. It follows from (4.11) and the properties of the projections that

$$\begin{aligned}\mathbb{B}^{(m|n)}(\bar t\,) = \sum_{\bar t\,'\Rightarrow \bar t\,'_{\rm I}\cup \bar t\,'_{\rm II}} \frac{\gamma_1(\bar t^1)}{f_{[2]}(\bar t^2_{\rm I},\bar t^1)} P^+_f\big(F_{2,1}(t^1_1)\cdots F_{2,1}(t^1_{r_1})P^-_f\big(\mathsf{F}(\bar t\,'_{\rm I})\big)\big)k^+_1(\bar t^1)\\ \times\frac{1}{f_{[2]}(\bar t^2_{\rm II},\bar t^1)}\frac{\prod_{s=2}^N\gamma_s(\bar t^s_{\rm II},\bar t^s_{\rm I})}{\prod_{s=2}^{N-1}f_{[s+1]}(\bar t^{s+1}_{\rm II},\bar t^s_{\rm I})}\mathbb{B}^{(m-1|n)}(\bar t\,'_{\rm II})\prod_{s=2}^N\lambda_s(\bar t^s_{\rm I}), \qquad (5.3)\end{aligned}$$

where we have identified $\bar t^1_{\rm I}$ with $\bar t^1$ and used the fact that the Cartan currents $k^+_1(z)$ commute with all the currents $F_s(t')$, $s=2,\dots,N$. Let

$$\mathscr{X}(\bar t\,) = \frac{\gamma_1(\bar t^1)}{f_{[2]}(\bar t^2,\bar t^1)}P^+_f\big(F_{2,1}(t^1_1)\cdots F_{2,1}(t^1_{r_1})P^-_f\big(\mathsf{F}(\bar t\,')\big)\big)k^+_1(\bar t^1), \qquad (5.4)$$

where $\bar t\,'=\{\bar t^2,\dots,\bar t^N\}$. Then the expression on the first line of the right-hand side of (5.3) is equal to

$$\mathscr{X}(\bar t^1,\bar t\,'_{\rm I}). \qquad (5.5)$$

To calculate the 'positive' projection of the product of currents

$$F_{2,1}(t^1_1)\cdots F_{2,1}(t^1_{r_1})$$

and the 'negative' projection $P^-_f\big(\mathsf{F}(\bar t\,')\big)$ in (5.4), we use the formulae (C.23) and (C.25) for different i, starting from larger i to smaller i. We use the first formula (C.23) for $i=m+1,\dots,N$, going from $i=N$ to $i=m+1$, and the second formula (C.25) for $i=2,\dots,m$, going from $i=m$ to $i=2$.

The results in Appendix C show that the sets $\bar t^\ell$ will always be further divided into subsets. To describe this, for each subset $\bar t^\ell$, $\ell=1,\dots,N$, we introduce the subdivision

$$\bar t^\ell \Rightarrow \{\bar t^\ell_\ell,\bar t^\ell_{\ell+1},\dots,\bar t^\ell_N\} \qquad (5.6)$$

such that the following constraints hold for the cardinalities of the subsets:

$$\#\bar t^\ell_q = \#\bar t^{\ell'}_q \quad\text{for all } \ell\neq\ell' \quad\text{and}\quad q=\max(\ell,\ell'),\dots,N. \qquad (5.7)$$

In (5.6) and (5.7) the superscripts of the subsets, as usual, describe the type of the Bethe parameters, while the subscripts count the subsets in the subdivision (5.6).

One should not confuse this notation with the notation $\bar{t}^\ell_i = \bar{t}^\ell \setminus \{t^\ell_i\}$ used in the previous section § 4.

Moreover, to get a non-trivial result in the calculation of the 'positive' projection in (5.4), we have to impose the following restrictions on the cardinalities of the subsets $\bar{t}^s$:

$$\#\bar{t}^1 \geqslant \#\bar{t}^2 \geqslant \cdots \geqslant \#\bar{t}^N \geqslant 0.$$

Appendix C shows how the Izergin determinant $K_{[i]}(\bar{y}|\bar{x})$ [29] (see (C.22)) arises in the calculation of the projections. It also shows how the result of these calculations can be rewritten in the form of sums over partitions of the sets of Bethe parameters into subsets. Let

$$\begin{aligned} K_0(\bar{y}|\bar{x}) &= K_{[i]}(\bar{y}|\bar{x}) && \text{for } i = 1,\ldots,m,\\ K_1(\bar{y}|\bar{x}) &= K_{[i]}(\bar{y}|\bar{x}) = K_0(\bar{x}|\bar{y}) && \text{for } i = m+1,\ldots,N. \end{aligned}$$

It is also convenient to introduce, for any sets $\bar{y}$ and $\bar{x}$ of the same cardinality, the following product of rational functions:

$$C(\bar{y}|\bar{x}) = g(\bar{y},\bar{x})h(\bar{x},\bar{x}). \tag{5.8}$$

We consider in greater detail the calculation of the projections, using the results obtained in Appendix C. The subdivision (5.6) can be represented using the table

$$\begin{array}{ccccccccccccccccc}
\bar{t}^1_1 & \cup & \bar{t}^1_2 & \cup & \cdots & \cup & \bar{t}^1_{m-1} & \cup & \bar{t}^1_m & \cup & \bar{t}^1_{m+1} & \cup & \cdots & \cup & \bar{t}^1_{N-1} & \cup & \bar{t}^1_N\\
 & & \bar{t}^2_2 & \cup & \cdots & \cup & \bar{t}^2_{m-1} & \cup & \bar{t}^2_m & \cup & \bar{t}^2_{m+1} & \cup & \cdots & \cup & \bar{t}^2_{N-1} & \cup & \bar{t}^2_N\\
 & & & & \ddots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots\\
 & & & & & & \bar{t}^{m-1}_{m-1} & \cup & \bar{t}^{m-1}_m & \cup & \bar{t}^{m-1}_{m+1} & \cup & \cdots & \cup & \bar{t}^{m-1}_{N-1} & \cup & \bar{t}^{m-1}_N\\
 & & & & & & & & \bar{t}^m_m & \cup & \bar{t}^m_{m+1} & \cup & \cdots & \cup & \bar{t}^m_{N-1} & \cup & \bar{t}^m_N\\
 & & & & & & & & & & \bar{t}^{m+1}_{m+1} & \cup & \cdots & \cup & \bar{t}^{m+1}_{N-1} & \cup & \bar{t}^{m+1}_N\\
 & & & & & & & & & & & & \ddots & & \vdots & & \vdots\\
 & & & & & & & & & & & & & & \bar{t}^{N-1}_{N-1} & \cup & \bar{t}^{N-1}_N\\
 & & & & & & & & & & & & & & & & \bar{t}^N_N
\end{array} \tag{5.9}$$

where the cardinalities of all the subsets in the same column are equal.

For any set $\bar{w}$ of cardinality $\#\bar{w} = d$ we introduce the following ordered product of composed (or simple, when $j = i+1$) currents:

$$\mathscr{F}_{j,i}(\bar{w}) = F_{j,i}(w_1)\cdot F_{j,i}(w_2)\cdots F_{j,i}(w_d). \tag{5.10}$$

Then the 'negative' projection $P^-_f\big(\mathsf{F}(\bar{t}\,')\big)$ in the definition of the element (5.4) can be written in the form

$$\frac{\prod_{s=2}^{N}\gamma_s(\bar{t}^s)}{\prod_{s=2}^{N-1} f_{[s+1]}(\bar{t}^{s+1},\bar{t}^s)} P^-_f\big(\mathscr{F}_{3,2}(\bar{t}^2)P^-_f(\ldots P^-_f(\mathscr{F}_{N,N-1}(\bar{t}^{N-1})$$
$$\times P^-_f(\mathscr{F}_{N+1,N}(\bar{t}^N)))\ldots)\big),$$

and in the first step we have to calculate

$$\frac{\gamma_{N-1}(\bar t^{N-1})\gamma_N(\bar t^N)}{f_{[N]}(\bar t^N,\bar t^{N-1})} P_f^-\big(\mathscr{F}_{N,N-1}(\bar t^{N-1})\cdot P_f^-\big(\mathscr{F}_{N+1,N}(\bar t^N)\big)\big), \tag{5.11}$$

using either (C.23) or (C.25), depending on the relation between m and N.

If $m<N$, and hence $[N]=1$, then we have to use (C.23) in order to obtain for the element (5.11) a sum over the partitions $\bar t^{N-1}\Rightarrow\{\bar t^{N-1}_{N-1}\cup\bar t^{N-1}_N\}$ such that $\#\bar t^{N-1}_N=\#\bar t^N_N$ (see the next-to-last line in the table above), where we identify the sets $\bar t^N_N\equiv\bar t^N$:

$$(-c)^{-\#\bar t^N_N}\gamma_{N-1}(\bar t^{N-1})\sum_{\bar t^{N-1}\Rightarrow\{\bar t^{N-1}_{N-1}\cup\bar t^{N-1}_N\}}\frac{f_1(\bar t^{N-1}_N,\bar t^{N-1}_{N-1})K_1(\bar t^N_N|\bar t^{N-1}_N)}{f_1(\bar t^N_N,\bar t^{N-1}_{N-1})f_1(\bar t^N_N,\bar t^{N-1}_N)}$$
$$\times P_f^-\big(\mathscr{F}_{N+1,N-1}(\bar t^{N-1}_N)\cdot\mathscr{F}_{N,N-1}(\bar t^{N-1}_{N-1})\big).$$

On the other hand, if $m=N$, and hence $[N]=0$ (this case corresponds to the algebra $\mathfrak{gl}(m|1)$), then we have to use (C.25) in order to obtain the element (5.11) again as a sum over the same partitions of $\bar t^{N-1}$:

$$c^{-\#\bar t^N_N}\gamma_{N-1}(\bar t^{N-1})\sum_{\bar t^{N-1}\Rightarrow\{\bar t^{N-1}_{N-1}\cup\bar t^{N-1}_N\}}\frac{f_0(\bar t^{N-1}_N,\bar t^{N-1}_{N-1})C(\bar t^N_N|\bar t^{N-1}_N)}{f_0(\bar t^N_N,\bar t^{N-1}_{N-1})f_0(\bar t^N_N,\bar t^{N-1}_N)}$$
$$\times\Delta_h(\bar t^{N-1}_N)^{-1}P_f^-\big(\mathscr{F}_{N+1,N-1}(\bar t^{N-1}_N)\cdot\mathscr{F}_{N,N-1}(\bar t^{N-1}_{N-1})\big).$$

The next step is to calculate the projections

$$\frac{\gamma_{N-2}(\bar t^{N-2})\gamma_{N-1}(\bar t^{N-1})}{f_{[N-1]}(\bar t^{N-1},\bar t^{N-2})}P_f^-\big(\mathscr{F}_{N-1,N-2}(\bar t^{N-2})$$
$$\times P_f^-\big(\mathscr{F}_{N+1,N-1}(\bar t^{N-1}_N)\cdot\mathscr{F}_{N,N-1}(\bar t^{N-1}_{N-1})\big)\big),$$

using (C.23) for $m<N-1$ and (C.25) for $m=N-1$. Continuing the calculation of the element (5.4) using first (C.23) and then (C.25), we eventually get that

$$\mathscr{X}(\bar t)=\sum_{\substack{\bar t^\ell\Rightarrow\{\bar t^\ell_\ell,\bar t^\ell_{\ell+1},\dots,\bar t^\ell_N\}\\ \ell=1,\dots,N}}\prod_{\ell=1}^{N-1}\prod_{\ell\leqslant q\leqslant q'\leqslant N}^{N}f_{[\ell+1]}(\bar t^{\ell+1}_{q'},\bar t^\ell_q)^{-1}$$
$$\times\prod_{\ell=1}^{N}\prod_{\ell\leqslant q<q'\leqslant N}\frac{f_{[\ell]}(\bar t^\ell_{q'},\bar t^\ell_q)}{h_{[\ell]}(\bar t^\ell_{q'},\bar t^\ell_q)^{\delta_{\ell,m}}}\prod_{q=2}^{m-1}\prod_{\ell=2}^{q}K_{[\ell]}(\bar t^\ell_q|\bar t^{\ell-1}_q)$$
$$\times\prod_{q=m+1}^{N}\prod_{\ell=m+1}^{q}K_{[\ell]}(\bar t^\ell_q|\bar t^{\ell-1}_q)\prod_{q=m}^{N}\prod_{\ell=2}^{m}C(\bar t^\ell_q|\bar t^{\ell-1}_q)\prod_{q=m}^{N}\Delta_h(\bar t^1_q)^{-1}$$
$$\times\gamma_1(\bar t^1)P_f^+\big(\mathscr{F}_{N+1,1}(\bar t^1_N)\cdots\mathscr{F}_{m+1,1}(\bar t^1_m)\cdot\mathscr{F}_{m,1}(\bar t^1_{m-1})\cdots\mathscr{F}_{2,1}(\bar t^1_1)\big)k_1^+(\bar t^1). \tag{5.12}$$

The projection in the last line in (5.12) can be calculated by the method in [13]. Being multiplied from the right by the product of the Cartan currents $k_1^+(\bar t^1)$, it can be expressed in terms of an ordered product of monodromy matrix elements $T_{1,\ell}(t)$, $\ell = 2, \dots, N+1$. This shows that the hierarchical relations which we have resolved by calculating the projections in (5.3) are compatible with the embedding of $DY(\mathfrak{gl}(m-1|n))$ in $DY(\mathfrak{gl}(m|n))$.

Finally, the element (5.4) is given as a multiple sum over partitions:

$$\begin{aligned}\mathscr{X}(\bar t) = &\sum_{\substack{\bar t^\ell \Rightarrow \{\bar t^\ell_\ell, \bar t^\ell_{\ell+1}, \dots, \bar t^\ell_N\} \\ \ell=1,\dots,N}} \prod_{\ell=1}^{N-1} \prod_{\ell \leqslant q \leqslant q' \leqslant N}^{N} f_{[\ell+1]}(\bar t^{\ell+1}_{q'}, \bar t^\ell_q)^{-1} \\ &\times \prod_{\ell=1}^{N} \prod_{\ell \leqslant q < q' \leqslant N} \frac{f_{[\ell]}(\bar t^\ell_{q'}, \bar t^\ell_q)}{h_{[\ell]}(\bar t^\ell_{q'}, \bar t^\ell_q)^{\delta_{\ell,m}}} \prod_{q=2}^{m-1} \prod_{\ell=2}^{q} K_{[\ell]}(\bar t^\ell_q | \bar t^{\ell-1}_q) \\ &\times \prod_{q=m+1}^{N} \prod_{\ell=m+1}^{q} K_{[\ell]}(\bar t^\ell_q | \bar t^{\ell-1}_q) \prod_{q=m}^{N} \prod_{\ell=2}^{m} C(\bar t^\ell_q | \bar t^{\ell-1}_q) \\ &\times \mathbb{T}_{1,N+1}(\bar t^1_N) \mathbb{T}_{1,N}(\bar t^1_{N-1}) \cdots \mathbb{T}_{1,m+1}(\bar t^1_m) \cdot T_{1,m}(\bar t^1_{m-1}) \cdots T_{1,2}(\bar t^1_1).\end{aligned} \tag{5.13}$$

Here we have used the notation

$$\mathbb{T}_{i,j}(\overline{w}) = \Delta_h(\overline{w})^{-1} T_{i,j}(w_1) T_{i,j}(w_2) \cdots T_{i,j}(w_{d-1}) T_{i,j}(w_d) \tag{5.14}$$

for any set $\overline{w}$ of cardinality $\#\overline{w} = d$ and for $[i]+[j]=1$. It is obvious that by the commutation relation (2.9) this product of odd matrix elements is symmetric with respect to permutations of the parameters w_i.

5.2. The Bethe vectors $\mathbb{B}(\bar t)$. We substitute the expression (5.13) with the subsets in (5.5) into the hierarchical relation (5.3), and then we repeat the same procedure for the Bethe vector $\mathbb{B}^{(m-1|n)}(\bar t'_{\mathrm{II}})$ in the second line of (5.3). In the end we will obtain an explicit expression for the Bethe vector $\mathbb{B}^{(m|n)}(\bar t)$ as a sum over multiple partitions of the set of Bethe parameters. Each term of this sum is a rational coefficient multiplied by symmetric products of monodromy matrix elements. To describe this expression it is necessary to introduce a more convenient indexing of the multiple partitions.

For all $\ell = 1, \dots, N$ we introduce the partition of the sets of Bethe parameters

$$\bar t^\ell = \bigcup_{q=1}^{\ell} \bigcup_{q'=\ell}^{N} \bar t^\ell_{q,q'}, \tag{5.15}$$

indexed by pairs of positive integers q, q' such that

$$1 \leqslant q \leqslant \ell \leqslant q' \leqslant N.$$

We also introduce ordering rules $\prec$ and $\preccurlyeq$ for these pairs according to the following convention:

$$q, q' \prec p, p' \quad \text{if} \quad q < p, \ \forall\, q', p' \quad \text{or} \quad q = p, \ q' < p', \tag{5.16}$$

and

$$q,q' \preccurlyeq p,p' \quad \text{if} \quad q<p,\ \forall\, q',p', \quad \text{or} \quad q=p,\ q'<p', \quad \text{or} \quad q=p,\ q'=p'.$$

Using this notation and combining (5.3) with (5.13), we obtain for the Bethe vector the expression

$$\mathbb{B}(\bar t\,) = \mathsf{B}(\bar t\,)|0\rangle,$$

where the pre-Bethe vector $\mathsf{B}(\bar t\,)$ is given by a sum over the partitions (5.15):

$$\begin{aligned}
\mathsf{B}(\bar t\,) = \sum_{\text{part}} \prod_{q,q'\preccurlyeq p,p'} \prod_{\ell=1}^{N-1} f_{[\ell+1]}(\bar t^{\ell+1}_{p,p'}, \bar t^{\ell}_{q,q'})^{-1} \prod_{q,q'\prec p,p'} g(\bar t^{m}_{p,p'}, \bar t^{m}_{q,q'}) \prod_{\substack{\ell=1\\ \ell\neq m}}^{N} f_{[\ell]}(\bar t^{\ell}_{p,p'}, \bar t^{\ell}_{q,q'}) \\
\times \prod_{q=1}^{m-2} \prod_{q'=q+1}^{m-1} \prod_{\ell=q+1}^{m-1} K_{[\ell]}(\bar t^{\ell}_{q,q'}|\bar t^{\ell-1}_{q,q'}) \prod_{q=1}^{m-1} \prod_{q'=m}^{N} \prod_{\ell=q+1}^{m} C(\bar t^{\ell}_{q,q'}|\bar t^{\ell-1}_{q,q'}) \\
\times \prod_{q=1}^{m-1} \prod_{q'=m+1}^{N} \prod_{\ell=m+1}^{q'} K_{[\ell]}(\bar t^{\ell}_{q,q'}|\bar t^{\ell-1}_{q,q'}) \prod_{q=m}^{N-1} \prod_{q'=q+1}^{N} \prod_{\ell=q+1}^{q'} K_{[\ell]}(\bar t^{\ell}_{q,q'}|\bar t^{\ell-1}_{q,q'}) \\
\times \overrightarrow{\prod_{1\leqslant q\leqslant m}} \Bigg(\overleftarrow{\prod_{N+1\geqslant q'\geqslant m+1}} \mathbb{T}_{q,q'}(\bar t^{q}_{q,q'-1}) \overleftarrow{\prod_{m\geqslant q'\geqslant q+1}} \mathrm{T}_{q,q'}(\bar t^{q}_{q,q'-1}) \Bigg) \\
\times \overrightarrow{\prod_{m+1\leqslant q\leqslant N}} \Bigg(\overleftarrow{\prod_{N+1\geqslant q'\geqslant q+1}} \mathrm{T}_{q,q'}(\bar t^{q}_{q,q'-1}) \Bigg) \prod_{\ell=2}^{N} \prod_{q=1}^{\ell-1} \prod_{q'=\ell}^{N} \mathrm{T}_{\ell,\ell}(\bar t^{\ell}_{q,q'}).
\end{aligned} \tag{5.17}$$

The partitions of the Bethe parameters can be pictured as an ordered table which is the following union of diagrams analogous to (5.9):

$$\overrightarrow{\bigcup_{\ell=1,\dots,N}} \quad \begin{matrix}
\bar t^{\ell}_{\ell,\ell} & \cup & \bar t^{\ell}_{\ell,\ell+1} & \cup & \cdots & \cup & \bar t^{\ell}_{\ell,N-1} & \cup & \bar t^{\ell}_{\ell,N} \\
 & & \bar t^{\ell+1}_{\ell,\ell+1} & \cup & \cdots & \cup & \bar t^{\ell+1}_{\ell,N-1} & \cup & \bar t^{\ell+1}_{\ell,N} \\
 & & & & \ddots & & \vdots & & \vdots \\
 & & & & & & \bar t^{N-1}_{\ell,N-1} & \cup & \bar t^{N-1}_{\ell,N} \\
 & & & & & & & & \bar t^{N}_{\ell,N}
\end{matrix} \tag{5.18}$$

The ordering means that if $\ell' < \ell$, then the diagram corresponding to ℓ' in (5.18) is on the left of the diagram corresponding to ℓ. The ordering rules (5.16) mean literally that if $q,q' \prec p,p'$, then the subset $\bar t^{\ell'}_{q,q'}$ in the ℓ'th row is located to the left of the subset $\bar t^{\ell}_{p,p'}$ in the ℓth row of the diagram. All the subsets in the same column have the same cardinality. The subsets which describe a partition of Bethe parameters of the same type are in the same row of the diagram (see examples of such tables in (5.19), (5.21), and (5.23)).

Example 5.1. Let us look at the formula (5.17) in some particular cases of small m and n.

The case $m = 2$ *and* $n = 1$. In this case $N = m + n - 1 = 2$ and the partitions of the sets $\bar{t}^1$ and $\bar{t}^2$ can be pictured by the following union of two diagrams:

$$\begin{array}{llllll} \bar{t}^1 : & \bar{t}^1_{1,1} & \cup & \bar{t}^1_{1,2} & & \\ \bar{t}^2 : & & & \bar{t}^2_{1,2} & \cup & \bar{t}^2_{2,2} \end{array} \tag{5.19}$$

In this case the formula (5.17) simplifies:

$$\mathsf{B}^{(2|1)}(\bar{t}^1, \bar{t}^2) = \sum_{\text{part}} f(\bar{t}^2, \bar{t}^1)^{-1} f(\bar{t}^1_{1,2}, \bar{t}^1_{1,1}) g(\bar{t}^2_{2,2}, \bar{t}^2_{1,2}) C(\bar{t}^2_{1,2}|\bar{t}^1_{1,2})$$
$$\times \mathbb{T}_{1,3}(\bar{t}^1_{1,2}) \mathrm{T}_{1,2}(\bar{t}^1_{1,1}) \mathbb{T}_{2,3}(\bar{t}^2_{2,2}) \mathrm{T}_{2,2}(\bar{t}^2_{1,2}). \tag{5.20}$$

After the identifications $\bar{t}^1_{1,1} \equiv \bar{u}_{\mathrm{II}}$, $\bar{t}^1_{1,2} \equiv \bar{u}_{\mathrm{I}}$, $\bar{t}^2_{2,2} \equiv \bar{v}_{\mathrm{II}}$, and $\bar{t}^2_{1,2} \equiv \bar{v}_{\mathrm{I}}$, we recover from (5.20) the expression (5.32) for the Bethe vector.

The case $m = 2$ *and* $n = 2$. The partitions (5.15) can be described by the following table:

$$\begin{array}{llllllllll} \bar{t}^1 : & \bar{t}^1_{1,1} & \cup & \bar{t}^1_{1,2} & \cup & \bar{t}^1_{1,3} & & & & \\ \bar{t}^2 : & & & \bar{t}^2_{1,2} & \cup & \bar{t}^2_{1,3} & \cup & \bar{t}^2_{2,2} & \cup & \bar{t}^2_{2,3} \\ \bar{t}^3 : & & & & & \bar{t}^3_{1,3} & & \cup & & \bar{t}^3_{2,3} \ \cup \ \bar{t}^3_{3,3} \end{array} \tag{5.21}$$

It corresponds to the union of three diagrams of the form (5.9). With this notation, (5.17) takes the form

$$\mathsf{B}^{(2|2)}(\bar{t}^1, \bar{t}^2, \bar{t}^3) = \sum_{\text{part}} f_0\big(\bar{t}^2_{1,2}, \{\bar{t}^1_{1,1} \cup \bar{t}^1_{1,2}\}\big)^{-1} f_0\big(\{\bar{t}^2_{1,3} \cup \bar{t}^2_{2,2} \cup \bar{t}^2_{2,3}\}, \bar{t}^1\big)^{-1}$$
$$\times f_1(\bar{t}^3_{1,3}, \{\bar{t}^2_{1,2} \cup \bar{t}^2_{1,3}\})^{-1} f_1\big(\{\bar{t}^3_{2,3} \cup \bar{t}^3_{3,3}\}, \bar{t}^2\big)^{-1}$$
$$\times f_0\big(\bar{t}^1_{1,3}, \{\bar{t}^1_{1,2} \cup \bar{t}^1_{1,1}\}\big) f_0(\bar{t}^1_{1,2}, \bar{t}^1_{1,1})$$
$$\times g\big(\bar{t}^2_{2,3}, \{\bar{t}^2_{2,2} \cup \bar{t}^2_{1,3} \cup \bar{t}^2_{1,2}\}\big) g\big(\bar{t}^2_{2,2}, \{\bar{t}^2_{1,3} \cup \bar{t}^2_{1,2}\}\big) g(\bar{t}^2_{1,3}, \bar{t}^2_{1,2})$$
$$\times f_1\big(\bar{t}^3_{3,3}, \{\bar{t}^3_{1,3} \cup \bar{t}^3_{2,3}\}\big) f_1(\bar{t}^3_{2,3}, \bar{t}^3_{1,3})$$
$$\times C(\bar{t}^2_{1,2}|\bar{t}^1_{1,2}) C(\bar{t}^2_{1,3}|\bar{t}^1_{1,3}) K_1(\bar{t}^3_{1,3}|\bar{t}^2_{1,3}) K_1(\bar{t}^3_{2,3}|\bar{t}^2_{2,3})$$
$$\times \mathbb{T}_{1,4}(\bar{t}^1_{1,3}) \mathbb{T}_{1,3}(\bar{t}^1_{1,2}) \mathrm{T}_{1,2}(\bar{t}^1_{1,1}) \mathbb{T}_{2,4}(\bar{t}^2_{2,3}) \mathbb{T}_{2,3}(\bar{t}^2_{2,2}) \mathrm{T}_{3,4}(\bar{t}^3_{3,3})$$
$$\times \mathrm{T}_{2,2}(\bar{t}^2_{1,2}) \mathrm{T}_{2,2}(\bar{t}^2_{1,3}) \mathrm{T}_{3,3}(\bar{t}^3_{1,3}) \mathrm{T}_{3,3}(\bar{t}^3_{2,3}). \tag{5.22}$$

There is a rule for constructing a pre-Bethe vector from any given table of partitions (5.15). We demonstrate this rule for the diagram (5.21), considering each line in (5.22) and explaining all the factors in this formula with the help of (5.21).

• For a given subset $\bar{t}^\ell_{i,j}$ in the ℓth row of the diagram (5.21) the first and second lines in (5.22) (which correspond to the values $\ell = 2, 3$) are products of the reciprocal functions $f_{[\ell]}\big(\bar{t}^\ell_{i,j}, \bar{t}^{\ell-1}_{k,l}\big)^{-1}$, where the subset $\bar{t}^{\ell-1}_{k,l}$ is either above or on the left of the starting subset $\bar{t}^\ell_{i,j}$.

• The third, fourth, and fifth lines in (5.22) correspond to certain products formed for each row of the diagram in accordance with the following rule. For the rows corresponding to $\bar{t}^\ell$ with $\ell < m$ (respectively, with $\ell = m$ or with $\ell > m$) we form products of the functions $f_0(\bar{x}, \bar{y})$ (respectively, $g(\bar{x}, \bar{y})$, or $f_1(\bar{x}, \bar{y})$). In these products the subset $\bar{x}$ is to the right of the subset $\bar{y}$ in each row of the diagram (5.21).

• The sixth line in (5.22) is a product of Cauchy determinants or Izergin determinants for neighbouring pairs of subsets $(\bar{t}^k_{i,j}, \bar{t}^{k-1}_{i,j})$ belonging to the same column of the diagram corresponding to some ℓ.

For $\ell = 1, \dots, m-1$ and any pair $(\bar{t}^k_{i,j}, \bar{t}^{k-1}_{i,j}) \equiv (\bar{x}, \bar{y})$, we use:
the Izergin determinant $K_0(\bar{x}|\bar{y})$ if $\ell + 1 \leqslant k \leqslant j \leqslant m-1$;
the normalized Cauchy determinant $C(\bar{x}|\bar{y})$ (5.8) if $\ell+1 \leqslant k \leqslant m \leqslant j \leqslant N$;
the Izergin determinant $K_1(\bar{x}|\bar{y})$ if $m+1 \leqslant k \leqslant j \leqslant N$.

For $\ell = m, \dots, N-1$ and any pair $(\bar{t}^k_{i,j}, \bar{t}^{k-1}_{i,j}) \equiv (\bar{x}, \bar{y})$, we use the Izergin determinant $K_1(\bar{x}|\bar{y})$ if $\ell + 1 \leqslant k \leqslant j \leqslant N$.

Note that the asymmetry between the cases $\ell < m$ and $\ell \geqslant m$ is due to the hierarchical relation (5.3), which is based on the series of inclusions $\mathfrak{gl}(m|n) \supset \mathfrak{gl}(m-1|n) \supset \cdots \supset \mathfrak{gl}(1|n) \supset \mathfrak{gl}(n)$.

In our example of the diagram (5.21) there are four such pairs

$$(\bar{t}^2_{1,2}, \bar{t}^1_{1,2}), \quad (\bar{t}^2_{1,3}, \bar{t}^1_{1,3}), \quad (\bar{t}^3_{1,3}, \bar{t}^2_{1,3}), \quad \text{and} \quad (\bar{t}^3_{2,3}, \bar{t}^2_{2,3}).$$

There are no $K_0(\bar{x}|\bar{y})$ determinants in this example, but they can appear for higher m. For instance, they appear in the Bethe vector for the algebra $\mathfrak{gl}(3|2)$ and are constructed for the pair of subsets $(\bar{t}^2_{1,2}, \bar{t}^1_{1,2})$ using the diagram in (5.23).

• The seventh line is an ordered product of monodromy matrix elements $\mathrm{T}_{i,j}$ with $i < j$ and depends on the subsets $\bar{t}^i_{i,j-1}$. It is the usual product for even matrix elements (that is, when $[i] + [j] = 0 \mod 2$) and the normalized product (5.14) otherwise. The order of the factors in the product is from top to bottom for the lines and from right to left within a line, as becomes clear upon comparing the seventh line in (5.22) and the diagram (5.21).

• The last line in (5.22) is the product of the diagonal matrix elements depending on the remaining subsets of Bethe parameters which were not used in the previous line. The index of a diagonal matrix element $\mathrm{T}_{i,i}$ coincides with the number of the line in the diagram. The order in this product is irrelevant, because the diagonal elements commute when the pre-Bethe vector (5.22) acts on the pseudo-vacuum vector $|0\rangle$.

The case $m = 3$ and $n = 2$. The Bethe vectors in this case can be constructed by the rules described above on the basis of the following table of partitions of the Bethe parameters $\bar{t}^1$, $\bar{t}^2$, $\bar{t}^3$, and $\bar{t}^4$:

$$\begin{array}{llllllllllll}
\bar{t}^1: & \bar{t}^1_{1,1} \cup \bar{t}^1_{1,2} & \cup\ \bar{t}^1_{1,3} & \cup\ \bar{t}^1_{1,4} & & & & & \\
\bar{t}^2: & \quad\ \bar{t}^2_{1,2} & \cup\ \bar{t}^2_{1,3} & \cup\ \bar{t}^2_{1,4} & \cup\ \bar{t}^2_{2,2} & \cup\ \bar{t}^2_{2,3} & \cup\ \bar{t}^2_{2,4} & & \\
\bar{t}^3: & & \quad\ \bar{t}^3_{1,3} & \cup\ \bar{t}^3_{1,4} & \cup & \quad\ \bar{t}^3_{2,3} & \cup\ \bar{t}^3_{2,4} & \cup\ \bar{t}^3_{2,2} \cup \bar{t}^3_{3,4} & \\
\bar{t}^4: & & & \quad\ \bar{t}^4_{1,4} & & \cup & \quad\ \bar{t}^4_{2,4} & \cup \quad \bar{t}^4_{3,4} \cup \bar{t}^4_{4,4} &
\end{array} \tag{5.23}$$

5.3. The Bethe vectors $\widehat{\mathbb{B}}(\bar{t})$. In a completely analogous way one can obtain for the Bethe vectors (3.22) defined by means of the second current realization of the Yangian double $DY(\mathfrak{gl}(m|n))$ hierarchical relations which are compatible with the embedding of $DY(\mathfrak{gl}(m|n-1))$ in $DY(\mathfrak{gl}(m|n))$. Another possibility for obtaining these hierarchial relations is to apply a special map to (5.3) and (5.13). This morphism was discussed in [30]. It maps the Bethe vectors $\mathbb{B}(\bar t)$ of $DY(\mathfrak{gl}(m|n))$ to the Bethe vectors $\widehat{\mathbb{B}}(\bar t)$ of $DY(\mathfrak{gl}(n|m))$ (see (5.26) and the discussion that follows). Thus, using this map and the exchange $m \leftrightarrow n$, we can obtain an explicit hierarchical relation for the Bethe vector $\widehat{\mathbb{B}}(\bar t)$. We do not give it here, but we give an analogue of (5.17) for $\widehat{\mathbb{B}}(\bar t)$.

Again, for all $\ell = 1,\dots,N$ we introduce a partition of the sets of Bethe parameters analogous to (5.15):

$$\bar t^{\ell} = \bigcup_{q=1}^{\ell} \bigcup_{q'=\ell}^{N} \bar t^{\ell}_{q',q}\,, \tag{5.24}$$

indexed by pairs of positive integers q, q' with

$$1 \leqslant q \leqslant \ell \leqslant q' \leqslant N.$$

We also introduce the ordering rules $\succ$ and $\succcurlyeq$ for these pairs according to the following conventions:

$$p',p \succ q',q \quad \text{if} \quad p' > q',\ \forall p,q \quad \text{or} \quad p' = q',\ p > q,$$

and

$$p,p' \succcurlyeq q,q' \quad \text{if} \quad p' > q',\ \forall p,q, \quad \text{or} \quad p' = q',\ p > q, \quad \text{or} \quad p' = q',\ p = q.$$

In this notation we have for the Bethe vector the expression

$$\widehat{\mathbb{B}}(\bar t) = \widehat{\mathsf{B}}(\bar t)|0\rangle,$$

where the pre-Bethe vector $\widehat{\mathsf{B}}(\bar t)$ is given by the sum over the partitions (5.24),

$$\begin{aligned}
\widehat{\mathsf{B}}(\bar t) = \sum_{\text{part}} \prod_{p',p\succcurlyeq q',q} \prod_{\ell=1}^{N-1} f_{[\ell+1]}(\bar t^{\ell+1}_{p',p}, \bar t^{\ell}_{q',q})^{-1} \\
\times \prod_{p',p\succ q',q} g(\bar t^{m}_{q',q}, \bar t^{m}_{p',p}) \prod_{\substack{\ell=1\\ \ell\neq m}}^{N} f_{[\ell+1]}(\bar t^{\ell}_{p',p}, \bar t^{\ell}_{q',q}) \\
\times \prod_{q'=m+2}^{N} \prod_{q=m+1}^{q'-1} \prod_{\ell=m+1}^{q'-1} K_{[\ell]}(\bar t^{\ell+1}_{q',q}|\bar t^{\ell}_{q',q}) \prod_{q'=m+1}^{N} \prod_{q=1}^{m} \prod_{\ell=m}^{q'-1} \widehat{C}(\bar t^{\ell+1}_{q',q}|\bar t^{\ell}_{q',q}) \\
\times \prod_{q'=m+1}^{N} \prod_{q=1}^{m-1} \prod_{\ell=q}^{m-1} K_{[\ell]}(\bar t^{\ell+1}_{q',q}|\bar t^{\ell}_{q',q}) \prod_{q'=2}^{m} \prod_{q=1}^{q'-1} \prod_{\ell=q}^{q'-1} K_{[\ell]}(\bar t^{\ell+1}_{q',q}|\bar t^{\ell}_{q',q}) \\
\times \overleftarrow{\prod_{N\geqslant q'\geqslant m}} \Bigg(\overrightarrow{\prod_{1\leqslant q\leqslant m}} \mathbb{T}_{q,q'+1}(\bar t^{q'}_{q',q}) \overrightarrow{\prod_{m<q\leqslant q'}} \mathrm{T}_{q,q'+1}(\bar t^{q'}_{q',q}) \Bigg) \\
\times \overleftarrow{\prod_{m>q'\geqslant 1}} \Bigg(\overrightarrow{\prod_{1\leqslant q\leqslant q'}} \mathrm{T}_{q,q'+1}(\bar t^{q'}_{q',q}) \Bigg) \prod_{\ell=1}^{N-1} \prod_{q'=\ell+1}^{N} \prod_{q=1}^{\ell} \mathrm{T}_{\ell+1,\ell+1}(\bar t^{\ell}_{q',q}),
\end{aligned} \tag{5.25}$$

and where in contrast to (5.8) we normalize the Cauchy determinant $\widehat{C}(\bar{y}|\bar{x})$ as follows:

$$\widehat{C}(\bar{y}|\bar{x}) = g(\bar{x},\bar{y})h(\bar{y},\bar{y}) = C(\bar{x}|\bar{y}).$$

The partitions of the Bethe parameters used in (5.25) also can be pictured using an ordered union of diagrams analogous to (5.9):

$$\overleftarrow{\bigcup_{\ell=N,\dots,1}}\quad \begin{array}{ccccccccc} \bar{t}^{\ell}_{\ell,\ell} & \cup & \bar{t}^{\ell}_{\ell,\ell-1} & \cup & \cdots & \cup & \bar{t}^{\ell}_{\ell,2} & \cup & \bar{t}^{\ell}_{\ell,1} \\ & & \bar{t}^{\ell-1}_{\ell,\ell-1} & \cup & \cdots & \cup & \bar{t}^{\ell-1}_{\ell,2} & \cup & \bar{t}^{\ell-1}_{\ell,1} \\ & & & & \ddots & & \vdots & & \vdots \\ & & & & & & \bar{t}^{2}_{\ell,2} & \cup & \bar{t}^{2}_{\ell,1} \\ & & & & & & & & \bar{t}^{1}_{\ell,1} \end{array}$$

The ordering here is opposite to the one used in the table (5.18). This means that a triangle for a smaller ℓ in (5.18) is to the right of a triangle for a larger ℓ. All the subsets in a given column again have the same cardinality. The subsets which describe partitions of the Bethe parameters of the same type are in the same row of the table (see examples of such tables in (5.27) and (5.29)).

We note that the two realizations (5.17) and (5.25) are related by the morphism φ defined in [30] by

$$\varphi\colon \begin{cases} DY(\mathfrak{gl}(m|n)) \to DY(\mathfrak{gl}(n|m)), \\ \mathrm{T}_{i,j}(x) \mapsto (-1)^{[i][j]+[j]+1}\widetilde{\mathrm{T}}_{j',i'}(x), \quad k' = m+n+1-k. \end{cases} \tag{5.26}$$

Indeed, starting from the pre-Bethe vector $\mathsf{B}(\bar{t}) \in DY(\mathfrak{gl}(m|n))$ and applying φ to it, we get the pre-Bethe vector $(-1)^{\#\bar{t}-\#\bar{t}^m}\widehat{\mathsf{B}}(\bar{s}) \in DY(\mathfrak{gl}(n|m))$, where the set $\bar{t}$ is divided into subsets $\bar{t}^{\ell}_{i,j}$ satisfying (5.15), while the set $\bar{s}$ is divided into subsets $\bar{s}^{\ell}_{i,j}$ satisfying (5.24). The relation between these partitions is given by $\bar{t}^{\ell}_{i,j} = \bar{s}^{\ell'-1}_{i'-1,j'-1}$, where $k' = m+n+1-k$ for any k. In particular, $\varphi\big(\mathsf{B}(\bar{t})\big) = (-1)^{\#\bar{t}-\#\bar{t}^m}\widehat{\mathsf{B}}(\bar{s})$ when $m=n$, as can be checked in the example $m=n=2$ described by (5.22) and (5.30).

Example 5.2. For $m=2$ and $n=1$ the partition (5.24) can be pictured using the table

$$\begin{array}{llllll} \bar{t}^2: & \bar{t}^2_{2,2} & \cup & \bar{t}^2_{2,1} & & \\ \bar{t}^1: & & & \bar{t}^1_{2,1} & \cup & \bar{t}^1_{1,1} \end{array} \tag{5.27}$$

and the formula (5.25) reduces to

$$\widehat{\mathsf{B}}^{(2|1)}(\bar{t}^1,\bar{t}^2) = \sum_{\text{part}} f(\bar{t}^2,\bar{t}^1)^{-1} f(\bar{t}^1_{2,1},\bar{t}^1_{1,1}) g(\bar{t}^2_{2,2},\bar{t}^2_{2,1}) K_0(\bar{t}^2_{2,1}|\bar{t}^1_{2,1}) \\ \times \mathbb{T}_{1,3}(\bar{t}^2_{2,1})\mathbb{T}_{2,3}(\bar{t}^2_{2,2})\mathrm{T}_{1,2}(\bar{t}^1_{1,1})\mathrm{T}_{2,2}(\bar{t}^1_{2,1}), \tag{5.28}$$

which implies (5.33) (see below) after the identifications $\bar{t}^1_{1,1} \equiv \bar{u}_{\mathrm{II}}$, $\bar{t}^1_{2,1} \equiv \bar{u}_{\mathrm{I}}$, $\bar{t}^2_{2,2} \equiv \bar{v}_{\mathrm{II}}$, and $\bar{t}^2_{2,1} \equiv \bar{v}_{\mathrm{I}}$.

In the case $m = 2$ and $n = 2$ the partitions (5.24) can be described using the following union of diagrams:

$$\begin{array}{llllllllllll} \bar t^3: & \bar t^3_{3,3} & \cup & \bar t^3_{3,2} & \cup & \bar t^3_{3,1} & & & & & & \\ \bar t^2: & & & \bar t^2_{3,2} & \cup & \bar t^2_{3,1} & \cup & \bar t^2_{2,2} & \cup & \bar t^2_{2,1} & & \\ \bar t^1: & & & & & \bar t^1_{3,1} & & \cup & & \bar t^1_{2,1} & \cup & \bar t^1_{1,1} \end{array} \tag{5.29}$$

According to this table, the formula (5.25) takes the form

$$\begin{aligned} \widehat{\mathsf{B}}^{(2|2)}(\bar t^1,\bar t^2,\bar t^3) = \sum_{\text{part}} & f_0(\bar t^2,\{\bar t^1_{2,1}\cup\bar t^1_{1,1}\})^{-1} f_0(\{\bar t^2_{3,2}\cup\bar t^2_{3,1}\},\bar t^1_{3,1})^{-1} \\ & \times f_1(\bar t^3_{3,1},\{\bar t^2_{3,1}\cup\bar t^2_{2,2}\cup\bar t^2_{2,1}\})^{-1} f_1(\{\bar t^3_{3,3}\cup\bar t^3_{3,2}\},\bar t^2)^{-1} \\ & \times f_0(\bar t^1_{3,1},\{\bar t^1_{2,1}\cup\bar t^1_{1,1}\}) f_0(\bar t^1_{2,1},\bar t^1_{1,1}) \\ & \times g(\{\bar t^2_{3,1}\cup\bar t^2_{2,2}\cup\bar t^2_{2,1}\},\bar t^2_{3,2}) g(\{\bar t^2_{2,2}\cup\bar t^2_{2,1}\},\bar t^2_{3,1}) g(\bar t^2_{2,1},\bar t^2_{2,2}) \\ & \times f_1(\{\bar t^3_{3,3}\cup\bar t^3_{3,2}\},\bar t^3_{3,1}) f_1(\bar t^3_{3,2},\bar t^3_{3,3}) \\ & \times K_0(\bar t^2_{2,1}|\bar t^1_{2,1}) K_0(\bar t^2_{3,1}|\bar t^1_{3,1}) \widehat{C}(\bar t^3_{3,1}|\bar t^2_{3,1}) \widehat{C}(\bar t^3_{3,2}|\bar t^2_{3,2}) \\ & \times \mathbb{T}_{1,4}(\bar t^3_{3,1}) \mathbb{T}_{2,4}(\bar t^3_{3,2}) \mathrm{T}_{3,4}(\bar t^3_{3,3}) \mathbb{T}_{1,3}(\bar t^2_{2,1}) \mathbb{T}_{2,3}(\bar t^2_{2,2}) \mathrm{T}_{1,2}(\bar t^1_{1,1}) \\ & \times \mathrm{T}_{2,2}(\bar t^1_{3,1}) \mathrm{T}_{2,2}(\bar t^1_{2,1}) \mathrm{T}_{3,3}(\bar t^2_{3,2}) \mathrm{T}_{3,3}(\bar t^2_{3,1}). \end{aligned} \tag{5.30}$$

Comparing (5.30) and the diagram (5.29), we can formulate the rules for associating with a partition diagram an explicit formula for the Bethe vector in a way similar to that in the previous subsection. We leave this as an exercise for the interested reader.

5.4. Dual Bethe vectors and examples for $DY(\mathfrak{gl}(2|1))$. In order to obtain explicit expressions for the dual Bethe vectors $\mathbb{C}(\bar t)$ and $\widehat{\mathbb{C}}(\bar t)$ we have to exploit the definition and the properties of the antimorphism (2.10), (2.11). It is clear that for even operators $\Psi(\mathrm{T}_{i,j}(\bar u)) = \mathrm{T}_{j,i}(\bar u)$. Consider an odd monodromy matrix element $\mathrm{T}_{i,j}(u)$ for $i < j$. This means that $[i] = 0$ and $[j] = 1$, and it follows from the commutation relations (2.9) that for any set $\bar u$ with cardinality $\#\bar u = a$ the product $\mathbb{T}_{i,j}(\bar u)$ given by (5.14) is symmetric with respect to permutations of the parameters u_i.

For an odd monodromy matrix element $\mathrm{T}_{i,j}(u)$ with $i > j$ and the set $\bar u$ we define the product

$$\mathbb{T}_{i,j}(\bar u) = \Delta'_h(\bar u)^{-1}\mathrm{T}_{i,j}(u_1)\mathrm{T}_{i,j}(u_2)\cdots\mathrm{T}_{i,j}(u_{a-1})\mathrm{T}_{i,j}(u_a),$$

which is also symmetric with respect to permutations in the set $\bar u$ due to the commutation relations (2.9).

Let us apply the antimorphism (2.10) to the product $\mathbb{T}_{i,j}(\bar u)$ with $i < j$. Using the property (2.11), we get for $i < j$ that

$$\begin{aligned} \Psi\big(\mathbb{T}_{i,j}(\bar u)\big) &= \Delta_h(\bar u)\Psi\big(\mathrm{T}_{i,j}(u_1)\mathrm{T}_{i,j}(u_2)\cdots\mathrm{T}_{i,j}(u_{a-1})\mathrm{T}_{i,j}(u_a)\big) \\ &= (-)^{a(a-1)/2}\Delta_h(\bar u)\Psi\big(\mathrm{T}_{i,j}(u_a)\big)\Psi\big(\mathrm{T}_{i,j}(u_{a-1})\big)\cdots\Psi\big(\mathrm{T}_{i,j}(u_2)\big)\Psi\big(\mathrm{T}_{i,j}(u_1)\big) \\ &= (-)^{a(a-1)/2}\Delta'_h(\bar u)\Psi\big(\mathrm{T}_{i,j}(u_1)\big)\Psi\big(\mathrm{T}_{i,j}(u_2)\big)\cdots\Psi\big(\mathrm{T}_{i,j}(u_{a-1})\big)\Psi\big(\mathrm{T}_{i,j}(u_a)\big) \\ &= (-)^{a(a-1)/2}\mathbb{T}_{j,i}(\bar u). \end{aligned}$$

Similarly, for $i<j$ we can calculate that

$$\Psi\big(\mathbb{T}_{j,i}(\overline{u})\big) = (-)^{a(a+1)/2}\mathbb{T}_{i,j}(\overline{u}) \tag{5.31}$$

by taking into account that in this case

$$\Psi\big(\mathrm{T}_{j,i}(u)\big) = (-)^{[j]([i]+1)}\mathrm{T}_{i,j}(u) = -\mathrm{T}_{i,j}(u).$$

The relation (5.31) shows that for any i and j such that $[i]+[j]=1$

$$\Psi\big(\Psi\big(\mathbb{T}_{i,j}(\overline{u})\big)\big) = (-)^{a}\mathbb{T}_{i,j}(\overline{u}),$$

and the antimorphism Ψ is an idempotent of fourth order.

Thus, we have described the action of Ψ on symmetric products of even and odd operators. Applying this action to the pre-Bethe vectors $\mathsf{B}(\bar{t}\,)$ in (5.17) and $\widehat{\mathsf{B}}(\bar{t}\,)$ in (5.25), we obtain explicit expressions for the dual pre-Bethe vectors $\mathsf{C}(\bar{t}\,)$ and $\widehat{\mathsf{C}}(\bar{t}\,)$, respectively. Up to a common sign factor they are still given by (5.17) and (5.25) with the opposite order of operator products and the replacement $\mathrm{T}_{i,j}\to\mathrm{T}_{j,i}$. Let us give explicit formulae for the particular case of the (dual) Bethe vectors $\mathbb{B}(\bar{t}\,)$, $\widehat{\mathbb{B}}(\bar{t}\,)$, $\mathbb{C}(\bar{t}\,)$, and $\widehat{\mathbb{C}}(\bar{t}\,)$ defined by (5.20) and (5.28) and connected with the Yangian double $DY(\mathfrak{gl}(2|1))$. In this case we have two sets of Bethe parameters $\bar{t}^{\ell}$ with cardinalities $\#\bar{t}^{\ell}=r_\ell$, $\ell=1,2$, which we rename as $\bar{t}^1\equiv\overline{u}$ and $\bar{t}^2\equiv\overline{v}$ with cardinalities $r_1=a$ and $r_2=b$. The formulae (3.14), (3.15), (3.22), and (3.23) for these Bethe vectors take the form

$$\begin{aligned}\mathbb{B}_{a,b}(\overline{u},\overline{v}) &= f(\overline{v},\overline{u})^{-1}\sum g(\overline{v}_{\mathrm{I}},\overline{u}_{\mathrm{I}})f(\overline{u}_{\mathrm{I}},\overline{u}_{\mathrm{II}})g(\overline{v}_{\mathrm{II}},\overline{v}_{\mathrm{I}})h(\overline{u}_{\mathrm{I}},\overline{u}_{\mathrm{I}})\\ &\qquad\times\mathbb{T}_{1,3}(\overline{u}_{\mathrm{I}})\mathrm{T}_{1,2}(\overline{u}_{\mathrm{II}})\mathbb{T}_{2,3}(\overline{v}_{\mathrm{II}})\lambda_2(\overline{v}_{\mathrm{I}})|0\rangle, &(5.32)\\ \widehat{\mathbb{B}}_{a,b}(\overline{u},\overline{v}) &= f(\overline{v},\overline{u})^{-1}\sum K_p(\overline{v}_{\mathrm{I}}|\overline{u}_{\mathrm{I}})f(\overline{u}_{\mathrm{I}},\overline{u}_{\mathrm{II}})g(\overline{v}_{\mathrm{II}},\overline{v}_{\mathrm{I}})\\ &\qquad\times\mathbb{T}_{1,3}(\overline{v}_{\mathrm{I}})\mathbb{T}_{2,3}(\overline{v}_{\mathrm{II}})\mathrm{T}_{1,2}(\overline{u}_{\mathrm{II}})\lambda_2(\overline{u}_{\mathrm{I}})|0\rangle, &(5.33)\\ \mathbb{C}_{a,b}(\overline{u},\overline{v}) &= (-)^{b(b-1)/2}f(\overline{v},\overline{u})^{-1}\sum g(\overline{v}_{\mathrm{I}},\overline{u}_{\mathrm{I}})f(\overline{u}_{\mathrm{I}},\overline{u}_{\mathrm{II}})g(\overline{v}_{\mathrm{II}},\overline{v}_{\mathrm{I}})h(\overline{u}_{\mathrm{I}},\overline{u}_{\mathrm{I}})\\ &\qquad\times\langle 0|\lambda_2(\overline{v}_{\mathrm{I}})\mathbb{T}_{3,2}(\overline{v}_{\mathrm{II}})\cdot\mathrm{T}_{2,1}(\overline{u}_{\mathrm{II}})\cdot\mathbb{T}_{3,1}(\overline{u}_{\mathrm{I}}), &(5.34)\\ \widehat{\mathbb{C}}_{a,b}(\overline{u},\overline{v}) &= (-)^{b(b-1)/2}f(\overline{v},\overline{u})^{-1}\sum K_p(\overline{v}_{\mathrm{I}}|\overline{u}_{\mathrm{I}})f(\overline{u}_{\mathrm{I}},\overline{u}_{\mathrm{II}})g(\overline{v}_{\mathrm{II}},\overline{v}_{\mathrm{I}})\\ &\qquad\times\langle 0|\lambda_2(\overline{u}_{\mathrm{I}})\mathrm{T}_{2,1}(\overline{u}_{\mathrm{II}})\mathbb{T}_{3,2}(\overline{v}_{\mathrm{II}})\mathbb{T}_{3,1}(\overline{v}_{\mathrm{I}}), &(5.35)\end{aligned}$$

where the sums run over partitions of the sets $\overline{u}\Rightarrow\{\overline{u}_{\mathrm{I}},\overline{u}_{\mathrm{II}}\}$ and $\overline{v}\Rightarrow\{\overline{v}_{\mathrm{I}},\overline{v}_{\mathrm{II}}\}$ such that $\#\overline{u}_{\mathrm{I}}=\#\overline{v}_{\mathrm{I}}=p\leqslant\min(a,b)$.

The formulae (5.32)–(5.35) were already used in the series of papers [18]–[20] to calculate the form factors of the monodromy matrix elements in the supersymmetric quantum integrable models associated with the super-Yangian $Y(\mathfrak{gl}(2|1))$.

Appendix A. Composed currents and Gauss coordinates

In the completed algebras $\overline{U}_F$, $\overline{U}_E$, $\widehat{\overline{U}}_F$, and $\widehat{\overline{U}}_E$ a product of total currents has some specific analytical properties. This means that if one performs the normal ordering of the current generators in these products, then one can see the pole structure of this product, which is encoded in the commutation relations of the total currents. This normal ordering procedure demonstrates that the products $F_i(u)F_{i+1}(v)$, $E_{i+1}(v)E_i(u)$, $\widehat{F}_{i+1}(v)\widehat{F}_i(u)$, and $\widehat{E}_i(u)\widehat{E}_{i+1}(v)$ have simple poles at $u=v$. We define the composed currents $F_{j,i}(u)$, $E_{i,j}(u)$, $\widehat{F}_{j,i}(u)$, and $\widehat{E}_{i,j}(u)$ for $1\leqslant i<j\leqslant m+n$ inductively as residues:

$$F_{j,i}(v)=\operatorname*{res}_{u=v}F_{a,i}(v)F_{j,a}(u)=-\operatorname*{res}_{u=v}F_{a,i}(u)F_{j,a}(v), \tag{A.1}$$

$$E_{i,j}(v)=\operatorname*{res}_{u=v}E_{a,j}(u)E_{i,a}(v)=-\operatorname*{res}_{u=v}E_{a,j}(v)E_{i,a}(u), \tag{A.2}$$

$$\widehat{F}_{j,i}(v)=\operatorname*{res}_{u=v}\widehat{F}_{j,a}(u)\widehat{F}_{a,i}(v)=-\operatorname*{res}_{u=v}\widehat{F}_{j,a}(v)\widehat{F}_{a,i}(u), \tag{A.3}$$

$$\widehat{E}_{i,j}(v)=\operatorname*{res}_{u=v}\widehat{E}_{i,a}(v)\widehat{E}_{a,j}(u)=-\operatorname*{res}_{u=v}\widehat{E}_{i,a}(u)\widehat{E}_{a,j}(v), \tag{A.4}$$

where $i<a<j$ and we have denoted the simple root currents as follows: $F_i(u)\equiv F_{i+1,i}(u)$, $E_i(u)\equiv E_{i,i+1}(u)$, $\widehat{F}_i(u)\equiv\widehat{F}_{i+1,i}(u)$, and $\widehat{E}_i(u)\equiv\widehat{E}_{i,i+1}(u)$.

Calculating the residues in (A.1)–(A.4) with the help of the commutation relations (2.26), (2.27), (2.39), and (2.40), respectively, we obtain

$$F_{j,i}(v)=c_{[i+1]}\cdots c_{[j-1]}F_{j,j-1}(v)F_{j-1,j-2}(v)\cdots F_{i+1,i}(v), \tag{A.5}$$

$$E_{i,j}(v)=c_{[i+1]}\cdots c_{[j-1]}E_{i,i+1}(v)E_{i+1,i+2}(v)\cdots E_{j-1,j}(v), \tag{A.6}$$

$$\widehat{F}_{j,i}(v)=c_{[i+1]}\cdots c_{[j-1]}\widehat{F}_{i+1,i}(v)\widehat{F}_{i+2,i+1}(v)\cdots \widehat{F}_{j,j-1}(v), \tag{A.7}$$

$$\widehat{E}_{i,j}(v)=c_{[i+1]}\cdots c_{[j-1]}\widehat{E}_{j-1,j}(v)\widehat{E}_{j-2,j-1}(v)\cdots \widehat{E}_{i,i+1}(v). \tag{A.8}$$

Let us prove one of these formulae, namely, (A.5). Consider (A.1) for $j=i+2$ and $a=i+1$. Since we know that the product $F_{i+1,i}(v)F_{i+2,i+1}(u)$ has a simple pole at $u=v$, we can calculate the residue in (A.1) as follows:

$$\begin{aligned}F_{i+2,i}(v)&=\operatorname*{res}_{u=v}F_{i+1,i}(v)F_{i+2,i+1}(u)=(u-v)F_{i+1,i}(v)F_{i+2,i+1}(u)\big|_{u=v}\\&=(u-v+c_{[i+1]})F_{i+2,i+1}(u)F_{i+1,i}(v)\big|_{u=v}=c_{[i+1]}F_{i+2,i+1}(v)F_{i+1,i}(v).\end{aligned}$$

Here we have used the commutation relation (2.26) in passing from the first line to the second line. Now we perform the analogous calculation in the case of the current $F_{i+3,i}(v)$, using the simple root current $F_{i+3,i+2}(u)$ and the composed current $F_{i+2,i}(v)$ that we just calculated. By the commutativity of $F_{i+3,i+2}(u)$ and $F_{i+1,i}(v)$ we get that

$$F_{i+3,i}(v)=c_{[i+1]}c_{[i+2]}F_{i+3,i+2}(v)F_{i+2,i+1}(v)F_{i+1,i}(v).$$

Iterating the calculation, we get the formula (A.5). The proof of the formulae (A.6)–(A.8) is completely analogous.

The composed currents are important in calculating the universal Bethe vectors using the formulae (3.14) and (3.22). In this section we show that the projections of composed currents discussed in § 4 coincide with the Gauss coordinates of the universal monodromy matrix (2.14)–(2.16) and (2.17)–(2.19) up to some unessential prefactors. To do this we rewrite the defining formulae for the composed currents in integral form.

Both equations in (A.1) can be expressed in terms of contour integrals:

$$\begin{aligned} F_{j,i}(v) &= -\oint_{C_0} du\, F_{a,i}(v)F_{j,a}(u) + \oint_{C_\infty} du\, \frac{u-v+c_{[a]}}{(u-v)_>} F_{j,a}(u)F_{a,i}(v) \\ &= -\oint_{C_\infty} du\, F_{a,i}(u)F_{j,a}(v) + \oint_{C_0} du\, \frac{u-v-c_{[a]}}{(u-v)_<} F_{j,a}(v)F_{a,i}(u), \end{aligned} \tag{A.9}$$

where C_0 and C_∞ are small closed contours around the points 0 and ∞ on the complex u-plane. The rational functions $1/(u-v)_{\lessgtr}$ are defined by the series in (2.33).

For any formal series $G(u) = \sum_{\ell\in\mathbb{Z}} G^{(\ell)}u^{-\ell-1}$ we define $G^{(\pm)}(u)$ by

$$G^{(\pm)}(u) = \pm \sum_{\substack{\ell\geqslant 0\\ \ell<0}} G^{(\ell)}u^{-\ell-1}. \tag{A.10}$$

It is obvious that the half-currents $F^{(\pm)}$ and $E^{(\pm)}$ coincide with the corresponding projections of currents only for the simple root currents $F_i(u)$ and $E_i(u)$. For the composed currents this is not the case, but nevertheless one can prove that

$$\begin{aligned} P_f^+\big(F_{j,i}^{(-)}(u)\cdot\mathscr{F}\big) &= 0, \qquad & P_f^-\big(\mathscr{F}\cdot F_{j,i}^{(+)}(u)\big) &= 0,\\ P_e^+\big(\mathscr{E}\cdot E_{i,j}^{(-)}(u)\big) &= 0, \qquad & P_e^-\big(E_{j,i}^{(+)}(u)\cdot\mathscr{E}\big) &= 0 \end{aligned} \tag{A.11}$$

for any elements $\mathscr{F}\in\overline{U}_F$ and $\mathscr{E}\in\overline{U}_E$. Similar properties can be formulated for the projections $\widehat{P}_f^{\pm}$ and $\widehat{P}_e^{\pm}$.

Using the notation (A.10) and calculating the formal contour integrals in (A.9) as

$$\oint_{C_0} du\, G(u) = \oint_{C_\infty} du\, G(u) = G^{(0)}, \tag{A.12}$$

we obtain the following expressions for the composed currents $F_{j,i}(v)$:

$$\begin{aligned} F_{j,i}(v) &= [F_{j,a}^{(0)}, F_{a,i}(v)] - c_{[a]}F_{j,a}^{(-)}(v)F_{a,i}(v)\\ &= [F_{j,a}(v), F_{a,i}^{(0)}] + c_{[a]}F_{j,a}(v)F_{a,i}^{(+)}(v). \end{aligned} \tag{A.13}$$

For the composed currents $E_{i,j}(v)$ defined by (A.2) we have

$$\begin{aligned} E_{i,j}(v) &= -\oint_{C_0} du\, E_{a,j}(u)E_{i,a}(v) + \oint_{C_\infty} du\, \frac{u-v+c_{[a]}}{(u-v)_>} E_{i,a}(v)E_{a,j}(u)\\ &= -\oint_{C_\infty} du\, E_{a,j}(v)E_{i,a}(u) + \oint_{C_0} du\, \frac{u-v-c_{[a]}}{(u-v)_<} E_{i,a}(u)E_{a,j}(v), \end{aligned} \tag{A.14}$$

or by using (A.12) we get for these composed currents that

$$\begin{aligned} E_{i,j}(v) &= [E_{i,a}(v), E_{a,j}^{(0)}] - c_{[a]}E_{i,a}(v)E_{a,j}^{(-)}(v)\\ &= [E_{i,a}^{(0)}, E_{a,j}(v)] + c_{[a]}E_{i,a}^{(+)}(v)E_{a,j}(v). \end{aligned} \tag{A.15}$$

Similarly, for the currents $\widehat{F}_{j,i}(v)$ defined by (A.3) we have

$$\begin{aligned}\widehat{F}_{j,i}(v) &= \oint_{C_\infty} du\, \widehat{F}_{j,a}(u)\widehat{F}_{a,i}(v) - \oint_{C_0} du\, \frac{u-v+c_{[a]}}{(u-v)_<}\widehat{F}_{a,i}(v)\widehat{F}_{j,a}(u)\\ &= \oint_{C_0} du\, \widehat{F}_{j,a}(v)\widehat{F}_{a,i}(u) - \oint_{C_\infty} du\, \frac{u-v-c_{[a]}}{(u-v)_>}\widehat{F}_{a,i}(u)\widehat{F}_{j,a}(v),\end{aligned} \tag{A.16}$$

or after calculating these formal contour integrals we get that

$$\begin{aligned}\widehat{F}_{j,i}(v) &= [\widehat{F}^{(0)}_{j,a}, \widehat{F}_{a,i}(v)] + c_{[a]}\widehat{F}_{a,i}(v)\widehat{F}^{(+)}_{j,a}(v)\\ &= [\widehat{F}_{j,a}(v), \widehat{F}^{(0)}_{a,i}] - c_{[a]}\widehat{F}^{(-)}_{a,i}(v)\widehat{F}_{j,a}(v).\end{aligned} \tag{A.17}$$

Finally, for the composed currents $\widehat{E}_{j,i}(v)$ defined by (A.4) we can calculate

$$\begin{aligned}\widehat{E}_{i,j}(v) &= \oint_{C_\infty} du\, \widehat{E}_{i,a}(v)E_{a,j}(u) - \oint_{C_0} du\, \frac{u-v+c_{[a]}}{(u-v)_<}\widehat{E}_{a,j}(u)\widehat{E}_{i,a}(v)\\ &= \oint_{C_0} du\, \widehat{E}_{i,a}(u)\widehat{E}_{a,j}(v) - \oint_{C_\infty} du\, \frac{u-v-c_{[a]}}{(u-v)_>}\widehat{E}_{a,j}(v)\widehat{E}_{i,a}(u),\end{aligned} \tag{A.18}$$

or, equivalently,

$$\begin{aligned}\widehat{E}_{i,j}(v) &= [\widehat{E}_{i,a}(v), \widehat{E}^{(0)}_{a,j}] + c_{[a]}\widehat{E}^{(+)}_{a,j}(v)\widehat{E}_{i,a}(v)\\ &= [\widehat{E}^{(0)}_{i,a}, \widehat{E}_{a,j}(v)] - c_{[a]}\widehat{E}_{a,j}(v)\widehat{E}^{(-)}_{i,a}(v).\end{aligned} \tag{A.19}$$

Projections of composed currents. The formulae (A.13), (A.15), (A.17), and (A.19) are very useful for calculating the projections of composed currents. Indeed, let us take $a = j-1$ in the first line of (A.13) and apply the 'positive' projection P^+_f defined by (3.9) to both sides of this equality. Similarly, we can consider the second line in (A.13) for $a = i+1$ and apply the 'negative' projection P^-_f to this equality. Using the properties of the projections (A.11), we have

$$\begin{aligned}P^+_f\big(F_{j,i}(v)\big) &= \big[F^{(0)}_{j,j-1}, P^+_f\big(F_{j-1,i}(v)\big)\big],\\ P^-_f\big(F_{j,i}(v)\big) &= \big[P^-_f\big(F_{j,i+1}(v)\big), F^{(0)}_{i+1,i}\big],\end{aligned} \tag{A.20}$$

where we have used the commutativity of the projections with the adjoint action of the zero modes of the simple root currents, which will be proved in Appendix B. Then the equations (A.20) can easily be iterated to obtain

$$\begin{aligned}P^+_f\big(F_{j,i}(v)\big) &= \mathscr{S}_{F^{(0)}_{j-1}}\mathscr{S}_{F^{(0)}_{j-2}}\cdots\mathscr{S}_{F^{(0)}_{i+1}}\big(\mathrm{F}^+_{i+1,i}(v)\big),\\ P^-_f\big(F_{j,i}(v)\big) &= (-)^{j-i}\mathscr{S}_{F^{(0)}_{i}}\mathscr{S}_{F^{(0)}_{i+1}}\cdots\mathscr{S}_{F^{(0)}_{j-2}}\big(\mathrm{F}^-_{j,j-1}(v)\big),\end{aligned} \tag{A.21}$$

where we have used the relation between the projections of the simple root currents and the Gauss coordinates: $P^\pm_f(F_{i+1,i}(v)) = \pm\mathrm{F}^\pm_{i+1,i}(v)$.

In a quite similar way we can get from (A.15), (A.17), and (A.19) that

$$\begin{aligned}P^+_e\big(E_{i,j}(v)\big) &= (-)^{j-i-1}\mathscr{S}_{E^{(0)}_{j-1}}\mathscr{S}_{E^{(0)}_{j-2}}\cdots\mathscr{S}_{E^{(0)}_{i+1}}\big(\mathrm{E}^+_{i,i+1}(v)\big),\\ P^-_e\big(E_{i,j}(v)\big) &= -\mathscr{S}_{E^{(0)}_{i}}\mathscr{S}_{E^{(0)}_{i+1}}\cdots\mathscr{S}_{E^{(0)}_{j-2}}\big(\mathrm{E}^-_{j-1,j}(v)\big),\end{aligned} \tag{A.22}$$

$$\begin{aligned}\widehat{P}^-_f\big(\widehat{F}_{j,i}(v)\big) &= -\mathscr{S}_{\widehat{F}^{(0)}_{j-1}}\mathscr{S}_{\widehat{F}^{(0)}_{j-2}}\cdots\mathscr{S}_{\widehat{F}^{(0)}_{i+1}}\big(\widehat{\mathrm{F}}^-_{i+1,i}(v)\big),\\ \widehat{P}^+_f\big(\widehat{F}_{j,i}(v)\big) &= (-)^{j-i-1}\mathscr{S}_{\widehat{F}^{(0)}_{i}}\mathscr{S}_{\widehat{F}^{(0)}_{i+1}}\cdots\mathscr{S}_{\widehat{F}^{(0)}_{j-2}}\big(\widehat{\mathrm{F}}^+_{j,j-1}(v)\big),\end{aligned} \tag{A.23}$$

$$\begin{aligned}\widehat{P}^-_e\big(\widehat{E}_{i,j}(v)\big) &= (-)^{j-i}\mathscr{S}_{\widehat{E}^{(0)}_{j-1}}\mathscr{S}_{\widehat{E}^{(0)}_{j-2}}\cdots\mathscr{S}_{\widehat{E}^{(0)}_{i+1}}\big(\widehat{\mathrm{E}}^-_{i,i+1}(v)\big),\\ \widehat{P}^+_e\big(\widehat{E}_{i,j}(v)\big) &= \mathscr{S}_{\widehat{E}^{(0)}_{i}}\mathscr{S}_{\widehat{E}^{(0)}_{i+1}}\cdots\mathscr{S}_{\widehat{E}^{(0)}_{j-2}}\big(\widehat{\mathrm{E}}^+_{j-1,j}(v)\big).\end{aligned} \tag{A.24}$$

In the rest of this section we are going to show that the 'positive' projections of composed currents given by the first lines in (A.21) and (A.22) and the second lines in (A.23) and (A.24) coincide with the Gauss coordinates of the universal monodromy operator $\mathrm{T}^+_{i,j}(v)$. To do this we consider the relation (2.8) for $i \to i$, $j \to j-1$, $k \to j-1$, $l \to j$, and $i < j-1$:

$$[\mathrm{T}^{\pm}_{i,j-1}(u), \mathrm{T}^{+}_{j-1,j}(v)] = \frac{c_{[j-1]}}{u-v}\big(\mathrm{T}^{\pm}_{i,j}(u)\mathrm{T}^{+}_{j-1,j-1}(v) - \mathrm{T}^{+}_{i,j}(v)\mathrm{T}^{\pm}_{j-1,j-1}(u)\big). \tag{A.25}$$

To obtain (A.25) from (2.7), we take into account that

$$(-)^{([i]+[j-1])([j-1]+[j])} = 1$$

for any i and j satisfying $i < j-1$, and the sign factor $(-)^{[j]([i]+[j-1])+[i][j-1]}$ is equal to $(-)^{[j-1]}$.

One can easily see from the Gauss decomposition and the mode expansions of the Gauss coordinates (2.3) that the zero modes of the monodromy matrix elements coincide with the zero modes of the corresponding currents:

$$\begin{aligned}\operatorname*{res}_{v\to\infty} v\mathrm{T}^+_{i,i+1}(v) &= (\mathrm{T}^+_{i,i+1})^{(0)} = (\mathrm{F}^+_{i+1,i})^{(0)} = (\widehat{\mathrm{F}}^+_{i+1,i})^{(0)} = F^{(0)}_i = \widehat{F}^{(0)}_i,\\ \operatorname*{res}_{v\to\infty} v\mathrm{T}^+_{i+1,i}(v) &= (\mathrm{T}^+_{i+1,i})^{(0)} = (\mathrm{E}^+_{i,i+1})^{(0)} = (\widehat{\mathrm{E}}^+_{i,i+1})^{(0)} = E^{(0)}_i = \widehat{E}^{(0)}_i.\end{aligned} \tag{A.26}$$

We multiply (A.25) by v and let $v \to \infty$. By (A.26), this relation becomes

$$c_{[j-1]}\mathrm{T}^{\pm}_{i,j}(u) = \mathscr{S}_{F^{(0)}_{j-1}}\big(\mathrm{T}^{\pm}_{i,j-1}(u)\big), \tag{A.27}$$

or, equivalently,

$$c_{[j-1]}\big(\mathrm{F}^{\pm}_{j,i}(u)k^{\pm}_i(u) + \cdots\big) = \mathscr{S}_{F^{(0)}_{j-1}}\big(\mathrm{F}^{\pm}_{j-1,i}(u)k^{\pm}_i(u) + \cdots\big), \tag{A.28}$$

where the dots denote the terms given by the Gauss decomposition (2.14). One can use weight arguments to prove that the contribution of these terms vanishes, and by the commutativity of the Cartan current $k^{\pm}_i(u)$ with the zero mode $F^{(0)}_{j-1}$ for $i < j-1$, we get from (A.28) that

$$c_{[j-1]}\mathrm{F}^{\pm}_{j,i}(u) = \mathscr{S}_{F^{(0)}_{j-1}}\big(\mathrm{F}^{\pm}_{j-1,i}(u)\big).$$

Iterating this relation for 'positive' Gauss coordinates, we obtain

$$c_{[i,j]}\mathrm{F}^{+}_{j,i}(u) = \mathscr{S}_{F^{(0)}_{j-1}}\cdots\mathscr{S}_{F^{(0)}_{i+1}}\big(\mathrm{F}^{+}_{i+1,i}(u)\big) = P^+_f\big(F_{j,i}(u)\big) \tag{A.29}$$

in accordance with the first line in (A.21), where we use the notation

$$c_{[i,j]} = c_{[i+1]}c_{[i+2]}\cdots c_{[j-2]}c_{[j-1]}. \tag{A.30}$$

In particular, we set $c_{[i,i+1]} = 1$.

The formula (A.29) describes the connection between the 'positive' projection of composed currents and the 'positive' Gauss coordinates. The connection between the 'negative' projection of composed currents and the 'negative' Gauss coordinates is more complicated. To find it we apply the 'negative' projection to the first equality in (A.13) for $a = j-1$ to obtain

$$P_f^-\big(F_{j,i}(u)\big) = \big(\mathscr{S}_{F^{(0)}_{j-1}} - c_{[j-1]}\mathrm{F}^-_{j,j-1}(u)\big)P_f^-\big(F_{j-1,i}(u)\big), \tag{A.31}$$

where we have used the equality $F^{(-)}_{j,j-1}(u) = \mathrm{F}^-_{j,j-1}(u)$ between 'negative' half-currents and 'negative' Gauss coordinates. Iterating (A.31), we obtain for the 'negative' projection an expression which uses only the zero-mode screening operators and the 'negative' Gauss coordinates:

$$\begin{aligned} P_f^-\big(F_{j,i}(u)\big) = -\big(\mathscr{S}_{F^{(0)}_{j-1}} - c_{[j-1]}\mathrm{F}^-_{j,j-1}(u)\big)\big(\mathscr{S}_{F^{(0)}_{j-2}} - c_{[j-2]}\mathrm{F}^-_{j-1,j-2}(u)\big) \times \cdots \\ \times \big(\mathscr{S}_{F^{(0)}_{i+1}} - c_{[i+1]}\mathrm{F}^-_{i+2,i+1}(u)\big)\mathrm{F}^-_{i+1,i}(u), \end{aligned}$$

where in the last step we have used the relation

$$P_f^-\big(F_{i+1,i}(u)\big) = -\mathrm{F}^-_{i+1,i}(u).$$

Multiplying out the parentheses in the equality above, we finally get that

$$P_f^-\big(F_{j,i}(u)\big) = -c_{[i,j]}\left(\mathrm{F}^-_{j,i}(u) + \sum_{\ell=1}^{j-i-1}(-)^\ell \sum_{j>i_\ell>\cdots>i_1>i} \mathrm{F}^-_{j,i_\ell}(u)\cdots\mathrm{F}^-_{i_2,i_1}(u)\mathrm{F}^-_{i_1,i}(u)\right). \tag{A.32}$$

This expression is very useful for calculating the action of the monodromy matrix elements on Bethe vectors.

On the other hand, we can establish a connection between the projection of the composed current given by the second line in (A.23), and the Gauss coordinate defined by the relation (2.17). To do this we consider (2.7) for $i \to i$, $j = k \to i+1$, $l \to j$, and $i < j-1$, which reduces to

$$[\mathrm{T}^+_{i,i+1}(u), \mathrm{T}^+_{i+1,j}(v)] = \frac{c_{[i+1]}}{u-v}\big(\mathrm{T}^+_{i+1,i+1}(v)\mathrm{T}^+_{i,j}(u) - \mathrm{T}^+_{i+1,i+1}(u)\mathrm{T}^+_{i,j}(v)\big). \tag{A.33}$$

As before, the factor $(-)^{([i]+[i+1])([i+1]+[j])}$ is equal to 1 for any i and j satisfying $i < j-1$, and the sign factor $(-)^{[i]([i+1]+[j])+[i+1][j]}$ is equal to $(-)^{[i+1]}$. Multiplying the equality (A.33) by u and letting $u \to \infty$, we obtain from (2.17) a relation between the Gauss coordinates:

$$c_{[i+1]}\widehat{\mathrm{F}}^+_{j,i}(v) = -\mathscr{S}_{\widehat{F}^{(0)}_i}\big(\widehat{\mathrm{F}}^+_{j,i+1}(v)\big). \tag{A.34}$$

Iterating this equality, we find that

$$c_{[i,j]}\widehat{\mathrm{F}}^+_{j,i}(u) = (-)^{j-i-1}\mathscr{S}_{\widehat{F}^{(0)}_i}\cdots\mathscr{S}_{\widehat{F}^{(0)}_{j-2}}\big(\widehat{\mathrm{F}}^+_{j,j-1}(u)\big) = \widehat{P}_f^+\big(\widehat{F}_{j,i}(v)\big). \tag{A.35}$$

For the relation between the 'negative' projection of composed currents and the 'negative' Gauss coordinates we have

$$\widehat{P}_f^-\big(\widehat{F}_{j,i}(u)\big) = -c_{[i,j]}\Bigg(\widehat{\mathrm{F}}^-_{j,i}(u) + \sum_{\ell=1}^{j-i-1}(-)^\ell \sum_{j>i_\ell>\cdots>i_1>i} \widehat{\mathrm{F}}^-_{i_1,i}(u)\widehat{\mathrm{F}}^-_{i_2,i_1}(u)\cdots\widehat{\mathrm{F}}^-_{j,i_\ell}(u)\Bigg). \tag{A.36}$$

Again starting from (2.8) for $i \to i+1$, $j \to i$, $k \to j$, $l \to i+1$, and $i < j-1$, we obtain a connection between the Gauss coordinate $\widehat{\mathrm{E}}^+_{i,j}(v)$ and the projection of the composed current $\widehat{P}_e^+\big(\widehat{E}_{i,j}(v)\big)$ by using analogous arguments and the Gauss decomposition (2.19):

$$c_{[i,j]}\widehat{\mathrm{E}}^+_{i,j}(v) = \mathscr{S}_{\widehat{E}_i^{(0)}}\cdots\mathscr{S}_{\widehat{E}_{j-2}^{(0)}}\big(\widehat{\mathrm{E}}^+_{j-1,j}(v)\big) = \widehat{P}_e^+\big(\widehat{E}_{i,j}(v)\big).$$

Finally, from the relation (2.7) for $i \to j-1$, $j \to i$, $k \to j$, $l \to j-1$, $i < j-1$ and (2.16) we get that

$$c_{[i,j]}\mathrm{E}^+_{i,j}(u) = (-)^{j-i-1}\mathscr{S}_{E_{j-1}^{(0)}}\cdots\mathscr{S}_{E_{i+1}^{(0)}}\big(\mathrm{E}^+_{i,i+1}(u)\big) = P_e^+\big(E_{i,j}(v)\big). \tag{A.37}$$

Summarizing the above considerations, we conclude that the 'positive' projections of composed currents coincide with the corresponding Gauss coordinates of the universal monodromy operator. The formulae for the connection for the 'negative' projections of composed currents are a bit more complicated, and one can also obtain formulae similar to (A.32) and (A.36) for the other two types of composed currents $E_{i,j}(u)$ and $\widehat{E}_{i,j}(u)$.

Appendix B. Commutativity of the projections and the screening operators

The adjoint actions by the zero modes of simple root currents $F_i^{(0)}$, $E_i^{(0)}$ and $\widehat{F}_i^{(0)}$, $\widehat{E}_i^{(0)}$ play an important role. For any elements $\mathscr{F} \in U_F$, $\mathscr{E} \in U_E$, $\widehat{\mathscr{F}} \in \widehat{U}_F$, and $\widehat{\mathscr{E}} \in \widehat{U}_E$ we introduce the screening operators

$$\begin{aligned} &\mathscr{S}_{F_i^{(0)}}(\mathscr{F}) \equiv [F_i^{(0)}, \mathscr{F}], \quad \mathscr{S}_{E_i^{(0)}}(\mathscr{E}) \equiv [E_i^{(0)}, \mathscr{E}], \\ &\mathscr{S}_{\widehat{F}_i^{(0)}}(\widehat{\mathscr{F}}) \equiv [\widehat{F}_i^{(0)}, \widehat{\mathscr{F}}], \quad \mathscr{S}_{\widehat{E}_i^{(0)}}(\widehat{\mathscr{E}}) \equiv [\widehat{E}_i^{(0)}, \widehat{\mathscr{E}}]. \end{aligned} \tag{B.1}$$

One can check that the intersections of standard and current Borel subalgebras are all stable under the corresponding action of the screening operators.

Let us check, for example, that the subalgebras $U_F^\pm$ defined by (3.6) are invariant under the adjoint action of the screening operators $\mathscr{S}_{F_i^{(0)}}$ for $i = 1, \ldots, N$. It follows from (3.10) that any element $\mathscr{F} \in U_F$ can be represented in the normal ordered form $\mathscr{F} = \sum_\ell \mathscr{F}_\ell^{(-)} \otimes \mathscr{F}_\ell^{(+)}$, where $\mathscr{F}_\ell^{(\pm)} \in U_F^\pm$ by definition. Then

$$\mathscr{S}_{F_i^{(0)}}(\mathscr{F}) = \sum_\ell \mathscr{S}_{F_i^{(0)}}\big(\mathscr{F}_\ell^{(-)}\big)\cdot\mathscr{F}_\ell^{(+)} + \sum_\ell \mathscr{F}_\ell^{(-)}\cdot\mathscr{S}_{F_i^{(0)}}\big(\mathscr{F}_\ell^{(+)}\big),$$

and by the definition (3.9) of the projection P_f^+ we have

$$P_f^+\big(\mathscr{S}_{F_i^{(0)}}(\mathscr{F})\big) = \sum_\ell \varepsilon\big(\mathscr{S}_{F_i^{(0)}}\big(\mathscr{F}_\ell^{(-)}\big)\big)\cdot\mathscr{F}_\ell^{(+)} + \sum_\ell \varepsilon\big(\mathscr{F}_\ell^{(-)}\big)\cdot\mathscr{S}_{F_i^{(0)}}\big(\mathscr{F}_\ell^{(+)}\big). \quad \text{(B.2)}$$

The first sum on the right-hand side of (B.2) vanishes because $\mathscr{S}_{F_i^{(0)}}\big(\mathscr{F}_\ell^{(-)}\big) \in U_F^-$ if $\varepsilon\big(\mathscr{F}_\ell^{(-)}\big) = 0$. It also vanishes if $\varepsilon\big(\mathscr{F}_\ell^{(-)}\big) = 1$ in view of the definition of the screening operators and the commutation relations

$$\mathscr{S}_{F_i^{(0)}}\big(k_i^-(u)\big) = c_{[i]}F_i^{(-)}(u)k_i^-(u) \quad \text{and} \quad \mathscr{S}_{F_i^{(0)}}\big(k_{i+1}^-(u)\big) = -c_{[i+1]}F_i^{(-)}(u)k_{i+1}^-(u),$$

which easily follow from (2.22). Since $\varepsilon\big(\mathscr{F}_\ell^{(-)}\big) \in \mathbb{C}$, the equality (B.2) can be rewritten in the form

$$P_f^+\big(\mathscr{S}_{F_i^{(0)}}(\mathscr{F})\big) = \mathscr{S}_{F_i^{(0)}}\bigg(\sum_\ell \varepsilon(\mathscr{F}_\ell^{(-)})\cdot(\mathscr{F}_\ell^{(+)})\bigg) = \mathscr{S}_{F_i^{(0)}}\big(P_f^+(\mathscr{F})\big),$$

which proves the assertion. The commutativity of the projections and the other relevant screening operators can be proved similarly.

Appendix C. Calculation of the projection

Let $\overline{v}$ be a set of variables with cardinality $\#\overline{v} = b$. Consider a product of composed currents (A.5)

$$F_{j_1,i}(v_1)\cdot F_{j_2,i}(v_2)\cdots F_{j_{b-1},i}(v_{b-1})\cdot F_{j_b,i}(v_b), \quad \text{(C.1)}$$

with the following restrictions on the indices of the composed currents:

$$j_1 \geqslant j_2 \geqslant \cdots \geqslant j_{b-1} \geqslant j_b \geqslant i+1. \quad \text{(C.2)}$$

In previous papers on the method of projections these products were called *strings*.

For any $\ell,\ell' = 1,\dots,N$ with $\ell \leqslant \ell'$ denote by $U_{\ell,\ell'}$ the subalgebra of $\overline{U}_F$ generated by the modes of the currents $F_\ell(t), F_{\ell+1}(t),\dots,F_{\ell'}(t)$. Then $U^\varepsilon_{\ell,\ell'} = U_{\ell,\ell'}\cap\operatorname{Ker}\varepsilon$ is the corresponding augmentation ideal.

Proposition C.1. *The commutation relations between composed currents imply the equality*

$$\begin{aligned}
&F_{i,i-1}(u_1)\cdots F_{i,i-1}(u_a)\cdot P_f^-\big(F_{j_1,i}(v_1)\cdot F_{j_2,i}(v_2)\cdots F_{j_{b-1},i}(v_{b-1})\cdot F_{j_b,i}(v_b)\big)\\
&= \frac{c_{[i]}^{-b}}{(a-b)!}\,\overline{\operatorname{Sym}}_{\overline{u}}\bigg[\prod_{\ell=1}^{b} g_{[i]}(v_\ell,u_\ell)\prod_{1\leqslant\ell<\ell'\leqslant b} f_{[i]}(u_\ell,u_{\ell'})\frac{f_{[i]}(v_{\ell'},u_\ell)}{f_{[i]}(v_{\ell'},v_\ell)}\prod_{\ell=1}^{b}\prod_{\ell'=b+1}^{a} f(u_\ell,u_{\ell'})\\
&\quad\times F_{j_1,i-1}(u_1)\cdot F_{j_2,i-1}(u_2)\cdots F_{j_b,i-1}(u_b)\cdot F_{i,i-1}(u_{b+1})\cdots F_{i,i-1}(u_a)\bigg]\\
&\quad \bmod P_f^-\big(U^\varepsilon_{i,j_1-1}\big)\cdot U_{i-1,j_1-1}. \qquad \text{(C.3)}
\end{aligned}$$

Proof. In what follows, equality of elements $\mathscr{A}_1$ and $\mathscr{A}_2$ in the subalgebra $\overline{U}_F$ modulo elements of the form $P_f^-(U_{i,j-1}^\varepsilon)\cdot U_{i-1,j-1}$ will be denoted by $\mathscr{A}_1 \sim_{i,j} \mathscr{A}_2$.

Let us prove (C.3) step by step. First of all, we observe that the 'negative' projection of the product of composed currents (C.1) with the restrictions (C.2) can be factorized [13], [14]:

$$\begin{aligned} &P_f^-\big(F_{j_1,i}(v_1)\cdot F_{j_2,i}(v_2)\cdots F_{j_{b-1},i}(v_{b-1})\cdot F_{j_b,i}(v_b)\big) \\ &\qquad = P_f^-\big(F_{j_1,i}(v_1;v_2,\dots,v_b)\big)\cdot P_f^-\big(F_{j_2,i}(v_2;v_3,\dots,v_b)\big)\cdots P_f^-\big(F_{j_b,i}(v_b)\big), \end{aligned}$$

where $F_{j,i}(v_1;v_2,\dots,v_b)$ is the linear combination

$$F_{j,i}(v_1;v_2,\dots,v_b) = F_{j,i}(v_1) - \sum_{\ell=2}^{b} h_{[i]}(v_\ell,v_1)^{-1} \prod_{\substack{\ell'=2\\ \ell'\neq\ell}}^{b} \frac{f_{[i]}(v_{\ell'},v_\ell)}{f_{[i]}(v_{\ell'},v_1)}\, F_{j,i}(v_\ell) \tag{C.4}$$

of composed currents of the same type. Next, we observe that due to the first relation for the composed currents in (A.13) we have

$$P_f^-\big(F_{j,i}(v)\big) + F_{j,i}^{(-)}(v) \sim_{i,j} \mathscr{S}_{F_{j-1}^{(0)}}\big(P_f^-\big(F_{j-1,i}(v)\big) + F_{j-1,i}^{(-)}(v)\big).$$

Iterating this relation, we find that

$$P_f^-\big(F_{j,i}(v)\big) + F_{j,i}^{(-)}(v) \sim_{i,j} \mathscr{S}_{F_{j-1}^{(0)}}\cdots\mathscr{S}_{F_{i+1}^{(0)}}\big(P_f^-\big(F_{i+1,i}(v)\big) + F_{i+1,i}^{(-)}(v)\big),$$

and since $P_f^-\big(F_{i+1,i}(v)\big) + F_{i+1,i}^{(-)}(v) = 0$, we arrive at the relation

$$P_f^-\big(F_{j,i}(v)\big) \sim_{i,j} -F_{j,i}^{(-)}(v).$$

This means that

$$\begin{aligned} &P_f^-\big(F_{j_1,i}(v_1)\cdot F_{j_2,i}(v_2)\cdots F_{j_{b-1},i}(v_{b-1})\cdot F_{j_b,i}(v_b)\big) \\ &\qquad \sim_{i,j} (-)^b F_{j_1,i}^{(-)}(v_1;v_2,\dots,v_b)\cdot F_{j_2,i}^{(-)}(v_2;v_3,\dots,v_b)\cdots F_{j_b,i}^{(-)}(v_b). \end{aligned} \tag{C.5}$$

Hence, by calculating the projection (5.2) one can move the terms of the form $P_f^-(U_{i+1,j-1}^\varepsilon)$ to the left through the product of currents $F_1(u)\cdots F_{i-1}(u)$, where they disappear under the action of the 'positive' projection P_f^+. This fact allows us to replace the product of currents and the 'negative' projection on the left-hand side of (C.3) by the product

$$(-)^b F_{i,i-1}(u_1)\cdots F_{i,i-1}(u_a)\cdot F_{j_1,i}^{(-)}(v_1;v_2,\dots,v_b)\cdot F_{j_2,i}^{(-)}(v_2;v_3,\dots,v_b)\cdots F_{j_b,i}^{(-)}(v_b).$$

The commutation relations between the product of the currents $F_{i,i-1}(u)$ and the 'negative' half-currents $F_{j,i}^{(-)}(v)$ can be calculated with the help of the relation

$$\begin{aligned} F_{i,i-1}(u)F_{j,i}^{(-)}(v) &= f_{[i]}(v,u)\big(F_{j,i}^{(-)}(v) - h_{[i]}(v,u)^{-1}F_{j,i}^{(-)}(u)\big)F_{i,i-1}(u) \\ &\quad + c_{[i]}^{-1} g_{[i]}(u,v)F_{j,i-1}(u). \end{aligned} \tag{C.6}$$

The latter equality is a consequence of the commutation relations

$$F_{i,i-1}(u)F_{j,i}(v) = f_{[i]}(v,u)\ F_{j,i}(v)F_{i,i-1}(u) - \delta(u,v)F_{j,i-1}(u)$$

between simple root currents and composed currents and the definition of the 'negative' half-current

$$F^{(-)}_{j,i}(v) = -\sum_{p<0} F^{(p)}_{j,i} u^{-p-1}.$$

Using the commutation relations (C.6), we get that

$$\begin{aligned} &F_{i,i-1}(u_1)\cdots F_{i,i-1}(u_a)\cdot F^{(-)}_{j,i}(v) \\ &\quad = f_{[i]}(v,\overline{u})\widetilde{F}^{(-)}_{j,i}(v;u_1,\ldots,u_a)\cdot F_{i,i-1}(u_1)\cdots F_{i,i-1}(u_a) \\ &\qquad + \sum_{q=1}^{a} c^{-1}_{[i]} g_{[i]}(u_q,v) \prod_{q'=q+1}^{a} \frac{(u_q-u_{q'})\epsilon_{i,m+1}+c_{[i]}}{(u_q-u_{q'})\epsilon_{i,m+1}-c_{[i]}} \\ &\qquad \times F_{i,i-1}(u_1)\cdots F_{i,i-1}(u_{q-1})\cdot F_{i,i-1}(u_{q+1})\cdots F_{i,i-1}(u_a)\cdot F_{j,i-1}(u_q), \end{aligned} \tag{C.7}$$

where

$$\widetilde{F}^{(-)}_{j,i}(v;u_1,\ldots,u_a) = F^{(-)}_{j,i}(v) - \sum_{\ell=1}^{a} h_{[i]}(v,u_\ell)^{-1} \prod_{\substack{q=1\\ q\neq\ell}}^{a} \frac{f_{[i]}(u_\ell,u_q)}{f_{[i]}(v,u_q)}\ F^{(-)}_{j,i}(u_\ell). \tag{C.8}$$

The linear combination of 'negative' half-currents (C.8) in the first term on the right-hand side of (C.7) commutes with all the products of currents

$$F_{i-2}(u),\ldots,F_1(u).$$

Therefore, this term eventually disappears under the action of the 'positive' projection in (5.2). To transform the sum over q on the right-hand side of (C.7), we move the composed current to the right using for $i\neq m+1$ the commutation relation

$$F_{j,i-1}(u_2)F_{i,i-1}(u_1) = f_{[i]}(u_1,u_2)^{-1}F_{i,i-1}(u_1)F_{j,i-1}(u_2) \tag{C.9}$$

and for $i=m+1$ the commutation relation

$$F_{j,m}(u_2)F_{m+1,m}(u_1) = -f_{[m+1]}(u_2,u_1)^{-1}F_{m+1,m}(u_1)F_{j,m}(u_2)$$

or, what is the same,

$$F_{j,m}(u_2)F_{m+1,m}(u_1) = -f(u_1,u_2)^{-1}F_{m+1,m}(u_1)F_{j,m}(u_2). \tag{C.10}$$

Here we have used the fact that $[m+1]=1$ and $f_1(u_2,u_1)=f(u_1,u_2)$. The two cases $i\neq m+1$ and $i=m+1$ can be combined into one formula, and by the definition (3.3) of the deformed symmetrization the sum in (C.7) can be written as

$$\begin{aligned} &F_{i,i-1}(u_1)\cdots F_{i,i-1}(u_a)\cdot F^{(-)}_{j,i}(v) \\ &\quad \sim_{i,j} \frac{c^{-1}_{[i]}}{(a-1)!}\,\overline{\mathrm{Sym}}_{\overline{u}}\big(g_{[i]}(u_a,v)F_{i,i-1}(u_1)\cdots F_{i,i-1}(u_{a-1})\cdot F_{j,i-1}(u_a)\big), \end{aligned} \tag{C.11}$$

or, equivalently,

$$F_{i,i-1}(u_1)\cdots F_{i,i-1}(u_a)\cdot F^{(-)}_{j,i}(v)$$
$$\sim_{i,j}\ \frac{c_{[i]}^{-1}}{(a-1)!}\,\overline{\mathrm{Sym}}_{\overline{u}}\big(g_{[i]}(u_1,v)f_{[i]}(u_1,\overline{u}_1)F_{j,i-1}(u_1)\cdot F_{i,i-1}(u_2)\cdots F_{i,i-1}(u_a)\big). \tag{C.12}$$

Here we have to use the commutation relations (C.9) and (C.10) in order to obtain (C.12) from (C.11).

By using the definition of the linear combination of half-currents (C.4) and the summation formula

$$g_{[i]}(u,v_1)f_{[i]}(\overline{v}_1,u)=g_{[i]}(u,v_1)f_{[i]}(\overline{v}_1,v_1)+\sum_{\ell=2}^{b}g_{[i]}(u,v_\ell)g_{[i]}(v_1,v_\ell)\prod_{\substack{\ell'=2\\ \ell'\neq\ell}}^{b}f_{[i]}(v_{\ell'},v_\ell)$$

we can now rewrite the equality (C.12) as

$$F_{i,i-1}(u_1)\cdots F_{i,i-1}(u_a)\cdot F^{(-)}_{j,i}(v_1,v_2,\dots,v_b)\sim_{i,j}\frac{c_{[i]}^{-1}}{(a-1)!}$$
$$\times\,\overline{\mathrm{Sym}}_{\overline{u}}\left(g_{[i]}(u_1,v_1)f_{[i]}(u_1,\overline{u}_1)\frac{f_{[i]}(\overline{v}_1,u_1)}{f_{[i]}(\overline{v}_1,v_1)}F_{j,i-1}(u_1)\cdot F_{i,i-1}(u_2)\cdots F_{i,i-1}(u_a)\right).$$

We can use this result for calculating the commutation of the product of currents $F_{i,i-1}(u_1)\cdots F_{i,i-1}(u_a)$ with the 'negative' projection (C.5) modulo terms which vanish under the action of the 'positive' projection in (5.1). The result gives us the proof of the relation (C.3). Note that the deformed symmetrization $\overline{\mathrm{Sym}}_{\overline{u}}$ over the set $\overline{u}$ becomes the usual antisymmetrization over this set for $i=m+1$. □

We stress the meaning of (C.3). Moving the 'negative' projection of the string (C.1) through the product of currents $F_{i,i-1}(u_1)\cdots F_{i,i-1}(u_a)$, we obtain linear combinations of analogous strings

$$F_{j_1,i-1}(u_1)\cdot F_{j_2,i-1}(u_2)\cdots F_{j_a,i-1}(u_a) \tag{C.13}$$

modulo terms which are irrelevant for calculation of the 'positive' projection in the definition of the Bethe vector (3.14), and with the restrictions

$$j_1\geqslant j_2\geqslant\cdots\geqslant j_{a-1}\geqslant j_a\geqslant i, \tag{C.14}$$

so that the first b indices j_ℓ, $\ell=1,\dots,b$, in the string (C.13) coincide with the corresponding indices in the string (C.1), and the remaining indices are equal to i: $j_{b+1}=\cdots=j_a=i$.

This linear combination is given by the deformed symmetrization over the set $\overline{u}$, which can be reduced to a sum over partitions of this set. We describe these partitions.

Let p_1 be the number of equal indices j_ℓ starting from j_1. Then let p_2 be the number of equal indices j_ℓ starting from j_{p_1+1}, and so on. Assume that the whole set of indices j_ℓ is divided into s subsets of identical indices with cardinalities p_l, $l = 1, \ldots, s$, and all $p_l > 0$. The integer s counts the number of groups of composed currents of the same type in the string (C.1). It is clear that $1 \leqslant s \leqslant b$, including the cases when all the currents are the same ($s = 1$) or all currents are different ($s = b$). The restriction on the indices in the product of composed currents (C.1) induces a natural decomposition

$$\overline{v} = \{v_1, v_2, \ldots, v_{b-1}, v_b\} \Rightarrow \{\overline{v}^1, \ldots, \overline{v}^s\} \tag{C.15}$$

of the set $\overline{v}$ into s disjoint subsets with cardinalities $\#\overline{v}^q = p_q$, $q = 1, \ldots, s$. Here we had to use a superscript to count these subsets, and this superscript should not be confused with the index which characterizes the type of Bethe parameters.

Assume that $a \geqslant b$. Let us decompose the set $\overline{u}$ into $s+1$ disjoint subsets

$$\overline{u} = \{u_1, u_2, \ldots, u_{a-1}, u_a\} \Rightarrow \{\overline{u}^1, \ldots, \overline{u}^s, \overline{u}^{s+1}\} \tag{C.16}$$

such that

$$\#\overline{u}^q = p_q > 0 \quad \text{and} \quad \#\overline{u}^{s+1} = a - b.$$

The last subset $\overline{u}^{s+1}$ can be empty for the terms with $a = b$ in (C.3). According to the definition of the sizes of the subsets $\overline{u}^q$, $q = 1, \ldots, s$, we have

$$j_1 = \cdots = j_{p_1} > j_{p_1+1} = \cdots = j_{p_2} > \cdots > j_{p_{s-1}+1} = \cdots = j_{p_s} > i.$$

Let

$$j_{p_{\ell-1}+1} = \cdots = j_{p_\ell} = j'_\ell$$

for $\ell = 1, \ldots, s$. Using the definition of the ordered product of composed or simple currents of the same type given by (5.10) and dividing the initial set of variables $\overline{v}$ in (C.15) into the subsets $\overline{v}^q$, $q = 1, \ldots, s$, we can transform the string (C.1) as follows:

$$\begin{aligned} &F_{j_1,i}(v_1) \cdot F_{j_2,i}(v_2) \cdots F_{j_{b-1},i}(v_{b-1}) \cdot F_{j_b,i}(v_b) \\ &\quad \to \mathscr{F}_{j'_1,i}(\overline{v}^1) \cdot \mathscr{F}_{j'_2,i}(\overline{v}^2) \cdots \mathscr{F}_{j'_{s-1},i}(\overline{v}^{s-1}) \cdot \mathscr{F}_{j'_s,i}(\overline{v}^s). \end{aligned} \tag{C.17}$$

Denote the ordered product of currents on the right-hand side of (C.17) by

$$\mathscr{F}_{\overline{\jmath}',i}(\overline{v}) = \mathscr{F}_{j'_1,i}(\overline{v}^1) \cdot \mathscr{F}_{j'_2,i}(\overline{v}^2) \cdots \mathscr{F}_{j'_{s-1},i}(\overline{v}^{s-1}) \cdot \mathscr{F}_{j'_s,i}(\overline{v}^s),$$

where $\overline{\jmath}' = \{j'_1, \cdots, j'_s\}$ and $j'_1 > j'_2 > \cdots > j'_s > i$.

Similarly, after dividing the set $\overline{u}$ into the subsets (C.16), we transform the string (C.13) into

$$\mathscr{F}_{\overline{\jmath},i-1}(\overline{u}) = \mathscr{F}_{j'_1,i-1}(\overline{u}^1) \cdot \mathscr{F}_{j'_2,i-1}(\overline{u}^2) \cdots \mathscr{F}_{j'_s,i-1}(\overline{u}^s) \cdot \mathscr{F}_{i,i-1}(\overline{u}^{s+1}), \tag{C.18}$$

where $\overline{\jmath} = \{j'_1, \ldots, j'_s, i\}$.

In order to rewrite the sum over permutations of the elements of the set $\overline{u}$ on the right-hand side of (C.3), we multiply both sides of (C.18) by the rational function $\Delta_{f_{[i]}}(\overline{u})\Delta_{h_{[i]}}(\overline{u})^{-\delta_{i,m+1}}$. Then using the fact that for any formal series $G(\overline{u})$ the deformed symmetrization (or antisymmetrization in the case when $i = m+1$) can be transformed into the usual symmetrization over $\overline{u}$, that is, using

$$\frac{\Delta_{f_{[i]}}(\overline{u})}{\Delta_{h_{[i]}}(\overline{u})^{\delta_{i,m+1}}}\,\overline{\mathrm{Sym}}_{\overline{u}}(G(\overline{u})) = \mathrm{Sym}_{\overline{u}}\left(\frac{\Delta_{f_{[i]}}(\overline{u})}{\Delta_{h_{[i]}}(\overline{u})^{\delta_{i,m+1}}}\,G(\overline{u})\right),$$

we can replace it by the sum over partitions (C.16) and by symmetrizations over the subsets in the partition:

$$\mathrm{Sym}_{\overline{u}}(\,\cdot\,) = \sum_{\overline{u}\Rightarrow\{\overline{u}^1,\dots,\overline{u}^s,\overline{u}^{s+1}\}} \mathrm{Sym}_{\overline{u}^1}\,\mathrm{Sym}_{\overline{u}^2}\cdots\mathrm{Sym}_{\overline{u}^s}\,\mathrm{Sym}_{\overline{u}^{s+1}}(\,\cdot\,).$$

Below we use the fact that after multiplication of both sides of (C.3) by the rational function $\Delta_{f_{[i]}}(\overline{u})\Delta_{h_{[i]}}(\overline{u})^{-\delta_{i,m+1}}$, we can sum over the symmetrizations in all the disjoint subsets $\overline{u}^q$, $q = 1,\dots,s+1$, on the right-hand side of (C.3).

For any composed current $F_{j,i}(u)$, $j > i$, we introduce its parity $\mu_{i,j}$ defined by

$$\mu_{i,j} = [i] + [j] = \begin{cases} 1, & i \leqslant m \leqslant j-1, \\ 0, & i > m \text{ or } m > j-1. \end{cases}$$

We refer to composed currents with parity 1 as *odd* and to those with parity 0 as *even*. Using the commutation relations for simple root currents, one can check that the commutation relations between even composed currents are the same as for even simple root currents, while odd composed currents anticommute:

$$\begin{aligned} (u-v-c_{[i]})F_{j,i}(u)F_{j,i}(v) &= (u-v+c_{[i]})F_{j,i}(v)F_{j,i}(u) \quad \text{for } \mu_{i,j}=0, \\ F_{j,i}(u)F_{j,i}(v) &= -F_{j,i}(v)F_{j,i}(u) \qquad\qquad\quad\; \text{for } \mu_{i,j}=1. \end{aligned} \tag{C.19}$$

If $m+1 < i \leqslant N$, then it is clear from the restrictions (C.2) and (C.14) that only even currents (simple and composed alike) appear in both sides of (C.3). Otherwise, for $i = m+1$ all the currents (again, simple and composed alike) on the right-hand side of (C.3) are odd. But if $1 < i \leqslant m$, then there are both odd and even currents on the right-hand side of (C.3), and according to the structure of the initial string (C.1) all the odd currents are to the left of all the even currents. In this case there are s' ($1 \leqslant s' < s$) factors in the string which are products of the same odd currents. In view of the commutation relations (C.19) for composed currents, the symmetrizations over the subsets $\overline{u}^q$ with $q = 1,\dots,s'$ and over those with $q = s'+1,\dots,s+1$ will be implemented differently. For $m+1 \leqslant i \leqslant N$ the symmetrizations over all the subsets $\overline{u}^q$ for $q = 1,\dots,s+1$ are the same. The number s' can be calculated as follows:

$$s' = \sum_{\ell=1}^{s} \mu_{i,j'_\ell}. \tag{C.20}$$

We first consider the case $m+1 \leqslant i \leqslant N$. Multiplying both sides of (C.3) by the function $\gamma_{i-1}(\overline{u})$, we get that

$$\gamma_{i-1}(\overline{u})\mathscr{F}_{i,i-1}(\overline{u}) \cdot P_f^-\left(\mathscr{F}_{\overline{\jmath}',i}(\overline{v})\right) \sim_{i,j_1} \frac{c_{[i]}^{-b}}{\Delta_{f_{[i]}}(\overline{v})} \sum_{\overline{u}\Rightarrow\{\overline{u}^1,\dots,\overline{u}^s,\overline{u}^{s+1}\}} \prod_{q<q'}^{s+1} f_{[i]}(\overline{u}^q,\overline{u}^{q'})$$
$$\times \prod_{q<q'}^{s} f_{[i]}(\overline{v}^{q'},\overline{u}^q)\gamma_{i-1}(\overline{u})\mathscr{F}_{\overline{\jmath},i-1}(\overline{u})$$
$$\times \prod_{q=1}^{s} \mathrm{Sym}_{\overline{u}^q}\left[\Delta'_{f_{[i]}}(\overline{u}^q)\prod_{\ell} g_{[i]}(v_\ell,u_\ell)\prod_{\ell<\ell'} f_{[i]}(v_{\ell'},u_\ell)\right]_{\substack{v_\ell,v_{\ell'}\in\overline{v}^q\\ u_\ell,u_{\ell'}\in\overline{u}^q}}, \tag{C.21}$$

where we have used the fact that the product of the function $\gamma_{i-1}(\overline{u})$ and the string (C.18) is symmetric with respect to permutations within each subset $\overline{u}^q$. In particular, this symmetry allows us to get rid of symmetrization over the subset $\overline{u}^{s+1}$ and cancel the combinatorial factor $\big((a-b)!\big)^{-1}$ in (C.3). Note that if $i = m+1$, then all the currents in the product $\mathscr{F}_{\overline{\jmath},m}(\overline{u})$ become odd, and the symmetry with respect to permutations of the variables in each subset $\overline{u}^q$ is ensured by the function $\gamma_m(\overline{u}) = \Delta_{g_{[m]}}(\overline{u})$.

The remaining symmetrization over each subset $\overline{u}^q, q=1,\dots,s$, is the well-known Izergin determinant [29] defined for two sets $\overline{y}$ and $\overline{x}$ with the same cardinality $\#\overline{y} = \#\overline{x} = p$ as follows:

$$K_{[i]}(\overline{y}|\overline{x}) = \mathrm{Sym}_{\overline{x}}\left[\Delta'_{f_{[i]}}(\overline{x})\prod_{\ell=1}^{p} g_{[i]}(y_\ell,x_\ell)\prod_{\ell<\ell'}^{p} f_{[i]}(y_{\ell'},x_\ell)\right]$$
$$= \Delta_{g_{[i]}}(\overline{y})\Delta'_{g_{[i]}}(\overline{x})h_{[i]}(\overline{y},\overline{x})\det\left[\frac{g_{[i]}(y_\ell,x_{\ell'})}{h_{[i]}(y_\ell,x_{\ell'})}\right]_{\ell,\ell'=1,\dots,p}. \tag{C.22}$$

Thus, we conclude that if the index i belongs to the interval $m+1 \leqslant i \leqslant N$, then (C.3) can be rewritten as a sum over partitions of $\overline{u}$ which is determined by the string $\mathscr{F}_{\overline{\jmath},i}(\overline{v})$:

$$\gamma_{i-1}(\overline{u})\mathscr{F}_{i,i-1}(\overline{u}) \cdot P_f^-\left(\mathscr{F}_{\overline{\jmath}',i}(\overline{v})\right)$$
$$\sim_{i,j_1} \frac{c_{[i]}^{-b}}{\Delta_{f_{[i]}}(\overline{v})} \sum_{\overline{u}\Rightarrow\{\overline{u}^1,\dots,\overline{u}^s,\overline{u}^{s+1}\}} \prod_{q<q'}^{s+1} f_{[i]}(\overline{u}^q,\overline{u}^{q'}) \prod_{q<q'}^{s} f_{[i]}(\overline{v}^{q'},\overline{u}^q)$$
$$\times \prod_{q=1}^{s} K_{[i]}(\overline{v}^q|\overline{u}^q)\gamma_{i-1}(\overline{u})\mathscr{F}_{\overline{\jmath},i-1}(\overline{u}). \tag{C.23}$$

Consider now the case when $1 < i \leqslant m$. As mentioned above, in this case the product of currents $\mathscr{F}_{\overline{\jmath},i-1}(\overline{u})$ contains both odd and even composed currents. Therefore, to perform symmetrization over the subsets $\overline{u}^q$ we have to use different approaches for odd and even currents.

Let s', $1 \leqslant s' \leqslant s$, be the number of products of the same odd currents on the right-hand side of (C.3), which is given by (C.20). Then the symmetrization over the subsets $\overline{u}^q$ for $s' < q \leqslant s+1$ in (C.21) is exactly the same as described above.

It leads to the appearance of Izergin determinants depending on the corresponding sets of variables. Since variables in the subsets $\overline{u}^q$ for $1 \leqslant q \leqslant s'$ become arguments of odd anticommuting currents, the relation (C.3) takes the following form after multiplication by the function in (3.1):

$$\begin{aligned}
&\gamma_{i-1}(\overline{u})\mathscr{F}_{i,i-1}(\overline{u}) \cdot P_f^-\big(\mathscr{F}_{\bar{\jmath}',i}(\overline{v})\big) \\
&\quad \sim_{i,j_1} \frac{c_{[i]}^{-b}}{\Delta_{f_{[i]}}(\overline{v})} \sum_{\overline{u}\Rightarrow\{\overline{u}^1,\dots,\overline{u}^s,\overline{u}^{s+1}\}} \prod_{q<q'}^{s+1} f_{[i]}(\overline{u}^q,\overline{u}^{q'}) \prod_{q<q'}^{s} f_{[i]}(\overline{v}^{q'},\overline{u}^q) \prod_{q=s'+1}^{s} K_{[i]}(\overline{v}^q|\overline{u}^q) \\
&\qquad \times \prod_{q=1}^{s'} \mathrm{Sym}_{\overline{u}^q}\Big[\Delta'_{g_{[i]}}(\overline{u}^q) \prod_{\ell} g_{[i]}(v_\ell,u_\ell) \prod_{\ell<\ell'} f_{[i]}(v_{\ell'},u_\ell)\Big]_{\substack{v_\ell,v_{\ell'}\in\overline{v}^q \\ u_\ell,u_{\ell'}\in\overline{u}^q}} \\
&\qquad \times \gamma_{i-1}(\overline{u}) \prod_{q=1}^{s'} \Delta'_{h_{[i]}}(\overline{u}^q)\mathscr{F}_{\bar{\jmath},i-1}(\overline{u}),
\end{aligned} \tag{C.24}$$

where we have used the factorization $\Delta'_{f_{[i]}}(\overline{u}^q) = \Delta'_{g_{[i]}}(\overline{u}^q)\Delta'_{h_{[i]}}(\overline{u}^q)$.

The fact that the products of odd currents on the right-hand side of (C.24) can be taken out from under the sign for symmetrization over the subsets $\overline{u}^q$, $q = 1,\dots,s'$, follows from the observation that for $1 < i \leqslant m$ the function $\gamma_{i-1}(\overline{u}) = \Delta_{f_{[i-1]}}(\overline{u})$ contains the factors $\Delta_{h_{[i-1]}}(\overline{u}^q)$ and $\Delta_{g_{[i-1]}}(\overline{u}^q)$. The first factor together with the function $\Delta'_{h_{[i]}}(\overline{u}^q)$ gives a function that is symmetric with respect to the variables in the subset $\overline{u}^q$:

$$\Delta_{h_{[i-1]}}(\overline{u}^q)\Delta'_{h_{[i]}}(\overline{u}^q) = \Delta_{h_{[i]}}(\overline{u}^q)\Delta'_{h_{[i]}}(\overline{u}^q) = h_{[i]}(\overline{u}^q,\overline{u}^q) \quad \text{for} \quad 1 < i \leqslant m,$$

while the second factor $\Delta_{g_{[i-1]}}(\overline{u}^q)$ makes symmetric the product of the odd currents depending on the variables in $\overline{u}^q$.

We denote the normalized symmetrization in the third line of (C.24) by $\mathscr{C}_{[i]}(\overline{v}|\overline{u})$:

$$\mathscr{C}_{[i]}(\overline{v}|\overline{u}) = \Delta'_{h_{[i]}}(\overline{u})\,\mathrm{Sym}_{\overline{u}}\Big[\Delta'_{g_{[i]}}(\overline{u}^q)\prod_{\ell} g_{[i]}(v_\ell,u_\ell)\prod_{\ell<\ell'} f_{[i]}(v_{\ell'},u_\ell)\Big]_{\substack{v_\ell,v_{\ell'}\in\overline{v} \\ u_\ell,u_{\ell'}\in\overline{u}}}.$$

This function is proportional to the Cauchy determinant, as follows from the chain of equalities

$$\begin{aligned}
\mathscr{C}_{[i]}(\overline{v}|\overline{u}) &= \Delta'_{h_{[i]}}(\overline{u})\Delta'_{h_{[g]}}(\overline{u})\,\mathrm{ASym}_{\overline{u}}\Big[\prod_{\ell} g_{[i]}(v_\ell,u_\ell)\prod_{\ell<\ell'} f_{[i]}(v_{\ell'},u_\ell)\Big]_{v_\ell,v_{\ell'}\in\overline{v};\ u_\ell\in\overline{u}} \\
&= \Delta'_{f_{[i]}}(\overline{u})\Delta_{f_{[i]}}(\overline{v})\,\mathrm{ASym}_{\overline{u}}\Big[\prod_{\ell} g_{[i]}(v_\ell,u_\ell)\Big]_{v_\ell\in\overline{v};\ u_\ell\in\overline{u}} \\
&= \frac{\Delta'_{f_{[i]}}(\overline{u})\Delta_{f_{[i]}}(\overline{v})}{\Delta'_{h_{[g]}}(\overline{u})\Delta_{g_{[i]}}(\overline{v})} g_{[i]}(\overline{v},\overline{u}) = \Delta'_{h_{[i]}}(\overline{u})\Delta_{h_{[i]}}(\overline{v})g_{[i]}(\overline{v},\overline{u}),
\end{aligned}$$

where the symbol $\mathrm{ASym}_{\overline{u}}$ means antisymmetrization with respect to the set $\overline{u}$.

Thus, for $1 < i \leqslant m$ the relation (C.3) can be represented as the following sum over partitions:

$$\gamma_{i-1}(\overline{u})\mathscr{F}_{i,i-1}(\overline{u}) \cdot P_f^-\bigl(\mathscr{F}_{\overline{\jmath}',i}(\overline{v})\bigr)$$
$$\sim_{i,j_1} \frac{c_{[i]}^{-b}}{\Delta_{f_{[i]}}(\overline{v})} \sum_{\overline{u}\Rightarrow\{\overline{u}^1,\dots,\overline{u}^s,\overline{u}^{s+1}\}} \prod_{q<q'}^{s+1} f_{[i]}(\overline{u}^q,\overline{u}^{q'}) \prod_{q<q'}^{s} f_{[i]}(\overline{v}^{q'},\overline{u}^q)$$
$$\times \prod_{q=1}^{s'} \mathscr{C}_{[i]}(\overline{v}^q|\overline{u}^q) \prod_{q=s'+1}^{s} K_{[i]}(\overline{v}^q|\overline{u}^q)\gamma_{i-1}(\overline{u})\mathscr{F}_{\overline{\jmath},i-1}(\overline{u}), \qquad \text{(C.25)}$$

where s' is given by (C.20).

Now we apply (C.23) and (C.25) to the calculation of the projection (5.2) and thereby obtain the recursion relation for the Bethe vectors (3.14).

We should add to (C.23) and (C.25) the rule for ordering the subsets $\overline{u}^q$. As we indicated in the definition of the string (C.18), the subsets with smaller indices occur in more complicated composed currents to the left in (C.18).

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Arthur A. Hutsalyuk
Moscow Institute of Physics and Technology
E-mail: hutsalyuk@gmail.com

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Translated by THE AUTHORS

Andrii Liashyk
Bogoliubov Institute for Theoretical Physics,
National Academy of Sciences of Ukraine;
National Research University
"Higher School of Economics";
Skolkovo Institute of Science and Technology
E-mail: a.liashyk@gmail.com

Stanislav Z. Pakuliak
Moscow Institute of Physics and Technology;
Joint Institute for Nuclear Research;
A. I. Alikhanov Institute for Theoretical
and Experimental Physics
E-mail: stanislav.pakuliak@jinr.ru

Eric Ragoucy
Laboratoire d'Annecy-le-Vieux de Physique
Théorique (LAPTH), Annecy-le-Vieux, France
E-mail: eric.ragoucy@lapth.cnrs.fr

Nikita A. Slavnov
Steklov Mathematical Institute
of Russian Academy of Sciences
E-mail: nslavnov@mi.ras.ru

Chapter 3

Scalar products of Bethe vectors in the models with $\mathfrak{gl}(m|n)$ symmetry

Introduction:

In this Chapter, using co-product properties of Bethe vectors we proved that the scalar product has bilinear structure with the rational coefficients. All the coefficients can be expressed in terms of the highest one. Using recurrence relations for Bethe vectors it was proven that the highest coefficient satisfies recurrence equations.

Contribution:

I proved that the scalar product has bilinear structure in λ_i's (Section 6.1). Using automorphism Ψ (3.20)-(3.23) I proved recurrence formulas (4.5) and co-product formula (A.4) for dual Bethe vectors. All these results are necessary for calculation of the scalar products.

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[b] *Nptdpx Jotujuvuf pg Ri fitjdt boe Ufdi opmphz- Epmhpqsveoz- Nptdpx sfh/- Svttjb*
[c] *Gbdi cfsfjdi D Ri fitjl - Cfshjtdi f Vojwfstju ua vqqfsubm 531: 9 a vqqfsubm Hfsn boz*
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B hf of sbrjfibujpo pg RJTN up ui f n pefrn x jui i jhi f s sbol tzn n fusz x bt hjwfo jo qbqf st]8˘ : ‘ x i fsf ui f oftufe brhfcsbjd Cfui f botbufi x bt efwfrpqfe/ Ui fsf b sfdvstjwf qspdfevsf x bt efwfrpqfe up dpotusvduCfui f wfdupst dpssftqpoejoh up ui f $gl(N)$ brhfcsb gspn ui f l opx o Cfui f wfdupst pg ui f $gl(N \cdot 2)$ brhfcsb/ Ui f qspcrfin pg ui f tdbrbs qspe vdut jo $SU(4)$.jowbsjboun pefrn x fsf tuvejfe jo]21‘- x i fsf bo bobrphpg ui f tvn gpsn vrb gps ui f tdbrbs qspe vdu x bt pcubjofe boe ui f opsn pg ui f usbotgfs n busjy fjhfotubuft x bt dpn qvufe/ Sfdfoum jo b tfsjft pg qbqf st]22˘ 27‘ ui f Cfui f wfdupst tdbrbs qspe vdut jo ui f n pefrn x jui $gl(4)$ boe $gl(3|2)$ tzn n fusjft x fsf joufotjwfm tuvejfe/ Ui fsf efufsn jobou sfqsftfoubujpot gps tpn f jn qpsubouqbsujdvmbs dbtft x fsf pcubjofe mfbejoh fwfouvbmz up ui f efufsn jobougpsn vrbt gps gpsn gbdupst pg mpdbmpqf sbupst jo ui f dpssftqpoejoh qi ztjdbm pefrn]28˘ 31‘/ B hf of sbrjfibujpo pg tpn f pg ui ptf sftvmut up ui f n pefrn x jui usjhpopn fusjd R.n busjy x bt hjwfo jo]32-33‘/

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Jo ui jt qbqf s x f tuvez ui f Cfui f wfdupst tdbrbs qspe vdut jo ui f n pefrn eftdsjcfe cz $gl(m|n)$ tvqf sbrhfcsbt/ I fodf jufodpn qbttft ui f dbtf pg $gl(m)$ brhfcsbt/ Jo tqjuf pg x f x psl x jui jo ui f gsbn fx psl pg ui f usbejujpobmbqqspbdi cbtfe po ui f oftufe brhfcsbjd Cfui f botbufi- x f fttfoujbmz vtf sfdfousftvmut pcubjofe jo]3: ‘ wjb ui f n fui pe pg qspkfdujpot gps dpotusvdujpo pg Cfui f wfdupst/ Ui jt n fui pe x bt qspqptfe jo ui f qbqf s]41‘/ Ju vtft ui f sfrbujpo cfux ffo ux p ejggfsfou sfbrjfib. ujpot pg ui f r vbou jfife I pqg brhfcsb $U_q(\widehat{gl}(N))$ bttpdjbufe x jui ui f bg–of brhfcsb $\widehat{gl}(N)$- pof jo ufsn t pg ui f vojwfstbm popespn z n busjy $T(z)$ boe RTT.dpn n vubujpo sfrbujpot boe tfdpoe jo ufsn t pg ui f upubmdvssfout- x i jdi bsf ef–ofe cz ui f Hbvtt efdpn qptjujpo pg ui f n popespn z n busjy $T(z)$]42‘/ Jo]3: ‘ x f hf of sbrjfife ui jt bqqspbdi up ui f dbtf pg ui f Zbohjbot pg $gl(m|n)$ tvqf sbrhfcsbt/ Bn poh ui f sftvmut pg]3: ‘ ui bubsf vtfe jo ui f qsftfouqbqf s- x f opuf ui f gpsn vrbt gps ui f bdujpo pg ui f n popespn z n busjy fousjft poup ui f Cfui f wfdupst- boe brtp ui f dpqspe vdu gpsn vrb gps ui f Cfui f wfdupst/

Ui f n bjo sftvmupg ui jt qbqf s jt ui f tvn gpsn vrb gps ui f tdbrbs qspe vdupg Cfui f wfdupst/ Jo pvs qsfwjpvt qvcrjdbujpot)tff f/h/]26-32‘* x f efsjwfe juvtjoh fyqrjdjugpsn vrbt pg ui f n popespn z n busjy frfin fout n vrjqrfi bdujpot poup ui f Cfui f wfdupst/ Ui jt n fui pe jt tusbjhi ugpsx bse- cvu ju cfdpn ft sbui fs dvn cfstpn f brsfbez gps $gl(4)$ boe $gl(3|2)$ cbtfe n pefrn/ Gvsui fsn psf- ui f qpttjcjrjuz pg jut bqqrjdbujpo up ui f n pefrn x jui i jhi f s sbol tzn n fusjft jt voefs r vftujpo/ Jotufbe- jo ui f qsftfouqbqf s x f vtf b n fui pe cbtfe po ui f dpqspe vdugpsn vrb gps ui f Cfui f wfdupst/ Bduvbmz- ui f tusvduvsf pg ui f tdbrbs qspe vdujt fodpefe jo ui f dpqspe vdugpsn vrb/ Ui fsf gpsf- ui jt n fui pe ejsfdumz rfibet up ui f tvn gpsn vrb- jo x i jdi ui f tdbrbs qspevdu jt hjwfo bt b tvn pwfs qbsujujpot pg Cfui f qbsbn fufst/

Ui f tvn gpsn vrb dpoubjot bo jn qpsubou pcklfdu dbmmfe ui f i jhi ftu dpfg–djfou)I D*]6'/ Jo ui f gl(3) cbtfe n pefm boe ui fjs q.efgpsn bujpo ui f I D dpjodjeft x jui b qbsujujpo gvodujpo pg ui f tjy.wfsufy n pefm x jui epn bjo x bmm cpvoebsz dpoejujpo/ Bo fyqmjdju sfqsftfoubujpo gps ju x bt gpvoe jo]43'/ Jo ui f n pefm x jui gl(4) tzn n fusz ui f I D bmtp dbo cf bttpdjbufe x jui b tqfdjbm qbsujujpo gvodujpo- i px fwfs- jut fyqmjdju gpsn jt n vdi n psf tpqi jtujdbufe)tff f/h/]22-24'*/ Pof dbo fyqfdu ui bu jo ui f dbtf pg i jhi fs sbol bmhfcsbt bo bobmphpvt fyqmjdju gpsn vrb gps ui f I D cfdpn ft upp dpn qmfy/ Ui fsfgpsf- jo ui jt qbqfs x f ep opu efsjwf tvdi gpsn vrbt- cvu jotufbe- x f pcubjo sfdvstjpot- x i jdi bmmpx pof up dpotusvdu ui f I D tubsujoh x jui ui f poft jo ui f n pefm x jui mpx fs sbol tzn n fusjft/ Ui ftf sfdvstjpot dbo cf efsjwfe gspn sfdvstjpot po ui f Cfui f wfdupst ui bu x f bmtp pcubjo jo ui jt qbqfs/

Bt x f i bwf bmsfbez n foujpofe- ui f Cfui f wfdupst tdbmbs qspevdut bsf pg hsfbu jn qpsubodf jo ui f qspcmfn pg dpssfmbujpo gvodujpot pg r vboum n joufhsbcmf n pefmt/ Dfsubjomz- ui f tvn gpsn vrb jt opu dpowfojfou gps jut ejsfdu bqqmjdbujpot- bt ju dpoubjot b cjh ovn cfs pg ufsn t- x i jdi hspx t fyqpofoujbmmz jo ui f ui fsn pezobn jd mjn ju/ I px fwfs- ju hjwft b l fz gps tuvezjoh qbsujdvmbs dbtft pg tdbmbs qspevdut- jo x i jdi ui f tvn pwfs qbsujujpot dbo cf sfevdfe up b tjohmf efufsn jobou/ Ui jt uzqf pg gpsn vrbt dbo cf vtfe gps dbmdvmbujoh gpsn gbdupst pg wbsjpvt joufhsbcmf n pefmt pg qi ztjdbm joufsftu- mjl f- gps jotubodf- ui f I vccbse n pefm]44'- ui f uK n pefm]45˘47' ps n vmuj.dpn qpofou Cptf0Gfsn j hbt]48'- opu up n foujpo tqjo di bjo n pefmt bt ui fz bsf opx bebzt uftufe jo dpoefotfe n bufs fyqfsjn fout]49'/ X f bmtp i pqf ui bu pvs sftvmut x jmm cf pg tpn f joufsftu jo ui f dpoufyu pg tvqfs.Zboh˘Njmmt ui fpsjft- x i fo tuvejfe jo ui f joufhsbcmf tztufn t gsbn fx psl / Joeffe- jo ui ftf ui fpsjft- ui f hfofsbm bqqspbdi sfmjft po b tqjo di bjo cbtfe po ui f psu(3, 3|5) tvqfsbmhfcsb/ X f cfmjfwf ui bu ui f qsftfou sftvmut x jmm dpousjcvuf up b cfuufs voefstuboejoh pg ui f ui fpsz/

Ui f bsujdmf jt pshbojafe bt gpmmpx t/ Jo tfdujpo 3 x f joupspevdf ui f n pefm voefs dpotjefsbujpo/ Ui fsf x f bmtp tqfdjgz pvs dpowfoujpot boe opubujpo/ Jo tfdujpo 4 x f eftdsjcf Cfui f wfdupst pg gl($m|n$).cbtfe n pefmt/ Tfdujpo 5 dpoubjot ui f n bjo sftvmut pg ui f qbqfs/ I fsf x f hjwf b tvn gps.n vmb gps ui f tdbmbs qspevdu pg hfofsjd Cfui f wfdupst boe sfdvstjpo sfmbujpot gps ui f Cfui f wfdupst boe ui f i jhi ftu dpfg–djfou/ Ui f sftu pg ui f qbqfs dpoubjot ui f qsppgt pg ui f sftvmut boopvodfe jo tfdujpo 5/ Jo tfdujpo 6 x f qspwf sfdvstjpo gpsn vmbt gps ui f Cfui f wfdupst/ Tfdujpo 7 dpoubjot b qsppg pg ui f tvn gpsn vmb gps ui f tdbmbs qspevdu/ Jo tfdujpo 8 x f tuvez i jhi ftu dpfg–djfou boe –oe b sfdvstjpo gps ju/ Qsppgt pg tpn f bvyjmjbsz tubufn fout bsf hbui fsfe jo bqqfoejdft/

31 Hftdskrukpo pg ui f n pefm

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Ui f R.n busjy pg gl($m|n$).cbtfe n pefmt bdut jo ui f ufotps qspevdu $\mathbf{D}^{m|n} \circ \mathbf{D}^{m|n}$- x i fsf $\mathbf{D}^{m|n}$ jt ui f $\mathbb{Z}_3$.hsbefe wfdups tqbdf x jui ui f hsbejoh $[i] = 1$ gps $2 \geq i \geq m$- $[i] = 2$ gps $m < i \geq m + n$/ I fsf- x f bttvn f ui bu $m \sim 2$ boe $n \sim 2$- cvu x f x bou up tusftt ui bu pvs dpotjefsbujpot bsf bq.qmjdbcmf up ui f dbtf $m = 1$ ps $n = 1$ bt x fmm- j/f/ up ui f opo.hsbefe bmhfcsbt/ Nbusjdft bdujoh jo ui jt tqbdf bsf bmtp hsbefe/ X f ef–of ui jt hsbejoh po ui f cbtjt pg fmfn foubsz vojut E_{ij} bt $[E_{ij}] = [i] + [j] \in \mathbb{Z}_3$)sfdbmm ui bu $(E_{ij})_{ab} = \eta_{ia}\eta_{jb}$*/ Ui f ufotps qspevdut pg $\mathbf{D}^{m|n}$ tqbdft bsf hsbefe bt gpmmpx t;

$$(\mathbf{2} \circ E_{ij}) \times (E_{kl} \circ \mathbf{2}) = (\cdot\ 2)^{([i]+[j])([k]+[l])} E_{kl} \circ E_{ij}. \qquad)3/2*$$

Ui f R.n busjy pg gl$(m|n)$.jowbsjboun pef rn i bt ui f gpsn

$$R(u,v) = \mathbb{I} + g(u,v)P, \qquad g(u,v) = \frac{c}{u \cdot \; v}. \qquad)3/3*$$

I fsf c jt b dpotubou- $\mathbb{I}$ boe P sftqfdujwfm bsf ui f jefoujuz n busjy boe ui f hsbefe qfsn vubujpo pqf sbups]4: ‘;

$$\mathbb{I} = \mathbf{2} \circ \; \mathbf{2} = \sum_{i,j=2}^{n+m} E_{ii} \circ \; E_{jj}, \qquad P = \sum_{i,j=2}^{n+m} (\cdot \; 2)^{[j]} E_{ij} \circ \; E_{ji}. \qquad)3/4*$$

Ui f l f z pckf dupg RJTN jt b r vbouwn n popespn z n busjy $T(u)$/ Jut n busjy fmfn fout $T_{i,j}(u)$ bsf hsbefe jo ui f tbn f x bz bt ui f n busjdf t $[E_{ij}]$; $[T_{i,j}(u)] = [i] + [j]$/ Ui f hsbejoh jt b n ps. qi jtn - j/f/ $[T_{i,j}(u) \times T_{k,l}(v)] = [T_{i,j}(u)] + [T_{k,l}(v)]$/ Ui fjs dpn n vubujpo sf mbujpot bsf hjwfo cz ui f RTT.sf mbujpo

$$R(u,v)\big)T(u) \circ \; \mathbf{2}\big[\big)\mathbf{2} \circ \; T(v)\big[= \big)\mathbf{2} \circ \; T(v)\big[\big)T(u) \circ \; \mathbf{2}\big[R(u,v). \qquad)3/5*$$

Fr vbujpo)3/5* i pmt jo ui f ufotps qspevdu $\mathbf{D}^{m|n} \circ \; \mathbf{D}^{m|n} \circ \; \mathcal{H}$- x i fsf $\mathcal{H}$ jt b I jmf sut qbdf pg ui f I bn jmpojbo voefs dpotjefsbujpo/ I fsf bmmi f ufotps qspevdut bsf hsbefe/

Ui f RTT.sf mbujpo)3/5* zjf mt b tf u pg dpn n vubujpo sf mbujpot gps ui f n popespn z n busjy fmfn fout

$$\begin{aligned} [T_{i,j}(u), T_{k,l}(v)\} &= (\cdot \; 2)^{[i]([k]+[l])+[k][l]} g(u,v)\Big)T_{k,j}(v)T_{i,l}(u) \cdot \; T_{k,j}(u)T_{i,l}(v)\Big[\\ &= (\cdot \; 2)^{[l]([i]+[j])+[i][j]} g(u,v)\Big)T_{i,l}(u)T_{k,j}(v) \cdot \; T_{i,l}(v)T_{k,j}(u)\Big[, \end{aligned} \qquad)3/6*$$

x i fsf x f joušpevdf ui f hsbefe dpn n vubups

$$[T_{i,j}(u), T_{k,l}(v)\} = T_{i,j}(u)T_{k,l}(v) \cdot \; (\cdot \; 2)^{([i]+[j])([k]+[l])} T_{k,l}(v)T_{i,j}(u). \qquad)3/7*$$

Ui f hsbefe usbotgf s n busjy jt ef–ofe bt ui f tvqf sušbdf pg ui f n popespn z n busjy

$$\mathcal{T}(u) = \text{tus}\, T(u) = \sum_{j=2}^{m+n} (\cdot \; 2)^{[j]} T_{j,j}(u). \qquad)3/8*$$

Pof dbo fbtjm di fdl]4: ‘ ui bu $[\mathcal{T}(u), \mathcal{T}(v)] = 1$/ Ui vt ui f usbotgf s n busjy dbo cf vtfe bt b hf of sbujoh gvodujpo pg jouf hsbm pg n pujpo pg bo jouf hsbcmf tztufn /

3/3/ Ppubujpo

Jo ui jt qbqf s x f vtf opubujpo boe dpowf oujpot pg ui f x psl]3: ‘/ Cftjeft ui f gvodujpo $g(u,v)$ x f joušpevdf ux p sbujpobm gvodujpot

$$\begin{aligned} f(u,v) &= 2 + g(u,v) = \frac{u \cdot \; v + c}{u \cdot \; v}, \\ h(u,v) &= \frac{f(u,v)}{g(u,v)} = \frac{u \cdot \; v + c}{c}. \end{aligned} \qquad)3/9*$$

Jo psefs up n bl f gpsn vmbt vojgpsn x f bmtp vtf ahsbefe(gvodujpot

$$g_{[i]}(u,v) = (\cdot\ 2)^{[i]} g(u,v) = \frac{(\cdot\ 2)^{[i]} c}{u \cdot\ v},$$

$$f_{[i]}(u,v) = 2 + g_{[i]}(u,v) = \frac{u \cdot\ v + (\cdot\ 2)^{[i]} c}{u \cdot\ v}, \qquad)3/:*$$

$$h_{[i]}(u,v) = \frac{f_{[i]}(u,v)}{g_{[i]}(u,v)} = \frac{(u \cdot\ v) + (\cdot\ 2)^{[i]} c}{(\cdot\ 2)^{[i]} c},$$

boe

$$\delta_i(u,v) = \frac{f_{[i]}(u,v)}{h(u,v)^{\eta_{i,m}}}, \qquad \hat\delta_i(u,v) = \frac{f_{[i+2]}(u,v)}{h(v,u)^{\eta_{i,m}}}. \qquad)3/21*$$

Pctfswf ui bu x f vtf ui f tvctdsjqu i gps ui f gvodujpot δ boe $\hat\delta$ jotufbe pg ui f tvctdsjqu [i]/ Ui jt jt cfdbvtf ui ftf gvodujpot bduvbmmz ubl f ui sff wbmvft/ Gps fybn qmf- $\delta_i(u,v) = f(u,v)$ gps $i < m$- $\delta_i(u,v) = g(u,v)$ gps $i = m$- boe $\delta_i(u,v) = f(v,u)$ gps $i > m$/ Jujt bmtp fbtz up tff ui bu $\hat\delta_i(u,v) = (\cdot\ 2)^{\eta_{i,m}}\delta_i(u,v)$/

Mfuvt gpsn vmbuf opx b dpowfoujpo po ui f opubujpo/ X f efopuf tfut pg wbsjbcmft cz cbs- gps fy. bn qmf- $\bar u$/ X i fo efbmjoh x jui tfwfsbmpgui fn - x f n bz fr vjq ui ftf tfut x jui beejujpobm tvqfstdsjqut $\bar s^i$- $\bar t^\sigma$- fud/ Joejwjevbmfmfn fout pgui f tfut ps tvctfut bsf efopufe cz Mbujo tvctdsjqut- gps jotubodf- u_j jt bo fmfn fou pg $\bar u$- t^i_k jt bo fmfn fou pg $\bar t^i$ fud/ Bt b svmf- ui f ovn cfs pg fmfn fout jo ui f tfut jt opu tipx o fyqmjdjumz jo ui f fr vbujpot- i px fwfs x f hjwf ui ftf dbsejobmjujft jo tqfdjbm dpn n fout up ui f gpsn vmbt/ X f bttvn f ui bu ui f fmfn fout jo fwfsz tvctfu pg wbsjbcmft bsf psefsfe jo tvdi b x bz ui bu ui f tfr vfodf pg ui fjs tvctdsjqut jt tusjdumz jodsfbtjoh; $\bar t^i = \{t^i_2, t^i_3, \ldots, t^i_{r_i}\}$/ X f dbmm ui jt psefsjoh ui f obuvsbm psefs/

X f vtf b ti psui boe opubujpo gps qspevdut pg ui f sbujpobm gvodujpot)3/9*´)3/21*/ Obn fmz- jg tpn f pg ui ftf gvodujpot efqfoet po b tfu pg wbsjbcmft)ps ux p tfut pg wbsjbcmft*- ui jt n fbot ui bu pof ti pvme ubl f ui f qspevdu pwfs ui f dpssftqpoejoh tfu)ps epvcmf qspevdu pwfs ux p tfut*/ Gps fybn qmf-

$$g(\bar u, v) = \prod_{u_j \in \bar u} g(u_j, v),$$

$$f_{[i]}(t^{i\cdot\ 2}_k, \bar t^i) = \prod_{t^i_\ell \in \bar t^i} f_{[i]}(t^{i\cdot\ 2}_k, t^i_\ell), \qquad)3/22*$$

$$\delta_\ell(\bar s^i, \bar t^\ell) = \prod_{s^i_j \in \bar s^i} \prod_{t^\ell_k \in \bar t^\ell} \delta_\ell(s^i_j, t^\ell_k).$$

Cz ef–ojujpo- boz qspevdu pwfs ui f fn quz tfu jt fr vbm up 2/ B epvcmf qspevdu jt fr vbm up 2 jg bu mfbtu pof pg ui f tfut jt fn quz/

Cfmpx x f x jmmfyufoe ui jt dpowfoujpo up ui f qspevdut pg n popespn z n busjy fousjft boe ui fjs fjhfowbmvft)tff)4/4* boe)4/5**/

41 Cfui f wfdupst

Cfui f wfdupst cfmpoh up ui f tqbdf $\mathcal{H}$ jo x i jdi ui f n popespn z n busjy fousjft bdu/ X f ep opu tqfdjgz ui jt tqbdf- i px fwfs- x f bttvn f ui bu ju dpoubjot b *qtfvepwbdvvn wfdups* $|1\rangle$- tvdi ui bu

$$T_{i,i}(u)|1\rangle = \nu_i(u)|1\rangle, \qquad i = 2, \ldots, m+n, \qquad)4/2*$$

$$T_{i,j}(u)|1\rangle = 1, \qquad i > j,$$

x i f sf $\nu_i(u)$ bsf tpn f tdbrbs gvodujpot/ Jo ui f gsbn f x psl pgui f hf of sbrjfife n pef m]6‘ dpotjefsfe jo ui jt qbqfs- ui fz sfn bjo gsff gvodujpobm qbsbn fufst/ Cfmpx juxjmcf dpowfojfouup efbmx jui sbujpt pgui ftf gvodujpot

$$\gamma_i(u) = \frac{\nu_i(u)}{\nu_{i+2}(u)}, \qquad i = 2, \ldots, m + n \cdot \ 2. \qquad)4/3*$$

Xf fyufoe ui f dpowfoujpo po ui f ti psui boe opubujpo)3/22* up ui f qspevdut pgui f gvodujpot joupspevdfe bcpwf- gps fybn qrfi-

$$\nu_k(\bar{u}) = \prod_{u_j \in \bar{u}} \nu_k(u_j), \qquad \gamma_i(\bar{t}^i) = \prod_{t^i_\ell \in \bar{t}^i} \gamma_i(t^i_\ell). \qquad)4/4*$$

Xf vtf ui f tbn f dpowfoujpo gps ui f qspevdut pgdpn n vujoh pqf sbupst

$$T_{i,j}(\bar{u}) = \prod_{u_j \in \bar{u}} T_{i,j}(u_j), \qquad \text{gps} \qquad [i] + [j] = 1, \qquad \text{n pe 3.} \qquad)4/5*$$

Gjobmmz- gps ui f qspevdupgpee pqf sbupst $T_{i,j}$ x jui $[i] + [j] = 2$ x f joupspevdf b tqf djbmopubujpo

$$\mathbb{T}_{i,j}(\bar{u}) = \frac{T_{i,j}(u_2) \ldots T_{i,j}(u_p)}{\prod_{2 \geq k < \ell \geq p} h(u_\ell, u_k)}, \qquad [i] + [j] = 2, \qquad i < j,$$
$$\mathbb{T}_{i,j}(\bar{u}) = \frac{T_{i,j}(u_2) \ldots T_{i,j}(u_p)}{\prod_{2 \geq k < \ell \geq p} h(u_k, u_\ell)}, \qquad [i] + [j] = 2, \qquad i > j. \qquad)4/6*$$

Evf up ui f dpn n vubujpo sf mbujpot)3/6* ui f pqf sbups qspevdut)4/6* bsf tzn n fusjd pwf s qf sn vub. ujpot pgui f qbsbn fufst $\bar{u}$/

4/2/ Dpmpsjoh

Jo qi ztjdbmn pef mt- wf dupst pgui f tqbdf $\mathcal{H}$ eftdsjcf tubuft x jui rvbtjqbsujdmft pg ejggf sfou uzqft)dpmpst*/ Jo $\mathfrak{gl}(m|n)$.cbtfe n pef mt rvbtjqbsujdmft n bz i bwf $N = m + n \cdot\ 2$ dpmpst/ Mfu $\{r_2, \ldots, r_N\}$ cf b tfu pg opo.ofhbujwf joufhf st/ Xf tbz ui bu b tubuf i bt dpmpsjoh $\{r_2, \ldots, r_N\}$- jg ju dpoubjot r_i rvbtjqbsujdmft pg ui f dpmps i/ B tubuf x jui b –yfe dpmpsjoh dbo cf pcubjofe cz tvddfttjwf bqqmjdbujpo pgui f dsfbujpo pqf sbupst $T_{i,j}$ x jui $i < j$ up ui f wf dups $|1\rangle$- x i jdi i bt ffsp dpmpsjoh/ Bdujoh po ui jt tubuf- bo pqf sbups $T_{i,j}$ beet rvbtjqbsujdmft x jui ui f dpmpst $i, \ldots, j \cdot\ 2$- pof qbsujdmf pgfbdi dpmps/ Jo qbsujdvmbs- ui f pqf sbups $T_{i,i+2}$ dsfbuft pof rvbtjqbsujdmf pgui f dpmps i- ui f pqf sbups $T_{2,n+m}$ dsfbuft N rvbtjqbsujdmft pg N ejggf sfoudpmpst/ Ui f ejbhpobmpqf sbupst $T_{i,i}$ bsf ofvusbmui f n busjy fmfn fout $T_{i,j}$ x jui $i > j$ qmbz ui f spmf pgbooji jmbujpo pqf sbupst/ Bdujoh po ui f tubuf pgb –yfe dpmpsjoh- ui f booji jmbujpo pqf sbups $T_{i,j}$ sfn pwf t gspn ui jt tubuf ui f rvbtjqbsujdmft x jui ui f dpmpst $j, \ldots, i \cdot\ 2$- pof qbsujdmf pg fbdi dpmps/ Jo qbsujdvmbs- jg $j \cdot\ 2 < k < i$- boe ui f booji jmbujpo pqf sbups $T_{i,j}$ bdut po b tubuf jo x i jdi ui f sf jt op qbsujdmft pgui f dpmps k- ui fo ui jt bdujpo wbojti ft/

Ui jt ef–ojujpo dbo cf gpsn bmjfife buui f mfwf mpgui f Zbohjbo ui spvhi ui f Dbsubo hf of sbupst pg ui f Mjf tvqf sbmhfcsb $\mathfrak{gl}(m|n)$/ Joeff e- ui f ffsp n peft

$$T_{ij}[1] = \lim_{u\to\infty} \frac{u}{c}\big) T_{ij}(u) \cdot\ \ \eta_{ij} \big[$$

gpsn b $\mathfrak{gl}(m|n)$ tvqf sbmhfcsb- x jui dpn n vubujpo sf mbujpot

$$[T_{ij}[1], T_{kl}[1]\} = (\cdot\ 2)^{[i]([k]+[l])+[k][l]}\Big)\eta_{il}\, T_{kj}[1]\cdot\ \ \eta_{jk}\, T_{il}[1]\Big[, \quad i,j,k,l = 2,\ldots,m+n.$$
)4/7*

Ui jt tvqfsbrhfcsb jt b tzn n fusz pg ui f hfofsbrjfife n pefmtjodf judpn n vuft xjui ui f usbotgfs n busjy- $[T_{ij}[1], \mathcal{T}(z)] = 1$- $i, j = 2, \ldots, m+n$/ Jo gbdu ui f n popespn z n busjy foujsft gpsn b sfqsftfoubujpo pg ui jt tvqfsbrhfcsb;

$$[T_{ij}[1], T_{kl}(z)\} = (\cdot\ 2)^{[i]([k]+[l])+[k][l]}\Big)\eta_{il}\, T_{kj}(z)\cdot\ \ \eta_{jk}\, T_{il}(z)\Big[, \quad i,j,k,l = 2,\ldots,m+n.$$
)4/8*

Jo qbsujdvrbs- gps ui f Dbsubo hfofsbupst $T_{jj}[1]$ xf pcubjo

$$[T_{jj}[1], T_{kl}(z)] = (\cdot\ 2)^{[j]}\big)\eta_{jl}\cdot\ \ \eta_{jk}\big[\, T_{kl}(z), \quad j,k,l = 2,\ldots,m+n. \qquad)4/9*$$

Ui fo- ui f dprpst dpssftqpoe up ui f fjhfowbrnft voefs ui f Dbsubo hfofsbupst[2]

$$h_j = \sum_{k=2}^{j} (\cdot\ 2)^{[k]}\, T_{kk}[1], \quad j = 2,\ldots,m+n\cdot\ 2. \qquad)4/{:}*$$

Joeffe- pof dbo di fdl ui bu

$$[h_j, T_{kl}(z)] = \varepsilon_j(k,l)\, T_{kl}(z) \quad \text{x jui} \quad \begin{cases} \varepsilon_j(k,l) = \cdot\ 2 & \text{jg} k \geq j < l \\ \varepsilon_j(k,l) = +2 & \text{jg} l \geq j < k \\ \varepsilon_j(k,l) = 1 & \text{pui fsx jtf} \end{cases} \qquad)4/21*$$

Ui ftf fjhfowbrnft kvtudpssftqpoe up dsfbujpo0booji jrbujpo pqfsbupst bt eftdsjcfe bcpwf/

Cfui f wfdupst bsf dfsubjo qprnopn jbrn jo ui f dsfbujpo pqfsbupst $T_{i,j}$ bqqrjfe up ui f wfdups $|1\rangle$/ Tjodf Cfui f wfdupst bsf fjhfowfdupst voefs ui f Dbsubo hfofsbupst $T_{kk}[1]$- ui fz bsf brtp fjhfowfd. upst pg ui f dprps hfofsbupst h_j- boe i fodf dpoubjo pomz ufsn t xjui ui f tbn f dprpsjoh/

Sfn bsl Jo wbsjpvt n pefrn pg qi ztjdbrjoufsftu ui f dprpsjoh pg ui f Cfui f wfdupst pcfzt dfsubjo dpotusbjout- gps jotubodf- $r_2 \sim r_3 \sim \cdots \sim r_N$/ Jo qbsujdvrbs- ui jt dbtf pddvst jg ui f n popespn z n busjy pg ui f n pefrjt hjwfo cz ui f qspevdupg ui f R.n busjdft)3/3*jo ui f gvoebn foubrsfqsftfo. ubujpo/ Xf ep opusftusjdupvstfrwft xjui ui jt qbsujdvrbs dbtf boe ep opujn qptf boz sftusjdujpo gps ui f dprpsjoh pg ui f Cfui f wfdupst/ Ui vt- jo x i bugprrpx t r_i bsf bscjusbsz opo.ofhbujwf joufhfst/

Jo ui jt qbqfs xf ep opuvtf bo fyqrjdjugpsn pg ui f Cfui f wfdupst- i px fwfs- ui f sfbefs dbo –oe jujo]3: ‘/ B hfofsjd Cfui f wfdups pg $\mathfrak{gl}(m|n)$.cbtfe n pefmefqfoet po $N = m+n\cdot\ 2$ tfut pg wbsjbcrft $\bar{t}^2, \bar{t}^3, \ldots, \bar{t}^N$ dbrrfe Cfui f qbsbn fufst/ Xf efopuf Cfui f wfdupst cz $\mathbb{B}(\bar{t})$- x i fsf

$$\bar{t}\ = \{t_2^2, \ldots, t_{r_2}^2; t_2^3, \ldots, t_{r_3}^3; \ldots; t_2^N, \ldots, t_{r_N}^N\}, \qquad)4/22*$$

boe ui f dbsejobrjujft r_i pg ui f tfut $\bar{t}^i$ dpjodjef xjui ui f dprpsjoh/ Ui vt- fbdi Cfui f qbsbn fufs t_k^i dbo cf bttpdjbufe xjui b r vbtjqbsujdrf pg ui f dprps i/

Cfui f wfdupst bsf tzn n fusjd pwfs qfsn vubujpot pg ui f qbsbn fufst t_k^i xjui jo ui f tfu $\bar{t}^i$- i px fwfs- ui fz bsf oputzn n fusjd pwfs qfsn vubujpot pwfs qbsbn fufst cfrpohjoh up ejggfsfoutfut $\bar{t}^i$ boe $\bar{t}^j$/ Gps hfofsjd Cfui f wfdupst ui f Cfui f qbsbn fufst t_k^i bsf hfofsjd dpn qrfiy ovn cfst/ Jg ui ftf qb. sbn fufst tbujtgz b tqfdjbmtztufn pg fr vbujpot)Cfui f fr vbujpot*- ui fo ui f dpssftqpoejoh wfdups

[2] Ui f rbtuhfofsbups h_{m+n} jt dfousbmtf)4/21*/

cf dpn ft bo fjhf owf dups pg ui f usbotgf s n busjy)3/8*/ Jo ui jt dbtf jujt dbmfie *po.ti fmmCfui f wfd. ups*/ Jo ui jt qbqfs x f dpotjefs hfofsjd Cfui f wfdupst- i px fwfs- tpn f gpsn vmbt)gps jotubodf- ui f tvn gpsn vmb gps ui f tdbmbs qspevdu)5/22*-)5/26** dbo cf tqfdj–fe up ui f dbtf pg po.ti fmmCfui f wfdupst bt x fmm/

Ui pvhi x f ep opu vtf ui f fyqmjdju gpsn pg ui f Cfui f wfdupst- x f ti pvme –y ui fjs opsn bmj{bujpo/ Xf i bwf bmsfbez n foujpofe ui bu b hfofsjd Cfui f wfdups i bt ui f gpsn pg b qpmzopn jbm jo $T_{i,j}$ x jui $i < j$ bqqmjfe up ui f qtfvepwbdvvn $|1\rangle$/ Bn poh bmm ui f ufsn t pg ui jt qpmzopn jbm ui fsf jt pof n popn jbm ui bu dpoubjot ui f pqfsbupst $T_{i,j}$ x jui $j \cdot i = 2$ pomz/ Mfu vt dbmm ui jt ufsn ui f *n bjo ufsn* boe efopuf ju cz $\widetilde{\mathbb{B}}(\bar{t})$/ Ui fo

$$\mathbb{B}(\bar{t}) = \widetilde{\mathbb{B}}(\bar{t}) + \dots . \qquad)4/23*$$

x i fsf fmmjqtjt n fbot bmm ui f ufsn t dpoubjojoh bu mfbtu pof pqfsbups $T_{i,j}$ x jui $j \cdot i > 2$/ Xf x jmm –y ui f opsn bmj{bujpo pg ui f Cfui f wfdupst cz –yjoh b dpn n fsjdbm dpf g–djfou pg ui f n bjo ufsn

$$\widetilde{\mathbb{B}}(\bar{t}) = \frac{\mathbb{T}_{2,3}(\bar{t}^2)\dots\mathbb{T}_{N,N+2}(\bar{t}^N)|1\rangle}{\prod_{i=2}^{N}\nu_{i+2}(\bar{t}^i)\prod_{i=2}^{N\cdot 2} f_{[i+2]}(\bar{t}^{i+2},\bar{t}^i)}, \qquad)4/24*$$

x i fsf

$$\mathbb{T}_{i,i+2}(\bar{t}^i) = \frac{T_{i,i+2}(t_2^i)\dots T_{i,i+2}(t_{r_i}^i)}{\Big)\prod_{2\geq j<k\geq r_i} h(t_k^i,t_j^i)\Big[^{\eta_{i,m}}}. \qquad)4/25*$$

Sfdbmm ui bu x f vtf i fsf ui f ti psu boe opubujpo gps ui f qspevdut pg ui f gvodujpot ν_{j+2} boe $f_{[j+2]}$/ Ui f opsn bmj{bujpo jo)4/24* jt ejggfsfou gspn ui f pof vtfe jo]3: ‘ cz ui f qspevdu $\prod_{j=2}^{N}\nu_{j+2}(\bar{t}^j)$/ Ui jt beejujpobm opsn bmj{bujpo gbdups jt dpowfojfou cfdbvtf jo ui jt dbtf ui f tdbmbs qspevdut pg ui f Cfui f wfdupst efqfoe po ui f sbujpt γ_i)4/3* pomz/

Tjodf ui f pqfsbupst $T_{i,i+2}$ boe $T_{j,j+2}$ ep opu dpn n vuf gps $i \neq j$- ui f n bjo ufsn dbo cf x sjuufo jo tfwfsbm gpsn t dpssftqpoejoh up ejggfsfou psefsjoh pg ui f n popespn z n busjy fousjft/ Ui f ps. efsjoh jo)4/24* pcwjpvtmz bsjtft jg x f dpotusvdu Cfui f wfdupst wjb ui f fn cfeejoh pg $\mathfrak{gl}(m\cdot 2|n)$ up $\mathfrak{gl}(m|n)$/

4/3/ Npsqi jtn pg Cfui f wfdupst

Zbohjbot $Y(\mathfrak{gl}(m|n))$ boe $Y(\mathfrak{gl}(n|m))$ bsf sfmbufe cz b n psqi jtn φ]51‘

$$\varphi : \begin{cases} Y(\mathfrak{gl}(m|n)) & \to \quad Y(\mathfrak{gl}(n|m)), \\ T_{i,j}^{m|n}(u) & \to \ (\cdot\ 2)^{[i][j]+[j]+2}\,T_{N+3\cdot\ j,N+3\cdot\ i}^{n|m}(u), \qquad i,j = 2,\dots,N+2, \end{cases} \qquad)4/26*$$

boe x f sfdbmm ui bu $N = m + n \cdot\ 2$/ I fsf x f bmtp i bwf fr vjqqfe ui f pqfsbupst T_{ij} x jui beej. ujpobm tvqfs tdsjqut ti px joh ui f dpssftqpoejoh Zbohjbot/ Ui jt n bqqjoh bmtp bdut po ui f wbdvvn fjhfowbmvft $\nu_i(u)$)4/2* boe ui fjs sbujpt $\gamma_i(u)$)4/3*

$$\varphi : \begin{cases} \nu_i(u) & \to \ \cdot\ \nu_{N+3\cdot\ i}(u), \qquad i = 2,\dots,N+2, \\ \gamma_i(u) & \to \ \frac{2}{\gamma_{N+2\cdot\ i}(u)}, \qquad i = 2,\dots,N. \end{cases} \qquad)4/27*$$

Npsqi jtn φ joevdft b n bqqjoh pg Cfui f wfdupst $\mathbb{B}^{m|n}$ pg $Y(\mathfrak{gl}(m|n))$ up Cfui f wfdupst $\mathbb{B}^{n|m}$ pg $Y(\mathfrak{gl}(n|m))$/ Up eftdsjcf ui jt n bqqjoh x f jouspevdf tqfdjbm psefsjoht pg ui f tfut pg Cfui f qbsbn fufst/ Obn fmz- mfu

$$\vec{t} = \{\bar{t}^2, \bar{t}^3, \ldots, \bar{t}^N\} \qquad \text{boe} \qquad \overleftarrow{t} = \{\bar{t}^N, \ldots, \bar{t}^3, \bar{t}^2\}. \qquad)4/28*$$

Ui f psefsjoh pgui f Cfui f qbsbn fufst x jui jo f wfsz tfu $\bar{t}^k$ jt opufttfoujbm/Ui fo

$$\varphi\Big)\mathbb{B}^{m|n}(\vec{t})\Big[= \frac{(\cdot\ 2)^{r_m}\mathbb{B}^{n|m}(t)}{\prod_{k=2}^{N}\gamma_{N+2\cdot\ k}(\bar{t}^k)}. \qquad)4/29*$$

Bqqmzjoh ui f n bqqjoh)4/29* up $\mathbb{B}^{m|n}$ boe ui fo sfqmbdjoh $m \leftrightarrow n$ x f pcubjo bo bmufsobujwf eftdsjqujpo pgui f Cfui f wfdupst dpssftqpoejoh up ui f fn cfeejoh pg gl$(m|n\cdot\ 2)$ up gl$(m|n)$/ Ui f vtf pg φ)4/29* bmmpx t pof up ftubcmjti jn qpsubou qspqfsujft pgui f Cfui f wfdupst tdbmbs qspevdut)tff tfdujpo 8/3*/

4/4/ EvbmCfui f wfdupst

EvbmCfui f wfdupst cfmpoh up ui f evbmtqbdf $\mathcal{H}^{\otimes}$- boe ui fz bsf qpmzopn jbmt jo $T_{i,j}$ x jui $i > j$ bqqmjfe gspn ui f sjhi u up ui f evbmqtfvepwbdvvn wfdups $\langle 1|$/ Ui jt wfdups qpttfttft qspqfsujft tjn jmbs up)4/2*

$$\begin{aligned} &\langle 1|T_{i,i}(u) = \nu_i(u)\langle 1|, && i = 2, \ldots, m+n, \\ &\langle 1|T_{i,j}(u) = 1\,, && i < j\,, \end{aligned} \qquad)4/2:*$$

x i fsf ui f gvodujpot $\nu_i(u)$ bsf ui f tbn f bt jo)4/2*/

Xf efopuf evbmCfui f wfdupst cz $\mathbb{C}(\bar{t})$- x i fsf ui f tfupgCfui f qbsbn fufst $\bar{t}$ dpotjtut pgtfwfsbm tfut $\bar{t}^i$ bt jo)4/22*/ Tjn jmbsmz up i px jux bt epof gps Cfui f wfdupst- x f dbo jouspevdf ui f dpmpsjoh pgui f evbmCfui f wfdupst/ Buui f tbn f ujn f ui f spmf pg dsfbujpo boe booji jmbujpo pqfsbupst bsf sfwfstfe/

Pof dbo pcubjo evbmCfui f wfdupst wjb b tqfdjbmbouj n psqi jtn pgui f bmhfcsb)3/5*]51‘

$$\Psi:\ T_{i,j}(u) \to (\cdot\ 2)^{[i]([j]+2)}T_{j,i}(u). \qquad)4/31*$$

Ui jt bouj n psqi jtn jt opui joh cvu b tvqfs)ps fr vjwbmfouum- hsbefe* usbotqptjujpo dpn qbujcmf x jui ui f opujpo pg tvqfsusbdf/ Jut bujt–ft b qspqfsuz

$$\Psi(A\, \times B) = (\cdot\ 2)^{[A][B]}\Psi(B)\, \times \Psi(A), \qquad)4/32*$$

x i fsf A boe B bsf bscjusbsz fmfn fout pgui f n popespn z n busjy/ Jg x f fyufoe ui f bdujpo pgui jt bouj n psqi jtn up ui f qtfvepwbdvvn wfdupst cz

$$\begin{aligned} &\Psi\big)|1\rangle\big[= \langle 1|, \qquad && \Psi\big)A|1\rangle\big[= \langle 1|\Psi\big)A\big[, \\ &\Psi\big)\langle 1|\big[= |1\rangle, && \Psi\big)\langle 1|A\big[= \Psi\big)A\big[|1\rangle, \end{aligned} \qquad)4/33*$$

ui fo juuvsot pvuui bu]3: ‘

$$\Psi\big)\mathbb{B}(\bar{t})\big[= \mathbb{C}(\bar{t}), \qquad \Psi\big)\mathbb{C}(\bar{t})\big[= (\cdot\ 2)^{r_m}\mathbb{B}(\bar{t}), \qquad)4/34*$$

x i fsf $r_m = '\,\bar{t}^m$/

Sfn bsl Juti pvme opucf tvsqsjtjoh ui bu $\Psi^3\big)\mathbb{B}(\bar{t})\big[\neq \mathbb{B}(\bar{t})$/ Ui f qpjoujt ui bu ui f bouj n psqi jtn Ψ jt jefn qpufoupg psefs 5 boe jut tr vbsf jt ui f qbsjuz pqfsbups)dpvoujoh ui f ovn cfs pg pee n popespn z n busjy fmfn fout n pevmp 3*/

Ui vt- evbmCfui f wfdupst bsf qpmzopn jbmt jo $T_{i,j}$ x jui $i > j$ bdujoh gspn ui f sjhi u pop $\langle 1|$/ Ui fz bmtp dpoubjo ui f n bjo ufsn $\widetilde{\mathbb{C}}(\bar{t})$- x i jdi opx dpotjtut pgui f pqfsbupst $T_{i,j}$ x jui $i\cdot\ j = 2$/ Ui f n bjo ufsn pgui f evbmCfui f wfdups dbo cf pcubjofe gspn)4/24* wjb ui f n bqqjoh Ψ;

$$\widetilde{\mathbb{C}}(\bar t) = \frac{(\cdot\ 2)^{r_m(r_m\cdot\ 2)/3}\langle 1|\mathbb{T}_{N+2,N}(\bar t^N)\dots\mathbb{T}_{3,2}(\bar t^2)}{\prod_{i=2}^{N}\nu_{i+2}(\bar t^i)\prod_{i=2}^{N\cdot\ 2} f_{[i+2]}(\bar t^{i+2},\bar t^i)}, \qquad)4/35*$$

x i fsf

$$\mathbb{T}_{i+2,i}(\bar t^i) = \frac{T_{i+2,i}(t_2^i)\dots T_{i+2,i}(t_{r_i}^i)}{\Big)\prod_{2\geq j<k\geq r_i} h(t_j^i,t_k^i)\Big[^{\eta_{i,m}}}. \qquad)4/36*$$

Gjobmmz- vtjoh ui f n psqi jtn φ x f pcubjo b sfmbujpo cfux ffo evbmCfui f wfdupst dpssftqpoejoh up ui f Zbohjbot $Y(\mathfrak{gl}(m|n))$ boe $Y(\mathfrak{gl}(n|m))$

$$\varphi\Big)\mathbb{C}^{m|n}(\vec t)\Big[= \frac{\mathbb{C}^{n|m}(t)}{\prod_{k=2}^{N}\gamma_{N+2\cdot\ k}(\bar t^k)}. \qquad)4/37*$$

51 P blo sftvmut

Jo ui jt tfdujpo x f qsftfou ui f n bjo sftvmut pg ui f qbqfs/ Ui fz bsf pg ui sff uzqft; sfdvstjpo gpsn vmbt gps Cfui f wfdupst @vn gpsn vmb gps ui f Cfui f wfdupst tdbmbs qspevdu@sfdvstjpo gpsn vmbt gps ui f tdbmbs qspevdu ui jhi ftu dpfg–djfout/ Sfdbmmui bu x f gpsn bmmz dpotjefs ui f dbtf $m,n \neq 1$/ I px fwfs- jo tvctfdujpo 5/4 x f tqfdjgz pvs sftvmut up ui f qbsujdvmbs dbtf pg $\mathfrak{gl}(m)$.cbtfe n pefmn- ui bujt- $n = 1$/ Ui f dbtf $m = 1$ dbo cf pcubjofe gspn ui f mbuufs wjb sfqmbdfn fou $c \to \cdot\ c$ jo ui f R.n busjy)3/3*/

5/2/ Sfdvstjpo gps Cfui f wfdupst

I fsf x f hjwf sfdvstjpot gps)evbm*Cfui f wfdupst/ Ui ftf sfdvstjpot bmmpx vt up dpotusvdu Cfui f wfdupst- l opx joh ui f poft efqfoejoh po b tn bmmfs ovn cfs pg qbsbn fufst/ Ui f dpssftqpoejoh qsppgt bsf hjwfo jo tfdujpo 6/

Rspr ptkukpo 5/2/1 *Cfui f wfdupst pg* $\mathfrak{gl}(m|n)$.*cbtfe n pefmt tbujtgz ui f sfdvstjpo*

$$\mathbb{B}(\}z,\bar t^{2|};\}\bar t^{k|}{}_3^N) = \sum_{j=3}^{N+2}\frac{T_{2,j}(z)}{\nu_3(z)}\sum_{\text{qbsu}(\bar t^3,\dots,\bar t^{j\cdot\ 2})}\mathbb{B}(\}\bar t^{2|};\}\bar t_{\text{II}}^{k|}{}_3^{j\cdot\ 2};\}\bar t^{k|}{}_j^N)$$

$$* \ \frac{\prod_{\sigma=3}^{j\cdot\ 2}\gamma_\sigma(\bar t_{\text{I}}^\sigma)g_{[\sigma]}(\bar t_{\text{I}}^\sigma,\bar t_{\text{I}}^{\sigma\cdot\ 2})\delta_\sigma(\bar t_{\text{II}}^\sigma,\bar t_{\text{I}}^\sigma)}{h(\bar t^2,z)^{\eta_{m,2}}\prod_{\sigma=2}^{j\cdot\ 2} f_{[\sigma+2]}(\bar t^{\sigma+2},\bar t_{\text{I}}^\sigma)}. \qquad)5/2*$$

I fsf gps $j > 3$ *ui f tfu pg Cfui f qbsbn fufst* $\bar t^3,\dots,\bar t^{j\cdot\ 2}$ *bsf ejwjefe joup ejtkpjou tvctfut* $\bar t_{\text{I}}^\sigma$ *boe* $\bar t_{\text{II}}^\sigma$ $)\sigma = 3,\dots,j\cdot\ 2$, *tvdi ui bu ui f tvctfu* $\bar t_{\text{I}}^\sigma$ *dpotjtut pg pof fmfn fou pomz*A’ $\bar t_{\text{I}}^\sigma = 2$/ *Ui f tvn jt ublfo pwfs bmmqbsujujpot pg ui jt uzqf/ a f tfu cz efgojujpo* $\bar t_{\text{I}}^2 \leq z$ *boe* $\bar t^{N+2} = \emptyset$/

Xf vtf ui f gpmmpx joh opubujpo jo Qspqptjujpo 5/2

$$\mathbb{B}(\}z,\bar t^{2|};\}\bar t^{k|}{}_3^N) = \mathbb{B}(\}z,\bar t^{2|};\bar t^3;\dots;\bar t^N),$$
$$\mathbb{B}(\}\bar t^{2|};\}\bar t_{\text{II}}^{k|}{}_3^{j\cdot\ 2};\}\bar t^{k|}{}_j^N) = \mathbb{B}(\bar t^2;\bar t_{\text{II}}^3;\dots;\bar t_{\text{II}}^{j\cdot\ 2};\bar t^j;\dots;\bar t^N). \qquad)5/3*$$

Ui jt boe tjn jmbs opubujpo x jmmcf vtfe ui spvhi pvu pg ui f qbqfs/

Sfn bsl Xf tusftt ui bufbdi pgui f tvctfut $\bar{t}^3_{\rm J}, \ldots, \bar{t}^N_{\rm J}$ jo)5/2* n vtudpotjtupgfybdumz pof fmfi. n fou/ I px fwfs- ui jt dpoejujpo jt opugfbtjcrfi- jgui f psjhjobmCfui f wfdups $\mathbb{B}(t)$ dpoubjot bo fn quz tfu $\bar{t}^k = \emptyset$ gps tpn f $k \in [3, \ldots, N]$/ Jo ui jt dbtf- ui f tvn pwfs j jo)5/2* csfbl t pgg bu $j = k$/ Joeffe- ui f bdujpo pgui f pqfsbupst $T_{2,j}(z)$ xjui $j > k$ po b Cfui f wfdups ofdfttbsjmz dsfbuft b r vbtjqbsujdrfi pgui f dprps k/ Tjodf ui jt r vbtjqbsujdrfi jt bctfoujo ui f mt pg)5/2*- x f dboopui bwf ui f pqfsbupst $T_{2,j}(z)$ xjui $j > k$ jo ui f si t/ Tjn jrbs dpotjefsbujpo ti px t ui bu jg $\mathbb{B}(t)$ dpoubjot tfwfsbmfn quz tfut $\bar{t}^{k_2}, \ldots, \bar{t}^{k_\ell}$- ui fo ui f tvn foet bu $j = $ n jo$(k_2, \ldots, k_\ell)$/

Sfn bsl Pof dbo opujdf ui bugps $m = 2$ bo beejujpobmgbdups $h(\bar{t}^2, z)^{\cdot\ 2}$ bqqfbst jo ui f sfdvstjpo/ Ui f qpjoujt ui bu x jui ui jt sfdvstjpo x f bee b r vbtjqbsujdrfi pgui f dprps 2 up ui f psjhjobmtfupg r vbtjqbsujdrfit wjb ui f bdujpot pgui f pqfsbupst $T_{2,j}$/ Gps $m = 2$ bmmui ftf pqfsbupst bsf pee- x i jdi fyqrbjot ui f bqqfbsbodf pgui f gbdups $h(\bar{t}^2, z)^{\cdot\ 2}$/ Ui jt ejggfsfodf dbo brtp cf tffo fyqrjdjumz jo ui f fybn qrfi pgsfdvstjpo gps ui f n bjo ufsn)4/24*

$$\widetilde{\mathbb{B}}(\}z, \bar{t}^2|; \}\bar{t}^k|^N_3) = \frac{T_{2,3}(z)\widetilde{\mathbb{B}}(\bar{t})}{h(\bar{t}^2, z)^{\eta_{m,2}} \nu_3(z) f_{[3]}(\bar{t}^3, z)} . \qquad)5/4*$$

Vtjoh ui f n bqqjoht)4/26* boe)4/31* pof dbo pcubjo pof n psf sfdvstjpo gps ui f Cfui f wfdupst boe ux p sfdvstjpot gps ui f evbmpoft/

Rspr ptkkpo 5B1 *Cfui f wfdupst pggl$(m|n)$.cbtfe n pefmt tbujtgfi b sfdvstjpo*

$$\mathbb{B}(\}\bar{t}^k|^{N\cdot\ 2}_2; \}z, \bar{t}^N|) = \sum_{j=2}^{N} \frac{T_{j,N+2}(z)}{\nu_{N+2}(z)} \sum_{\text{qbsu}(\bar{t}^j,\ldots,\bar{t}^{N\cdot\ 2})} \mathbb{B}(\}\bar{t}^k|^{j\cdot\ 2}_2; \}\bar{t}^k_{\rm JJ}|^{N\cdot\ 2}_j; \bar{t}^N)$$

$$* \frac{\prod_{\sigma=j}^{N\cdot\ 2} g_{[\sigma+2]}(\bar{t}^{\sigma+2}_{\rm J}, \bar{t}^\sigma_{\rm J})\delta_\sigma(\bar{t}^\sigma_{\rm J}, \bar{t}^\sigma_{\rm JJ})}{h(\bar{t}^N, z)^{\eta_{m,N}} \prod_{\sigma=j}^{N} f_{[\sigma]}(\bar{t}^\sigma_{\rm J}, \bar{t}^{\sigma\cdot\ 2})} . \qquad)5/5*$$

I fsf gps $j < N$ ui f tfut pgCfui f qbsbn fufst $\bar{t}^j, \ldots, \bar{t}^{N\cdot\ 2}$ bsf ejwjefe jouprejtkpjoutvctfut $\bar{t}^\sigma_{\rm J}$ boe $\bar{t}^\sigma_{\rm JJ}$)$\sigma = j, \ldots, N \cdot\ 2$, tvdi ui bu ui f tvctfu $\bar{t}^\sigma_{\rm J}$ dpotjtut pgpof fmfn fouA' $\bar{t}^\sigma_{\rm J} = 2$/ Ui f tvn jt ublfo pwfs bmmqbsujujpot pgui jt uzqf/ a f tfucfi ef–ojujpo $\bar{t}^N_{\rm J} \leq z$ boe $\bar{t}^1 = \emptyset$/

Sfn bsl Jg ui f Cfui f wfdups $\mathbb{B}(t)$ dpoubjot tfwfsbmfn quz tfut $\bar{t}^{k_2}, \ldots, \bar{t}^{k_\ell}$- ui fo ui f tvn pwfs j jo)5/5* cfhjot xjui $j = $ n by$(k_2, \ldots, k_\ell) + 2$/

Bdujoh xjui boujn psqi jtn)4/31* poup fr vbujpot)5/2* boe)5/5* x f jn n fejbufmz bssjwf bu sfdvstjpot gps ui f evbmCfui f wfdupst/

Dpspmbsfi 5B21 *EvbmCfui f wfdupst pggl$(m|n)$.cbtfe n pefmt tbujtgfi sfdvstjpot*

$$\mathbb{C}(\}z, \bar{s}^2|; \}\bar{s}^k|^N_3) = \sum_{j=3}^{N+2} \sum_{\text{qbsu}(\bar{s}^3,\ldots,\bar{s}^{j\cdot\ 2})} \mathbb{C}(\}\bar{s}^2|; \}\bar{s}^k_{\rm JJ}|^{j\cdot\ 2}_3; \}\bar{s}^k|^N_j) \frac{T_{j,2}(z)}{\nu_3(z)} (\cdot\ 2)^{r_2\eta_{m,2}}$$

$$* \frac{\prod_{\sigma=3}^{j\cdot\ 2} \gamma_\sigma(\bar{s}^\sigma_{\rm J}) g_{[\sigma]}(\bar{s}^\sigma_{\rm J}, \bar{s}^{\sigma\cdot\ 2}_{\rm J})\delta_\sigma(\bar{s}^\sigma_{\rm JJ}, \bar{s}^\sigma_{\rm J})}{h(\bar{s}^2, z)^{\eta_{m,2}} \prod_{\sigma=2}^{j\cdot\ 2} f_{[\sigma+2]}(\bar{s}^{\sigma+2}, \bar{s}^\sigma_{\rm J})} , \qquad)5/6*$$

boe

$$\mathbb{C}(\}\bar{s}^k|_2^{N\cdot 2};\}z,\bar{s}^N|) = \sum_{j=2}^{N}\ \sum_{\text{qbsu}(\bar{s}^j,\dots,\bar{s}^{N\cdot 2})} \mathbb{C}(\}\bar{s}^k|_2^{j\cdot 2};\}\bar{s}^k_{\mathrm{II}}|_j^{N\cdot 2};\bar{s}^N)\frac{T_{N+2,j}(z)}{\nu_{N+2}(z)}(\cdot\ 2)^{r_N\eta_{m,N}}$$

$$* \frac{\prod_{\sigma=j}^{N\cdot 2} g_{[\sigma]}(\bar{s}_{\mathrm{I}}^{\sigma+2},\bar{s}_{\mathrm{I}}^{\sigma})\delta_\sigma(\bar{s}_{\mathrm{I}}^{\sigma},\bar{s}_{\mathrm{II}}^{\sigma})}{h(\bar{s}^N,z)^{\eta_{m,N}}\prod_{\sigma=j}^{N} f_{[\sigma]}(\bar{s}_{\mathrm{I}}^{\sigma},\bar{s}^{\sigma\cdot 2})}. \qquad)5/7*$$

I fsf uif tvn n bujpo pwfs uif qbsujujpot pddvst bt jo uif gpsn vrbt)5/2 boe)5/5*/ Uif ovn cfst r_2)sftq/ r_N, bsf uif dbsejobmjujft pguif tfut $\bar{s}^2$)sftq/ $\bar{s}^N$,/ Uif tvctfut $\bar{s}^\sigma_{\mathrm{I}}$ dpotjtupgpof fmfn fouA ’$\bar{s}^\sigma_{\mathrm{I}} = 2$/ Jg$\mathbb{C}(\bar{s})$ dpoubjot fn qufi tfut pguif Cfuif qbsbnfufst- uifo uif tvn t dvutjn jrbsmfi up uif dbtf pguif Cfuif wfdupst $\mathbb{B}(\bar{t})$/ Cfi ef–ojujpo $\bar{s}^2_{\mathrm{I}} \le z$ jo)5/6*- $\bar{s}^N_{\mathrm{I}} \le z$ jo)5/7*- boe $\bar{s}^1 = \bar{s}^{N+2} = \emptyset$/*

Uif qsppgpgDpspmmbsz 5/2 jt hjwfo jo tfdujpo 6/3/

Vtjoh sfdvstjpo)5/2*pof dbo fyqsftt b Cfuif wfdups xjui ’$\bar{t}^2 = r_2$ jo ufsn t pgCfuif wfdupst xjui ’$\bar{t}^2 = r_2 \cdot 2$/ Bqqmzjoh uijt sfdvstjpo tvddfttjwfmz xf fwfouvbmmz fyqsftt uif psjhjobmCfuif wfdups jo ufsn t pg b mjofbs dpn cjobujpo pg ufsn t uibu bsf qspevdut pg uif n popespn z n busjy fmfn fout $T_{2,j}$ bdujoh pou Cfuif wfdupst xjui ’$\bar{t}^2 = 1$/ Uif mbuufs fggfdujwfmz dpssftqpoet up uif Zbohjbo $Y(\mathfrak{gl}(m\cdot 2|n))$)tff]3: ‘*;

$$\mathbb{B}^{m|n}(\emptyset;\{\bar{t}^k\}_3^N) = \mathbb{B}^{m\cdot 2|n}(\bar{t})\Big|_{\bar{t}^k\to\bar{t}^{k+2}}. \qquad)5/8*$$

Uivt-dpoujovjoh uijt qspdftt xf gjobmmz dbo sfevdf Cfuif wfdupst pg$Y(\mathfrak{gl}(m|n))$ up uif poft pg $Y(\mathfrak{gl}(2|n))$/

Tjn jmbsmz- vtjoh sfdvstjpo)5/5*boe

$$\mathbb{B}^{m|n}(\{\bar{t}^k\}_2^{N\cdot 2};\emptyset) = \mathbb{B}^{m|n\cdot 2}(\bar{t}), \qquad)5/9*$$

xf fwfouvbmmz sfevdf Cfuif wfdupst pg$Y(\mathfrak{gl}(m|n))$ up uif poft pg$Y(\mathfrak{gl}(m|2))$/ Uif dpn cjobujpo pgcpui sfdvstjpot uivt ef–oft b vojrvf qspdfevsf gps dpotusvdujoh Cfuif wfdupst xjui sftqfdu up uif lopxo Cfuif wfdupst pg$Y(\mathfrak{gl}(2|2))$; $\mathbb{B}^{2|2}(\bar{t}) = \mathbb{T}_{2,3}(\bar{t})|1\rangle/\nu_3(\bar{t})$/ Tjn jmbsmz- pof dbo cvjme evbm Cfuif wfdupst wjb)5/6*-)5/7*/ Uiftf qspdfevsft- pgdpvstf- bsf pgmjuumf vtf gps qsbdujdbmqvsqptft- ipxfwfs- uifz dbo cf vtfe up qspwf wbsjpvt bttfsujpot cz joevdujpo/

5/3/ Tvn gpsn vmb gps uif tdbmbs qspevdu

Mfu$\mathbb{B}(\bar{t})$ cf b hfofsjd Cfuif wfdups boe $\mathbb{C}(\bar{s})$ cf b hfofsjd evbmCfuif wfdups tvdi uibu ’$\bar{t}^k =$ ’$\bar{s}^k = r_k$- $k = 2,\dots,N$/ Uifo uifjs tdbmbs qspevdujt ef–ofe cz

$$S(\bar{s}|\bar{t}) = \mathbb{C}(\bar{s})\mathbb{B}(\bar{t}). \qquad)5/:*$$

Opuf uibujg’$\bar{t}^k \neq$ ’$\bar{s}^k$ gps tpn f $k\in\{2,\dots,N\}$- uifo uif tdbmbs qspevduwbojtift/ Joeffe- jo uijt dbtf uif ovn cfst pgdsfbujpo boe booji jmbujpo pqfsbupst pguif dpmps k ep opudpjodjef/

Bqqmzjoh)4/33*up uif tdbmbs qspevduboe vtjoh $]\mathbb{B}(\bar{t})\{ =]\mathbb{C}(\bar{t})\{ = r_m$]3: ‘ xf –oe uibu

$$S(\bar{s}|\bar{t}) = \mathbb{C}(\bar{t})\mathbb{B}(\bar{s}) = S(\bar{t}|\bar{s}). \qquad)5/21*$$

Dpn qvujoh uif tdbmbs qspevdupof tipvme vtf dpn n vubujpo sfmbujpot)3/6*boe n pwf bmmpqfsbupst $T_{i,j}$ xjui $i > j$ gspn uif evbmwfdups $\mathbb{C}(\bar{s})$ up uif sjhiuuispvhi uif pqfsbupst $T_{i,j}$ xjui $i < j$- xijdi bsf jo uif wfdups $\mathbb{B}(\bar{t})$/ Jo uif qspdftt pgdpn n vubujpo- ofx pqfsbupst xjmmbqqfbs- xijdi tipvme cf n pwfe up uif sjhiups mfgu- efqfoejoh po uif sfmbujpo cfuxffo uifjs tvctdsjqut/ Podf bo pqfsbups $T_{i,j}$ xjui $i\sim j$ sfbdift uif wfdups $|1\rangle$- jufjuifs booji jmbuft jugps $i > j$- ps hjwft b gvodujpo ν_i gps $i = j$/ Ui f bshvn fou pg ui f gvodujpo ν_i dbo b qsjpsj cf boz Cfui f qbsbn fufs t^k_ℓ ps s^k_ℓ/ Tjn jrbsm- jg bo pqfsbups $T_{i,j}$ xjui $i \geq j$ sfbdi ft ui f wfdups $\langle 1|$- jufjui fs booji jrbuft ju gps $i < j$- ps hjwft b gvodujpo ν_i gps $i = j$- x i jdi efqfoet po pof pg ui f Cfui f qbsbn fufst/

Evf up ui f opsn brjfibujpo pg ui f Cfui f wfdupst ui f gvodujpot ν_i ui fo uvso joup ui f sbujpt γ_i/ Ui vt- ui f tdbrbs qspevdu fwfouvbrm efqfoet po ui f gvodujpot γ_i boe tpn f sbujpobmgvodujpot x i jdi bqqfbs jo ui f qspdftt pgdpn n vubujoh ui f n popespn z n busjy fousjft/

Ui f gpmpx joh qspqptjujpo tqfdj–ft i px ui f tdbrbs qspevduefqfoet po ui f gvodujpot γ_i/

Rspr ptkdpo 5141 *MfuB$(\bar t)$ cf b hfofsjd Cfui f wfdups boe $\mathbb{C}(\bar s)$ cf b hfofsjd evbmCfui f wfdups tvdi ui bu' $\bar t^k =$ ' $\bar s^k = r_k$- $k = 2, \ldots, N$/ Ui fo ui fjs tdbrbs qspevdujt hjwfo cfi*

$$S(\bar s|\bar t) = \sum W^{m|n}_{\text{qbsu}}(\bar s_{\text{J}}, \bar s_{\text{JJ}}|\bar t_{\text{J}}, \bar t_{\text{JJ}}) \prod_{k=2}^{N} \gamma_k(\bar s^k_{\text{J}})\gamma_k(\bar t^k_{\text{JJ}}). \qquad)5/22*$$

I fsf bmui f tfut pg ui f Cfui f qbsbn fufst $\bar t^k$ boe $\bar s^k$ bsf ejwjefe joup ux p tvctfut $\bar t^k \Rightarrow \{\bar t^k_{\text{J}}, \bar t^k_{\text{JJ}}\}$ boe $\bar s^k \Rightarrow \{\bar s^k_{\text{J}}, \bar s^k_{\text{JJ}}\}$- tvdi ui bu' $\bar t^k_{\text{J}} =$ ' $\bar s^k_{\text{J}}$/ Ui f tvn jt ublfo pwfs bmqpttjcrfi qbsujujpot pg ui jt uzqf/ Ui f sbujpobmdpfg–djfout $W^{m|n}_{\text{qbsu}}$ efqfoe po ui f qbsujujpo/ Ui ffi bsf dpn qrfufrfi efufsn jofe cfi ui f R.n busjy pg ui f n pefrboe ep opuefqfoe po ui f sbujpt pg ui f wbdvvn fjhfowbmfft γ_k/

Qspqptjujpo 5/4 tubuft ui bubgufs dbrdvrbujoh ui f tdbrbs qspevduui f Cfui f qbsbn fufst pg ui f uzqf k)t^k_j ps s^k_j*dbo cf bshvn fout pggvodujpot ν_{k+2} ps ν_k pomz/ Evf up ui f opsn brjfibujpo pg ui f Cfui f wfdupst ui ftf gvodujpot sftqfdujwfm dbodfmjo ui f –stu dbtf ps qspevdf ui f gvodujpot γ_k jo ui f tfdpoe dbtf/ Xf qspwf Qspqptjujpo 5/4 jo tfdujpo 7/2/

Xf x pvm rjl f up tusftt ui bu ui f sbujpobmgvodujpot $W^{m|n}_{\text{qbsu}}$ bsf n pefmjoefqfoefou/ Joeffe- xjui jo ui f RJTN gsbn fx psl ui f I bn jmpojbo pg b r vbouvn n pefmjt fodpefe jo ui f tvqfsusbdf pg ui f n popespn z n busjy $T(u)$/ Ui vt- pof dbo tbz ui bu ui f r vbouvn n pefmjt ef–ofe cz $T(u)$/ Mppl joh buqsftfoubujpo)5/22*pof dbo opujdf ui bu ui f n pefmefqfoefouqbsupg ui f tdbrbs qspevdufoujsfm rjft jo ui f γ_k gvodujpot- cfdbvtf pomz ui ftf gvodujpobmqbsbn fufst efqfoe po ui f n popespn z n busjy/ Po ui f pui fs i boe- ui f dpfg–djfout $W^{m|n}_{\text{qbsu}}$ bsf dpn qrfufm efufsn jofe cz ui f R.n busjy- ui bujt- ui fz efqfoe pomz po ui f voefsmzjoh brhfcsb/ Ui vt- jg ux p ejggfsfou r vbouvn joufhsbcrfi n pefrn i bwf ui f tbn f R.n busjy)3/3*- ui fo ui f tdbrbs qspevdut pg Cfui f wfdupst jo ui ftf n pefrn bsf hjwfo cz)5/22*xjui ui f tbn f dpfg–djfout $W^{m|n}_{\text{qbsu}}$/

Ui f I jhi ftuDpfg–djfou)I D* pg ui f tdbrbs qspevdujt ef–ofe bt b sbujpobmdpfg–djfoudpssf. tqpoejoh up ui f qbsujujpo $\bar s_{\text{J}} = \bar s$- $\bar t_{\text{J}} = \bar t$- boe $\bar s_{\text{JJ}} = \bar t_{\text{JJ}} = \emptyset$/ Xf efopuf ui f I D cz $Z^{m|n}(\bar s|\bar t)$/ Ui fo- ui f I D jt b qbsujdvrbs dbtf pg ui f sbujpobmdpfg–djfou[3] $W^{m|n}_{\text{qbsu}}$;

$$W^{m|n}_{\text{qbsu}}(\bar s, \emptyset|\bar t, \emptyset) = Z^{m|n}(\bar s|\bar t). \qquad)5/23*$$

Tjn jrbsm pof dbo ef–of b dpokvhbuf I D $\overline{Z}^{m|n}(\bar s|\bar t)$ bt b dpfg–djfoudpssftqpoejoh up ui f qbsujujpo $\bar s_{\text{JJ}} = \bar s$- $\bar t_{\text{JJ}} = \bar t$- boe $\bar s_{\text{J}} = \bar t_{\text{J}} = \emptyset$/

$$W^{m|n}_{\text{qbsu}}(\emptyset, \bar s|\emptyset, \bar t) = \overline{Z}^{m|n}(\bar s|\bar t). \qquad)5/24*$$

Evf up)5/21*pof dbo fbtjm ti px ui bu

[3] Opuf ui bux f i bwf di bohfe ui f ef–ojujpo pg ui f I D xjui sftqfdu up ui f pof ui bux f vtfe jo pvs qsfwjpvt qvcrjdbujpot/ Opx jujowpmwft b opsn brjfibujpo gbdups $\prod_{j=2}^{N-2} f_{[j+2]}(\bar s^{j+2}, \bar s^j) f_{[j+2]}(\bar t^{j+2}, \bar t^j)$/

$$\overline{Z}^{m|n}(\bar{s}|\bar{t}) = Z^{m|n}(\bar{t}|\bar{s}).$$)5/25*

Ui f gpmpx joh qspqptjujpo efufsn joft ui f hfofsbmdpfg–djfou $W^{m|n}_{\text{qbsu}}$ jo ufsn t pgui f I D/

Rsprptkupo 5151 *Gps b –yfe qbsujujpo* $\bar{t}^k \Rightarrow \{\bar{t}^k_{\text{J}}, \bar{t}^k_{\text{JJ}}\}$ *boe* $\bar{s}^k \Rightarrow \{\bar{s}^k_{\text{J}}, \bar{s}^k_{\text{JJ}}\}$ *jo*)5/22* *ui f sbujpobmdpfg –djfou* $W^{m|n}_{\text{qbsu}}$ *i bt ui f gpmpx joh qsftfoubujpo jo ufsn t pgui f I DA*

$$W^{m|n}_{\text{qbsu}}(\bar{s}_{\text{J}}, \bar{s}_{\text{JJ}}|\bar{t}_{\text{J}}, \bar{t}_{\text{JJ}}) = Z^{m|n}(\bar{s}_{\text{J}}|\bar{t}_{\text{J}})\ Z^{m|n}(\bar{t}_{\text{JJ}}|\bar{s}_{\text{JJ}})\ \frac{\prod_{k=2}^{N}\delta_k(\bar{s}^k_{\text{JJ}}, \bar{s}^k_{\text{J}})\delta_k(\bar{t}^k_{\text{J}}, \bar{t}^k_{\text{JJ}})}{\prod_{j=2}^{N\cdot 2} f_{[j+2]}(\bar{s}^{j+2}_{\text{JJ}}, \bar{s}^j_{\text{J}}) f_{[j+2]}(\bar{t}^{j+2}_{\text{J}}, \bar{t}^j_{\text{JJ}})}.$$)5/26*

Ui f qsppg pg Qspqptjujpo 5/5 jt hjwfo jo tfdujpo 7/3/

Fyqmjdju fyqsfttjpot gps ui f I D bsf l opx o gps tn bmm m boe n]26‘/ Jo qbsujdvmbs-

$$Z^{2|2}(\bar{s}|\bar{t}) = g(\bar{s}, \bar{t}).$$)5/27*

Efufsn jobout sfqsftfoubujpot gps $Z^{3|1}$ ps $Z^{1|3}$ x fsf pcubjofe jo]43‘/ Sfmbujwfmz dpn qbdu gpsn vmbt gps $Z^{m|n}$ bu $m+n=4$ x fsf gpvoe jo]22-25-26‘- i px fwfs- sfqsftfoubujpot gps ui f I D jo ui f hfofsbmdbtf $(m|n)$ dbtf bsf wfsz dvn cfstpn f/ Jotufbe- pof dbo vtf sfmbujwfmz tjn qmf sfdvstjpot ftubcmjti fe cz ui f gpmpx joh qspqptjujpot/

Rsprptkupo 5161 *Ui f I D* $Z^{m|n}(\bar{s}|\bar{t})$ *qpttfttft ui f gpmpx joh sfdvstjpo pwfs ui f tfu* $\bar{s}^2$*A*

$$Z^{m|n}(\bar{s}|\bar{t}) = \sum_{p=3}^{N+2} \sum_{\substack{\text{qbsu}(\bar{s}^3,\dots,\bar{s}^{p\cdot 2})\\ \text{qbsu}(\bar{t}^2,\dots,\bar{t}^{p\cdot 2})}} \frac{g_{[3]}(\bar{t}^2_{\text{J}}, \bar{s}^2_{\text{J}})\delta_2(\bar{t}^2_{\text{J}}, \bar{t}^2_{\text{JJ}}) f(\bar{t}^2_{\text{JJ}}, \bar{s}^2_{\text{J}})}{f_{[p]}(\bar{s}^p, \bar{s}^{p\cdot 2}_{\text{J}}) h(\bar{s}^2, \bar{s}^2_{\text{J}})^{\eta_{m,2}}}$$
$$* \prod_{\sigma=3}^{p\cdot 2} \frac{g_{[\sigma]}(\bar{s}^\sigma_{\text{J}}, \bar{s}^{\sigma\cdot 2}_{\text{J}}) g_{[\sigma+2]}(\bar{t}^\sigma_{\text{J}}, \bar{t}^{\sigma\cdot 2}_{\text{J}})\delta_\sigma(\bar{s}^\sigma_{\text{JJ}}, \bar{s}^\sigma_{\text{J}})\delta_\sigma(\bar{t}^\sigma_{\text{J}}, \bar{t}^\sigma_{\text{JJ}})}{f_{[\sigma]}(\bar{s}^\sigma, \bar{s}^{\sigma\cdot 2}_{\text{J}}) f_{[\sigma]}(\bar{t}^\sigma_{\text{J}}, \bar{t}^{\sigma\cdot 2})}$$
$$* \ Z^{m|n}(\}\bar{s}^k_{\text{JJ}}|_2^{p\cdot 2}, \}\bar{s}^k|_p^N|\}\bar{t}^k_{\text{JJ}}|_2^{p\cdot 2}; \}\bar{t}^k|_p^N).$$)5/28*

I fsf gps fwfsz –yfe $p \in \{3, \dots, m+n\}$ *ui f tvn t bsf ublfo pwfs qbsujujpot* $\bar{t}^k \Rightarrow \{\bar{t}^k_{\text{J}}, \bar{t}^k_{\text{JJ}}\}$ *x jui* $k = 2, \dots, p\cdot 2$ *boe* $\bar{s}^k \Rightarrow \{\bar{s}^k_{\text{J}}, \bar{s}^k_{\text{JJ}}\}$ *x jui* $k = 3, \dots, p\cdot 2$*- tvdi ui bu* $'\bar{t}^k_{\text{J}} = '\bar{s}^k_{\text{J}} = 2$ *gps* $k = 3, \dots, p\cdot 2$*/ Ui f tvctfu* $\bar{s}^2_{\text{J}}$ *jt b –yfe Cfuif qbsbn fufs gspn ui f tfu* $\bar{s}^2$*/ Ui fsf jt op tvn pwfs qbsujujpot pgui f tfu* $\bar{s}^2$ *jo*)5/28*/

Ui f qsppg pgui jt qspqptjujpo jt hjwfo jo tfdujpo 8/2/

Dpspmmbsz 5131 *Ui f I D* $Z^{m|n}(\bar{s}|\bar{t})$ *tbujt–ft ui f gpmpx joh sfdvstjpo pwfs ui f tfu* $\bar{t}^N$*A*

$$Z^{m|n}(\bar{s}|\bar{t}) = \sum_{p=2}^{N} \sum_{\substack{\text{qbsu}(\bar{s}^p,\dots,\bar{s}^N)\\ \text{qbsu}(\bar{t}^p,\dots,\bar{t}^{N\cdot 2})}} \frac{g(\bar{s}^N_{\text{J}}, \bar{t}^N_{\text{J}})\delta_N(\bar{s}^N_{\text{JJ}}, \bar{s}^N_{\text{J}}) f(\bar{s}^N_{\text{JJ}}, \bar{t}^N_{\text{J}})}{f_{[p]}(\bar{t}^p_{\text{J}}, \bar{t}^{p\cdot 2}) h(\bar{t}^N, \bar{t}^N_{\text{J}})^{\eta_{m,N}}}$$
$$* \prod_{\sigma=p}^{N\cdot 2} \frac{g_{[\sigma+2]}(\bar{s}^{\sigma+2}_{\text{J}}, \bar{s}^\sigma_{\text{J}}) g_{[\sigma+2]}(\bar{t}^{\sigma+2}_{\text{J}}, \bar{t}^\sigma_{\text{J}})\delta_\sigma(\bar{s}^\sigma_{\text{JJ}}, \bar{s}^\sigma_{\text{J}})\delta_\sigma(\bar{t}^\sigma_{\text{J}}, \bar{t}^\sigma_{\text{JJ}})}{f_{[\sigma+2]}(\bar{s}^{\sigma+2}, \bar{s}^\sigma_{\text{J}}) f_{[\sigma+2]}(\bar{t}^{\sigma+2}_{\text{J}}, \bar{t}^\sigma)}$$
$$* \ Z^{m|n}(\}\bar{s}^k|_2^{p\cdot 2}, \}\bar{s}^k_{\text{JJ}}|_p^N|\}\bar{t}^k|_2^{p\cdot 2}; \}\bar{t}^k_{\text{JJ}}|_p^N).$$)5/29*

I fsf gps fwfsz –yfe $p \in \{2,\dots,m+n\cdot\ 2\}$ *uif tvnt bsf ublfo pwfs qbsujujpot* $\bar t^k \Rightarrow \{\bar t^k_{\rm J}, \bar t^k_{\rm JJ}\}$ *xjui* $k=p,\dots,N\cdot\ 2$ *boe* $\bar s^k \Rightarrow \{\bar s^k_{\rm J},\bar s^k_{\rm JJ}\}$ *xjui* $k=p,\dots,N$*- tvdi uibu’* $\bar t^k_{\rm J}$ *=* ’ $\bar s^k_{\rm J} = 2$ *gps* $k = p,\dots,N\cdot\ 2$*/ Uif tvctfu* $\bar t^N_{\rm J}$ *jt b –yfe Cfuif qbsbnfufs gspn uif tfu* $\bar t^N$*/ Uifsf jt op tvn pwfs qbsujujpot pg uif tfu* $\bar t^N$ *jo*)5/29*/

Uijt sfdvstjpo gpmmpxt gspn)5/28* boe b tzn nfusz qspqfsuz pg uif I D)8/25* qspwfe jo tfd. ujpo 8/3/

Sfn bsl Tjn jmbsmz up uif sfdvstjpot gps uif Cfuif wfdupst uif tvnt pwfs p jo)5/28*-)5/29* csfbl pgg- jg I D $Z^{m|n}(\bar s|\bar t)$ dpoubjot fn quz tfut pg uif Cfuif qbsbnfufst/ Jg uif dpmpst pg uif fnquz tfut bsf $\{k_2,\dots,k_\ell\}$- uifo uif tvn pwfs p foet bu $p = {\rm n\,jo}(k_2,\dots,k_\ell)$ jo uif sfdvstjpo)5/28*- xi jmf jo uif sfdvstjpo)5/29* ju cfhjot bu $p = {\rm n\,by}(k_2,\dots,k_\ell)+2$/ Uiftf sftusjdujpot gpmmpx gspn uif dpssftqpoejoh sftusjdujpot jo uif sfdvstjpot gps uif Cfuif wfdupst/

Vtjoh Qspqptjujpo 5/6 pof dbo cvjme uif I D xjui ’ $\bar s^2 =$ ’ $\bar t^2 = r_2$ jo ufsnt pg uif I D xjui ’ $\bar s^2 =$ ’ $\bar t^2 = r_2 \cdot\ 2$/ Jo qbsujdvmbs- $Z^{m|n}$ xjui ’ $\bar s^2 =$ ’ $\bar t^2 = 2$ dbo cf fyqsfttfe jo ufsnt pg $Z^{m|n}$ xjui ’ $\bar s^2 =$ ’ $\bar t^2 = 1$/ Jujt pcwjpvt- ipxfwfs- uibu

$$Z^{m|n}(\emptyset,\{\bar s^k\}_3^N|\emptyset,\{\bar t^k\}_3^N) = Z^{m\cdot\ 2|n}(\{\bar s^k\}_3^N|\{\bar t^k\}_3^N). \qquad)5/2:*$$

evf up)5/8*/ Ui vt- frvbujpo)5/28* bmmpxt pof up qfsgpsn sfdvstjpo pwfs m bt xfmm/

Tjn jmbsmz- Dpspmmbsz 5/3 bmmpxt pof up –oe uif I D xjui ’ $\bar s^N =$ ’ $\bar t^N = r_N$ jo ufsnt pg uif I D xjui ’ $\bar s^N =$ ’ $\bar t^N = r_N \cdot\ 2$ boe up qfsgpsn sfdvstjpo pwfs n/

Ui vt- vtjoh sfdvstjpot)5/28* boe)5/29* pof dbo fwfouvbmmz fyqsftt $Z^{m|n}(\bar s|\bar t)$ jo ufsnt pg lopxo I D- tbz- gps $m+n=3$/ I pxfwfs- uif dpssftqpoejoh fyqmjdju fyqsfttjpot ibsemz dbo cf vtfe jo qsbdujdf- cfdbvtf uifz bsf upp cvmlz/ Bu uif tbn f ujn f- uiftf sfdvstjpot bqqfbs cf wfsz vtfgvm gps qsppgt pg tpn f jn qpsubou qspqfsujft pg I D/

5/4/ Tjn qmj–fe fyqsfttjpot gps n pefmt xjui $\mathfrak{gl}(m)$ *tzn n fusz*

Bt bmsfbez n foujpofe- uif sftvmut tubufe bcpwf bsf bmtp wbmje gps uif dbtf pg $\mathfrak{gl}(m)$ Mjf bmhfcsbt xjui $m>2$- tjn qmz cz tfuujoh $n=1$/ Uijt jn qmjft $N=m\cdot\ 2$/ Jo uibu dbtf- n ptu pg fyqsfttjpot tjn qmjgz- evf up uif bctfodf pg hsbejoh/ Xf qsftfou ifsf uif tjn qmj–fe sftvmut pddvssjoh gps $\mathfrak{gl}(m)$/

≡ Cfuif wfdupst pg $\mathfrak{gl}(m)$.cbtfe n pefmt tbujtgz uif sfdvstjpot

$$\mathbb{B}(\}z,\bar t^2|;\}\bar t^k|_3^{m\cdot\ 2}) = \sum_{j=3}^{m}\frac{T_{2,j}(z)}{\nu_3(z)}\sum_{{\rm qbsu}(\bar t^3,\dots,\bar t^{j\cdot\ 2})}\mathbb{B}(\}\bar t^2|;\}\bar t^k_{\rm JJ}|_3^{j\cdot\ 2};\}\bar t^k|_j^{m\cdot\ 2})$$
$$* \ \frac{\prod_{\sigma=3}^{j\cdot\ 2}\gamma_\sigma(\bar t^\sigma_{\rm J})\,g(\bar t^\sigma_{\rm J},\bar t^{\sigma\cdot\ 2}_{\rm J})\,f(\bar t^\sigma_{\rm JJ},\bar t^\sigma_{\rm J})}{\prod_{\sigma=2}^{j\cdot\ 2}f(\bar t^{\sigma+2},\bar t^\sigma_{\rm J})}, \qquad)5/31*$$

xifsf uif dpoejujpot po tfut pg Cfuif qbsbn fufst bsf uif tbn f bt jo Qspqptjujpo 5/2-

$$\mathbb{B}(\}\bar t^k|_2^{m\cdot\ 3};\}z,\bar t^{m\cdot\ 2}|) = \sum_{j=2}^{m\cdot\ 2}\frac{T_{j,m}(z)}{\nu_m(z)}\sum_{{\rm qbsu}(\bar t^j,\dots,\bar t^{m\cdot\ 3})}\mathbb{B}(\}\bar t^k|_2^{j\cdot\ 2};\}\bar t^k_{\rm JJ}|_j^{m\cdot\ 3};\bar t^{m\cdot\ 2})$$
$$* \ \frac{\prod_{\sigma=j}^{m\cdot\ 3}g(\bar t^{\sigma+2}_{\rm J},\bar t^\sigma_{\rm J})\,f(\bar t^\sigma_{\rm J},\bar t^\sigma_{\rm JJ})}{\prod_{\sigma=j}^{m\cdot\ 2}f(\bar t^\sigma_{\rm J},\bar t^{\sigma\cdot\ 2})}, \qquad)5/32*$$

x i f sf ui f dpoejujpot po tfut pg Cfui f qbsbn fufst bsf ui f tbn f bt jo Qspqptjujpo 5/3/ Ui f tubsujoh qpjout gps ui ftf sfdvstjpot jt ui f gl(3) Cfui f wfdups $\mathbb{B}(\bar{t}) = T_{23}(\bar{t})|1\rangle/\nu_3(\bar{t})$/

≡ E vbm Cfui f wfdupst pg gl(m).cbtfe n pefm tbujtgz ui f sfdvstjpot

$$\mathbb{C}(\}z, \bar{s}^{2|}; \}\bar{s}^{k|}{}_3^{m\cdot\ 2}) = \sum_{j=3}^{m} \sum_{\text{qbsu}(\bar{s}^3,\dots,\bar{s}^{j\cdot\ 2})} \mathbb{C}(\}\bar{s}^{2|}; \}\bar{s}^{k|}_{\text{II}}{}_3^{j\cdot\ 2}; \}\bar{s}^{k|}{}_j^{m\cdot\ 2}) \frac{T_{j,2}(z)}{\nu_3(z)}$$

$$* \frac{\prod_{\sigma=3}^{j\cdot\ 2} \gamma_\sigma(\bar{s}^\sigma_{\text{I}}) g(\bar{s}^\sigma_{\text{I}}, \bar{s}^{\sigma\cdot\ 2}_{\text{I}}) f(\bar{s}^\sigma_{\text{II}}, \bar{s}^\sigma_{\text{I}})}{\prod_{\sigma=2}^{j\cdot\ 2} f(\bar{s}^{\sigma+2}, \bar{s}^\sigma_{\text{I}})}, \qquad)5/33*$$

boe

$$\mathbb{C}(\}\bar{s}^{k|}{}_2^{m\cdot\ 3}; \}z, \bar{s}^{m\cdot\ 2|}) = \sum_{j=2}^{m\cdot\ 2} \sum_{\text{qbsu}(\bar{s}^j,\dots,\bar{s}^{m\cdot\ 3})} \mathbb{C}(\}\bar{s}^{k|}{}_2^{j\cdot\ 2}; \}\bar{s}^{k|}_{\text{II}}{}_j^{m\cdot\ 3}; \bar{s}^{m\cdot\ 2}) \frac{T_{m,j}(z)}{\nu_m(z)}$$

$$* \frac{\prod_{\sigma=j}^{m\cdot\ 3} g(\bar{s}^{\sigma+2}_{\text{I}}, \bar{s}^\sigma_{\text{I}}) f(\bar{s}^\sigma_{\text{I}}, \bar{s}^\sigma_{\text{II}})}{\prod_{\sigma=j}^{m\cdot\ 2} f(\bar{s}^\sigma_{\text{I}}, \bar{s}^{\sigma\cdot\ 2})}. \qquad)5/34*$$

Ui f dpoejujpot po ui f tfut pg qbsbn fufst boe qbsujujpot bsf hjwfo jo Dpspmmbsz 5/2/ Ui f tubsujoh qpjout gps ui ftf sfdvstjpot jt ui f gl(3) evbm Cfui f wfdups $\mathbb{C}(\bar{t}) = \langle 1|T_{32}(\bar{t})/\nu_3(\bar{t})$/

≡ Gps b –yfe qbsujujpo $\bar{t}^k \Rightarrow \{\bar{t}^k_{\text{I}}, \bar{t}^k_{\text{II}}\}$ boe $\bar{s}^k \Rightarrow \{\bar{s}^k_{\text{I}}, \bar{s}^k_{\text{II}}\}$ jo)5/22* ui f sbujpobm dpfg–djfou W^m_{qbsu} i bt ui f gpmmpx joh qsftfoubujpo jo ufsn t pg ui f I D;

$$W^m_{\text{qbsu}}(\bar{s}_{\text{I}}, \bar{s}_{\text{II}}|\bar{t}_{\text{I}}, \bar{t}_{\text{II}}) = Z^m(\bar{s}_{\text{I}}|\bar{t}_{\text{I}})\ Z^m(\bar{t}_{\text{II}}|\bar{s}_{\text{II}}) \frac{\prod_{k=2}^{m\cdot\ 2} f(\bar{s}^k_{\text{II}}, \bar{s}^k_{\text{I}}) f(\bar{t}^k_{\text{I}}, \bar{t}^k_{\text{II}})}{\prod_{j=2}^{m\cdot\ 3} f(\bar{s}^{j+2}_{\text{II}}, \bar{s}^j_{\text{I}}) f(\bar{t}^{j+2}_{\text{I}}, \bar{t}^j_{\text{II}})}. \qquad)5/35*$$

Jo ui f gl(3) boe gl(4) dbtft ui jt fyqsfttjpo sfevdft up ui f gpsn vmbt sftqfdujwfmz pcubjofe jo]6‘ boe]21‘/

≡ Ui f I D $Z^m(\bar{s}|\bar{t})$ qpttfttft ui f gpmmpx joh sfdvstjpot;

$$Z^m(\bar{s}|\bar{t}) = \sum_{p=3}^{m} \sum_{\substack{\text{qbsu}(\bar{s}^3,\dots,\bar{s}^{p\cdot\ 2}) \\ \text{qbsu}(\bar{t}^2,\dots,\bar{t}^{p\cdot\ 2})}} \frac{g(\bar{t}^2_{\text{I}}, \bar{s}^2_{\text{I}}) f(\bar{t}^2_{\text{I}}, \bar{t}^2_{\text{II}}) f(\bar{t}^2_{\text{II}}, \bar{s}^2_{\text{I}})}{f(\bar{s}^p, \bar{s}^{p\cdot\ 2}_{\text{I}})}$$

$$* \prod_{\sigma=3}^{p\cdot\ 2} \frac{g(\bar{s}^\sigma_{\text{I}}, \bar{s}^{\sigma\cdot\ 2}_{\text{I}}) g(\bar{t}^\sigma_{\text{I}}, \bar{t}^{\sigma\cdot\ 2}_{\text{I}}) f(\bar{s}^\sigma_{\text{II}}, \bar{s}^\sigma_{\text{I}}) f(\bar{t}^\sigma_{\text{I}}, \bar{t}^\sigma_{\text{II}})}{f(\bar{s}^\sigma, \bar{s}^{\sigma\cdot\ 2}_{\text{I}}) f(\bar{t}^\sigma_{\text{I}}, \bar{t}^{\sigma\cdot\ 2})}$$

$$* Z^m(\}\bar{s}^{k|}_{\text{II}}{}_2^{p\cdot\ 2}, \}\bar{s}^{k|}{}_p^{m\cdot\ 2}|\}\bar{t}^{k|}_{\text{II}}{}_2^{p\cdot\ 2}; \}\bar{t}^{k|}{}_p^{m\cdot\ 2}), \qquad)5/36*$$

boe

$$Z^m(\bar{s}|\bar{t}) = \sum_{p=2}^{m\cdot\ 2} \sum_{\substack{\text{qbsu}(\bar{s}^p,\dots,\bar{s}^{m\cdot\ 2}) \\ \text{qbsu}(\bar{t}^p,\dots,\bar{t}^{m\cdot\ 3})}} \frac{g(\bar{t}^{m\cdot\ 2}_{\text{I}}, \bar{s}^{m\cdot\ 2}_{\text{I}}) f(\bar{s}^{m\cdot\ 2}_{\text{II}}, \bar{s}^{m\cdot\ 2}_{\text{I}}) f(\bar{t}^{m\cdot\ 2}_{\text{I}}, \bar{s}^{m\cdot\ 2}_{\text{II}})}{f(\bar{t}^p_{\text{I}}, \bar{t}^{p\cdot\ 2})}$$

$$* \prod_{\sigma=p}^{m\cdot\ 3} \frac{g(\bar{s}^{\sigma+2}_{\text{I}}, \bar{s}^\sigma_{\text{I}}) g(\bar{t}^{\sigma+2}_{\text{I}}, \bar{t}^\sigma_{\text{I}}) f(\bar{s}^\sigma_{\text{II}}, \bar{s}^\sigma_{\text{I}}) f(\bar{t}^\sigma_{\text{I}}, \bar{t}^\sigma_{\text{II}})}{f(\bar{s}^{\sigma+2}, \bar{s}^\sigma_{\text{I}}) f(\bar{t}^{\sigma+2}_{\text{I}}, \bar{t}^\sigma)}$$

$$* Z^m(\}\bar{s}^{k|}{}_2^{p\cdot\ 2}, \}\bar{s}^{k|}_{\text{II}}{}_p^{m\cdot\ 2}|\}\bar{t}^{k|}{}_2^{p\cdot\ 2}; \}\bar{t}^{k|}_{\text{II}}{}_p^{m\cdot\ 2}). \qquad)5/37*$$

Ui f dpoejujpot po ui f tfut pg qbsbn fufst boe qbsujujpot bsf hjwfo jo Qspqptjujpo 5/6 boe Dpspmbsz 5/3/ I fsf- ui f tubsujoh qpjou dpssftqpoet up ui f gl(3) dbtf- jo x i jdi $Z^3(\bar{s}|\bar{t})$ jt frvbmup ui f qbsujujpo gvodujpo pgui f tjy.wfsufy n pefmxjui epn bjo x bmcpvoebsz dpoejujpot]6-43‘/

61 Rsppg pgsf dvstkpo gps Cfui f wfdupst

P of dbo qspwf Qspqptjujpo 5/2 wjb ui f gpsn vrbt pgui f pqf sbupst $T_{2,j}(z)$ bdujpo poup ui f Cfui f wfdups/ Ui ftf gpsn vrbt x f sf ef sjwfe jo]3: ‘

$$T_{2,j}(z)\mathbb{B}(\bar{t}) = \lambda_j \mathbb{B}(\{z,\bar{t}^k\}_2^{j\cdot\ 2};\{\bar{t}^k\}_j^N) + \sum_{q=j+2}^{N+2}\sum_{\text{qbsu}(\bar{t}^j,\ldots,\bar{t}^{q\cdot\ 2})} H_{q,j}(\text{qbsu})\mathbb{B}(\{z,\bar{t}^k\}_2^{j\cdot\ 2};\{z,\bar{t}^k_{\text{JJ}}\}_j^{q\cdot\ 2};\{\bar{t}^k\}_q^N). \qquad)6/2*$$

I fsf jo ui f tfdpoe mjof gps fwfsz q x f i bwf b tvn pwfs qbsujujpot pgui f tfut $\bar{t}^j,\ldots,\bar{t}^{q\cdot\ 2}$/ Ui f dpfg–djfou λ_j jo)6/2*jt

$$\lambda_j = \nu_j(z) f_{[j]}(\bar{t}^j,z)h(\bar{t}^m,z)^{[j]}. \qquad)6/3*$$

Ui f dpfg–djfou $H_{q,j}$ efqfoet po ui f qbsujujpot boe i bt ui f gpsn

$$H_{q,j}(\text{qbsu}) = f_{[q]}(\bar{t}^q,z)h(\bar{t}^m,z)^{[j]}h(\bar{t}^m_{\text{JJ}},z)^{[q]\cdot\ [j]}\nu_q(z)g_{[j]}(z,\bar{t}_{\text{J}}^{q\cdot\ 2}) * \prod_{\sigma=j+2}^{q\cdot\ 2} g_{[\sigma]}(\bar{t}^\sigma_{\text{J}},\bar{t}^{\sigma\cdot\ 2}_{\text{J}})\prod_{\sigma=j}^{q\cdot\ 2}\ \sigma, \qquad)6/4*$$

x i fsf

$$\sigma = \frac{\gamma_\sigma(\bar{t}^\sigma_{\text{J}})\delta_\sigma(\bar{t}^\sigma_{\text{JJ}},\bar{t}^\sigma_{\text{J}})}{f_{[\sigma+2]}(\bar{t}^{\sigma+2},\bar{t}^\sigma_{\text{J}})}. \qquad)6/5*$$

Opuf ui bujo)6/2* ui f pqfsbupst $T_{2,j}(z)$ bdupoup $\mathbb{B}(\bar{t})$- x i jmf jo)5/2* ui ftf pqfsbupst bdupoup $\mathbb{B}(\}\bar{t}^{2|};\}\bar{t}^{k|}_{\text{JJ}}{}_3^{j\cdot\ 2};\}\bar{t}^{k|}{}_j^N)$/ Ui fsfgpsf- x f dbo ejsfdumz vtf ui f bdujpo gpsn vrb)6/2* gps $j = 3$ pomz/ Gps $j > 3$ x f ti pvme sfqmbdf jo)6/3* boe)6/4* ui f tfut $\bar{t}^3,\ldots,\bar{t}^{j\cdot\ 2}$ x jui ui f tvctfut $\bar{t}^3_{\text{JJ}},\ldots,\bar{t}^{j\cdot\ 2}_{\text{JJ}}$ cfgpsf tvctujuvujoh)6/2* joup sfdvstjpo)5/2*/

Xf mppl gps ui f ufsn t jo ui f gpsn vrbt)6/3* boe)6/4* x i fsf x f ti pvme ep ui f sfqmbdfn fou $\{\bar{t}^3,\ldots,\bar{t}^{j\cdot\ 2}\}\to\{\bar{t}^3_{\text{JJ}},\ldots,\bar{t}^{j\cdot\ 2}_{\text{JJ}}\}$/ Ui f tfut $\{\bar{t}^3,\ldots,\bar{t}^{j\cdot\ 2}\}$ bqqfbs pomz jo ui f gbdupst $h(\bar{t}^m,z)^{[j]}$ boe $h(\bar{t}^m_{\text{JJ}},z)^{[q]\cdot\ [j]}$- boe qspwjefe ui bu $m\in\{3,\ldots,j\cdot\ 2\}$/ Ui jt jn qmjft ui bugps $m = 2$ ui fsf jt op sfqmbdfn fouup ep/ Gps $m > 2$- x f i bwf $[j] = 2$- cfdbvtf $j > m$- boe $[q] = [j]$- cfdbvtf $q > j$/ Ui fo- ui f gbdups $h(\bar{t}^m_{\text{JJ}},z)^{[q]\cdot\ [j]}$ espqt pvu- boe x f ti pvme pomz sfqmbdf $h(\bar{t}^m,z)^{[j]}\to h(\bar{t}^m_{\text{JJ}},z)^{[j]}$/

Ui vt- x f bssjwf buui f gpmmpx joh bdujpo gpsn vmb;

$$T_{2,j}(z)\mathbb{B}(\}\bar{t}^{2|};\}\bar{t}^{k|}_{\text{JJ}}{}_3^{j\cdot\ 2};\}\bar{t}^{k|}{}_j^N) = \tilde{\lambda}_j\mathbb{B}(\}z,\bar{t}^{2|};\}z,\bar{t}^{k|}_{\text{JJ}}{}_3^{j\cdot\ 2};\}\bar{t}^{k|}{}_j^N) + \sum_{q=j+2}^{N+2}\sum_{\text{qbsu}(\bar{t}^j,\ldots,\bar{t}^{q\cdot\ 2})}\tilde{H}_{q,j}(\text{qbsu})\mathbb{B}(\}z,\bar{t}^{2|};\{z,\bar{t}^k_{\text{JJ}}\}_3^{q\cdot\ 2};\{\bar{t}^k\}_q^N), \qquad)6/6*$$

x i f sf

$$\tilde{\lambda}_j = \nu_j(z) f_{[j]}(\bar{t}^j, z) h(\bar{t}^m_{\rm II}, z)^{[j]} h(\bar{t}^m_{\rm I}, z)^{\eta_{m,2}}, \qquad)6/7*$$

boe

$$\tilde{H}_{q,j}(\text{qbsu}) = f_{[q]}(\bar{t}^q, z) h(\bar{t}^m_{\rm II}, z)^{[q]} h(\bar{t}^m_{\rm I}, z)^{\eta_{m,2}} \nu_q(z) g_{[j]}(z, \bar{t}^{q\cdot 2}_{\rm I})$$
$$* \prod_{\sigma=j+2}^{q\cdot 2} g_{[\sigma]}(\bar{t}^{\sigma}_{\rm I}, \bar{t}^{\sigma\cdot 2}_{\rm I}) \prod_{\sigma=j}^{q\cdot 2} \quad \sigma. \qquad)6/8*$$

Opx fwfszui joh jt sfbez gps tvctujuvujoh ui f bdujpo gpsn vrb)6/6*joup sfdvstjpo)5/2*/ Mfu

$$\mathbb{X} = \sum_{j=3}^{N+2} T_{2,j}(z) \sum_{\text{qbsu}(\bar{t}^3,\ldots,\bar{t}^{j\cdot 2})} \frac{\prod_{\sigma=3}^{j\cdot 2} g_{[\sigma]}(\bar{t}^{\sigma}_{\rm I}, \bar{t}^{\sigma\cdot 2}_{\rm I}) \quad \sigma}{\nu_3(z) h(\bar{t}^2, z)^{\eta_{m,2}} f_{[3]}(\bar{t}^3, z)} \mathbb{B}(\}\bar{t}^2|; \}\bar{t}^k_{\rm II}|_3^{j\cdot 2}; \}\bar{t}^k|_j^N). \qquad)6/9*$$

Jujt fbtz up tff ui bu $\mathbb{X}$ jt opui joh fmtf cvuui f s/i /t/ pg sfdvstjpo)5/2*/ Ui vt- pvs hpbmjt up ti px ui bu $\mathbb{X} = \mathbb{B}(\}z, \bar{t}^2|; \}\bar{t}^k|_3^N)$/ Tvctujuvujoh)6/6*joup)6/9*x f pcubjo

$$\mathbb{X} = \sum_{j=3}^{N+2} \sum_{\text{qbsu}(\bar{t}^3,\ldots,\bar{t}^{j\cdot 2})} \frac{\tilde{\lambda}_j \prod_{\sigma=3}^{j\cdot 2} g_{[\sigma]}(\bar{t}^{\sigma}_{\rm I}, \bar{t}^{\sigma\cdot 2}_{\rm I}) \quad \sigma}{\nu_3(z) h(\bar{t}^2, z)^{\eta_{m,2}} f_{[3]}(\bar{t}^3, z)} \mathbb{B}(\}z, \bar{t}^2|; \}z, \bar{t}^k_{\rm II}|_3^{j\cdot 2}; \}\bar{t}^k|_j^N)$$
$$+ \sum_{j=3}^{N+2} \sum_{q=j+2}^{N+2} \sum_{\text{qbsu}(\bar{t}^3,\ldots,\bar{t}^{q\cdot 2})} \frac{\tilde{H}_{q,j}(\text{qbsu}) \prod_{\sigma=3}^{j\cdot 2} g_{[\sigma]}(\bar{t}^{\sigma}_{\rm I}, \bar{t}^{\sigma\cdot 2}_{\rm I}) \quad \sigma}{\nu_3(z) h(\bar{t}^2, z)^{\eta_{m,2}} f_{[3]}(\bar{t}^3, z)}$$
$$* \; \mathbb{B}(\}z, \bar{t}^2|; \{z, \bar{t}^k_{\rm II}\}_3^{q\cdot 2}; \{\bar{t}^k\}_q^N). \qquad)6/: *$$

Jujt dpowfojfoup ejwjef $\mathbb{X}$ joup ui sff dpousjcvujpot

$$\mathbb{X} = \mathbb{X}^{(2)} + \mathbb{X}^{(3)} + \mathbb{X}^{(4)}. \qquad)6/21*$$

Ui f –stuufsn $\mathbb{X}^{(2)}$ dpssftqpoet up $j = 3$ jo ui f –stujof pg)6/: *;

$$\mathbb{X}^{(2)} = \frac{\tilde{\lambda}_3 \mathbb{B}(\}z, \bar{t}^2|; \}\bar{t}^k|_3^N)}{\nu_3(z) h(\bar{t}^2, z)^{\eta_{m,2}} f_{[3]}(\bar{t}^3, z)}. \qquad)6/22*$$

Tvctujuvujoh i fsf $\tilde{\lambda}_3$ x f tff ui bu

$$\mathbb{X}^{(2)} = \mathbb{B}(\}z, \bar{t}^2|; \}\bar{t}^k|_3^N). \qquad)6/23*$$

Ui f dpousjcvujpo $\mathbb{X}^{(3)}$ jodmveft ui f ufsn t x jui $j > 3$ gspn ui f –stujof pg)6/: */ Ui f dpousjcv. ujpo $\mathbb{X}^{(4)}$ dpn ft gspn ui f tfdpoe ijof pg)6/: */ Dpotjefs $\mathbb{X}^{(4)}$ di bohjoh ui f psefs pg tvn n bujpo boe tvctujuvujoh ui fsf)6/8*/ Xf i bwf

$$\mathbb{X}^{(4)} = \sum_{q=4}^{N+2} \sum_{j=3}^{q\cdot 2} \sum_{\text{qbsu}(\bar{t}^3,\ldots,\bar{t}^{q\cdot 2})} \frac{\nu_q(z) f_{[q]}(\bar{t}^q, z) h(\bar{t}^m_{\rm II}, z)^{[q]} h(\bar{t}^m_{\rm I}, z)^{\eta_{m,2}}}{\nu_3(z) h(\bar{t}^2, z)^{\eta_{m,2}} f_{[3]}(\bar{t}^3, z)}$$
$$* \; \frac{g(z, \bar{t}^{q\cdot 2}_{\rm I})}{g(\bar{t}^j_{\rm I}, \bar{t}^{j\cdot 2}_{\rm I})} \left(\prod_{\sigma=3}^{q\cdot 2} g_{[\sigma]}(\bar{t}^{\sigma}_{\rm I}, \bar{t}^{\sigma\cdot 2}_{\rm I}) \quad \sigma \right) \mathbb{B}(\}z, \bar{t}^2|; \{z, \bar{t}^k_{\rm II}\}_3^{q\cdot 2}; \{\bar{t}^k\}_q^N). \qquad)6/24*$$

Ui f tvn pwfs j dbo cf fbtjmz dpn qvufe

$$\sum_{j=3}^{q\cdot 2}\frac{2}{g(\bar t_J^{j},\bar t_J^{j\cdot 2})}=\frac{2}{c}\sum_{j=3}^{q\cdot 2}(\bar t_J^{j}\cdot\ \bar t_J^{j\cdot 2})=\frac{2}{c}(\bar t_J^{q\cdot 2}\cdot\ \bar t_J^{2})=\cdot\ 2/g(z,\bar t_J^{q\cdot 2}), \qquad)6/25*$$

boe x f sfdbmmi bucz ef–ojujpo $\bar t_J^2 = z$/ Ui vt-

$$\mathbb{X}^{(4)}=\cdot\sum_{q=4}^{N+2}\sum_{\text{qbsu}(\bar t^3,\dots,\bar t^{q\cdot 2})}\frac{\nu_q(z)f_{[q]}(\bar t^q,z)h(\bar t_{JJ}^m,z)^{[q]}}{\nu_3(z)h(\bar t_{JJ}^2,z)^{\eta_{m,2}}f_{[3]}(\bar t^3,z)}\prod_{\sigma=3}^{q\cdot 2}g_{[\sigma]}(\bar t_J^{\sigma},\bar t_J^{\sigma\cdot 2})\quad\sigma$$
$$*\ \mathbb{B}(\}z,\bar t^{2|};\{z,\bar t_{JJ}^k\}_3^{q\cdot 2};\{\bar t^k\}_q^N). \qquad)6/26*$$

P o ui f pui fs i boe- ui f dpousjcvujpo $\mathbb{X}^{(3)}$ jt

$$\mathbb{X}^{(3)}=\sum_{j=4}^{N+2}\sum_{\text{qbsu}(\bar t^3,\dots,\bar t^{j\cdot 2})}\frac{\nu_j(z)f_{[j]}(\bar t^j,z)h(\bar t_{JJ}^m,z)^{[j]}}{\nu_3(z)h(\bar t_{JJ}^2,z)^{\eta_{m,2}}f_{[3]}(\bar t^3,z)}\prod_{\sigma=3}^{j\cdot 2}g_{[\sigma]}(\bar t_J^{\sigma},\bar t_J^{\sigma\cdot 2})\quad\sigma$$
$$*\ \mathbb{B}(\}z,\bar t^{2|};\{z,\bar t_{JJ}^k\}_3^{j\cdot 2};\{\bar t^k\}_j^N). \qquad)6/27*$$

Dpn qbsjoh)6/27* boe)6/26* x f tff ui bu ui fz dbodfm fbdi pui fs/ Ui vt- $\mathbb{X}=\mathbb{B}(\}z,\bar t^{2|};\}\bar t^{k|}{}_3^N)$/ □

6/2/ Rsppg pg Rspqptjujpo 5/3

Mfu vt efsjwf opx sfdvstjpo)5/5* tubsujoh x jui)5/2* boe vtjoh n psqi jtn)4/26*/ Tjodf ui f n bqqjoh)4/26* sfmbuft ux p ejggfsfou Zbohjbot $Y(\mathfrak{gl}(m|n))$ boe $Y(\mathfrak{gl}(n|m))$- x f vtf i fsf beej. ujpobm vqfst dsjqut gps ui f gvodujpot $g(u,v)$- $f(u,v)$- $\delta(u,v)$- boe $\hat\delta(u,v)$/ Gps fybn qmf- opubujpo $f_{[\sigma]}^{m|n}(u,v)$ n fbot ui bu ui f gvodujpo $f_{[\sigma]}(u,v)$ jt ef–ofe x jui sftqfduup $Y(\mathfrak{gl}(m|n))$;

$$f_{[\sigma]}^{m|n}(u,v)=\begin{cases}f(u,v), & \sigma\ge m,\\ f(v,u), & \sigma> m.\end{cases} \qquad)6/28*$$

Buui f tbn f ujn f ui f opubujpo $f_{[\sigma]}^{n|m}(u,v)$ n fbot ui bu ui f gvodujpo $f_{[\sigma]}(u,v)$ jt ef–ofe x jui sftqfdu up $Y(\mathfrak{gl}(n|m))$;

$$f_{[\sigma]}^{n|m}(u,v)=\begin{cases}f(u,v), & \sigma\ge n,\\ f(v,u), & \sigma> n.\end{cases} \qquad)6/29*$$

Ui f pui fs sbujpobm gvodujpot ti pvme cf voefstppe tjn jmbsmz/ Ujt fbtz up tff ui bu

$$\begin{aligned} g_{[\sigma]}^{m|n}(u,v)&=g_{[N+3\cdot\ \sigma]}^{n|m}(v,u),\\ f_{[\sigma]}^{m|n}(u,v)&=f_{[N+3\cdot\ \sigma]}^{n|m}(v,u),\\ \delta_{\sigma}^{m|n}(u,v)&=\hat\delta_{N+2\cdot\ \sigma}^{n|m}(v,u).\end{aligned} \qquad)6/2:*$$

Mfu vt bdu x jui φ pour)5/2*/ Evf up)4/26*`)4/29* x f i bwf

$$\varphi\left(\frac{T_{2,j}^{m|n}(z)}{\nu_3(z)}\right)=(\cdot\ 2)^{[j]}\frac{T_{N+3\cdot\ j,N+2}^{n|m}(z)}{\nu_N(z)}, \qquad)6/31*$$

$$\varphi\Big)\mathbb{B}^{m|n}(\}z,\bar t^{2|};\}\bar t^{k|}{}_3^N)\Big[=(\cdot\ 2)^{r_m+\eta_{m,2}}\frac{\mathbb{B}^{n|m}(\}\bar t^{k|}{}_N^3;\}z,\bar t^{2|})}{\gamma_N(z)\prod_{k=2}^N\gamma_{N+2\cdot\ k}(\bar t^k)}, \qquad)6/32*$$

boe

$$\varphi\left(\mathbb{B}^{m|n}(\}\bar{t}^{2|};\}\bar{t}_{\text{II}}^{k|}{}_{3}^{j\cdot\ 2};\}\bar{t}^{k|}{}_{j}^{N})\prod_{\sigma=3}^{j\cdot\ 2}\gamma_\sigma(\bar{t}_{\text{I}}^{\sigma})\right)=(\cdot\ 2)^{r_m+\eta_{m,2}+[j]}\frac{\mathbb{B}^{n|m}(\}\bar{t}^{k|}{}_{N}^{j};\}\bar{t}_{\text{II}}^{k|}{}_{j\cdot\ 2}^{3};\bar{t}^{2})}{\prod_{k=2}^{N}\gamma_{N+2\cdot\ k}(\bar{t}^{k})}$$)6/33*

Ui vt-ui f bdujpo pgui f n psqi jtn φ poup)5/2*hjwft

$$\mathbb{B}^{n|m}(\}\bar{t}^{k|}{}_{N}^{3};\}z,\bar{t}^{2|})=\sum_{j=3}^{N+2}\frac{T_{N+3\cdot\ j,N+2}(z)}{\nu_{N+2}(z)}\sum_{\text{qbsu}(\bar{t}^{3},\dots,\bar{t}^{j\cdot\ 2})}\mathbb{B}^{n|m}(\}\bar{t}^{k|}{}_{N}^{j};\}\bar{t}_{\text{II}}^{k|}{}_{j\cdot\ 2}^{3};\bar{t}^{2})$$
$$*\ \frac{\prod_{\sigma=3}^{j\cdot\ 2}g_{[\sigma]}^{m|n}(\bar{t}_{\text{I}}^{\sigma},\bar{t}_{\text{I}}^{\sigma\cdot\ 2})\delta_{\sigma}^{m|n}(\bar{t}_{\text{II}}^{\sigma},\bar{t}_{\text{I}}^{\sigma})}{h(\bar{t}^{2},z)^{\eta_{m,2}}\prod_{\sigma=2}^{j\cdot\ 2}f_{[\sigma+2]}^{m|n}(\bar{t}^{\sigma+2},\bar{t}_{\text{I}}^{\sigma})}.$$)6/34*

Vtjoh ui f sfmbujpot)6/2: *boe ui f usjwjbmjefoujuz $\eta_{m,2}=\eta_{n,N}$ x f sfdbtu)6/34*bt

$$\mathbb{B}^{n|m}(\}\bar{t}^{k|}{}_{N}^{3};\}z,\bar{t}^{2|})=\sum_{j=3}^{N+2}\frac{T_{N+3\cdot\ j,N+2}(z)}{\nu_{N+2}(z)}\sum_{\text{qbsu}(\bar{t}^{3},\dots,\bar{t}^{j\cdot\ 2})}\mathbb{B}^{n|m}(\}\bar{t}^{k|}{}_{N}^{j};\}\bar{t}_{\text{II}}^{k|}{}_{j\cdot\ 2}^{3};\bar{t}^{2})$$
$$*\ \frac{\prod_{\sigma=3}^{j\cdot\ 2}g_{[N+3\cdot\ \sigma]}^{n|m}(\bar{t}_{\text{I}}^{\sigma\cdot\ 2},\bar{t}_{\text{I}}^{\sigma})\delta_{N+2\cdot\ \sigma}^{n|m}(\bar{t}_{\text{I}}^{\sigma},\bar{t}_{\text{II}}^{\sigma})}{h(\bar{t}^{2},z)^{\eta_{n,N}}\prod_{\sigma=2}^{j\cdot\ 2}f_{[N+2\cdot\ \sigma]}^{n|m}(\bar{t}_{\text{I}}^{\sigma},\bar{t}^{\sigma+2})}.$$)6/35*

Gjobmmz- sfmbcfmjoh ui f tfut pgui f Cfui f qbsbn fufst $\bar{t}^{k}\to\bar{t}^{N+2\cdot\ k}$ boe di bohjoh $\sigma\to N+2\cdot\ \sigma$ x f pcubjo

$$\mathbb{B}^{n|m}(\}\bar{t}^{k|}{}_{2}^{N\cdot\ 2};\{z,\bar{t}^{N}\})=\sum_{j=2}^{N}\frac{T_{j,N+2}(z)}{\nu_{N+2}(z)}\sum_{\text{qbsu}(\bar{t}^{j},\dots,\bar{t}^{N\cdot\ 2})}\mathbb{B}^{n|m}(\}\bar{t}^{k|}{}_{2}^{j\cdot\ 2};\}\bar{t}_{\text{II}}^{k|}{}_{j}^{N\cdot\ 2};\bar{t}^{N})$$
$$*\ \frac{\prod_{\sigma=j}^{N\cdot\ 2}g_{[\sigma+2]}^{n|m}(\bar{t}_{\text{I}}^{\sigma+2},\bar{t}_{\text{I}}^{\sigma})\delta_{\sigma}^{n|m}(\bar{t}_{\text{I}}^{\sigma},\bar{t}_{\text{II}}^{\sigma})}{h(\bar{t}^{N},z)^{\eta_{n,N}}\prod_{\sigma=j}^{N}f_{[\sigma]}^{n|m}(\bar{t}_{\text{I}}^{\sigma},\bar{t}^{\sigma\cdot\ 2})}.$$)6/36*

Jusfn bjot up sfqmbdf $m\leftrightarrow n$- boe x f bssjwf bu)5/5*/ □

6/3/ Rsppgpgsfdvstjpo gps evbmCfui f wfdupst

Up pcubjo sfdvstjpo gps evbmCfui f wfdupst jujt fopvhi up bdux jui boujn psqi jtn)4/31*poup sfdvstjpot)5/2*boe)5/5*/ Dpotjefs jo efubjmui f bdujpo pg Ψ poup)5/2*/

Bdujoh x jui Ψ po ui f mit pg)5/2*x f pcubjo b evbmwfdups $\mathbb{C}(\}z,\bar{t}^{2|};\}\bar{t}^{k|}{}_{3}^{N})$ evf up)4/34*/ Jo ui f si t x f i bwf

$$\Psi(T_{2,j}\mathbb{B})=(\cdot\ 2)^{[j][\mathbb{B}]}\mathbb{C}\,T_{j,2}.$$)6/37*

Ui f qbsjuz pgui f Cfui f wfdups dbo cf efufsn jofe wjb ui f dpmpsjoh bshvn fout/ Sfdbmmui buCfui f wfdupst bsf qpmzopn jbmtjo ui f pqfsbupst $T_{i,j}$ bdujoh po ui f wfdups $|1\rangle$- boe bmmui f ufsn t pgui ftf qpmzopn jbmti bwf ui f tbn f dpmpsjoh/ Evf up ui f hfofsbmmsvmf- b r vbtjqbsujdmf pgui f dpmps m dbo cf dsfbufe cz ui f pqfsbupst $T_{i,j}$ x jui $i\geq m$ boe $j>m$/ I fodf- bmmui ftf pqfsbupst bsf pee- cfdbvtf $[i]=1$ gps $i\geq m$ boe $[j]=2$ gps $j>m$/ Po ui f pui fs i boe- ui f bdujpo pgbo fwfo pqfsbups $T_{i,j}$ dboopudsfbuf b r vbtjqbsujdmf pgui f dpmps m evf up tjn jmbs bshvn fout/ Ui vt- jgb Cfui f wfdups i bt b dpmsjoh $\{r_2, \dots, r_N\}$- ui fo bmmi f ufsn t pgui f qpmzopn jbmjo $T_{i,j}$ dpoubjo fybdumz r_m pee pqf sbupst- x i fsf $r_m = '\,\bar{t}^m$/ Ui vt- $]\mathbb{B}(\bar{t})\{ = r_m,$ n pe 3/

Jo ui f dbtf voefs dpotjefsbujpo x f ti pvme –oe ui f ovn cfs r'_m pgui f pee pqfsbupst jo ui f Cfui f wfdups $\mathbb{B}(\}\bar{t}^{2|};\}\bar{t}^{k|}_{\text{II}}\,{}^{j\cdot\,2}_{3};\}\bar{t}^{k|}\,{}^{N}_{j})$/ Mfu $r_m = '\,\bar{t}^m$ jo ui f psjhjobmwfdups $\mathbb{B}(\bar{t})$/ Jg $m = 2$- ui fo $r'_m = r_m$/ Jg $2 < m < j$- ui fo $r'_m = r_m \cdot\ 2$/ Gjobmmz- jg $m \sim j$- ui fo $r'_m = r_m$/ Bmmui ftf dbtft dbo cf eftdsjcfe cz ui f gpsn vmb $r'_m = r_m \cdot\ [j] + \eta_{m,2}$/ Ui vt- x f pcubjo

$$\mathbb{C}(\}z,\bar{t}^{2|};\}\bar{t}^{k|}\,{}^{N}_{3}) = \sum_{j=3}^{N+2}\ \sum_{\text{qbsu}(\bar{t}^3,\dots,\bar{t}^{j\cdot\,2})} \mathbb{C}(\}\bar{t}^{2|};\}\bar{t}^{k|}_{\text{II}}\,{}^{j\cdot\,2}_{3};\}\bar{t}^{k|}\,{}^{N}_{j})\frac{T_{j,2}(z)}{\nu_3(z)}(\cdot\ 2)^{[j]r'_m}$$

$$* \ \frac{\prod_{\sigma=3}^{j\cdot\,2}\gamma_\sigma(\bar{t}^\sigma_{\text{I}})g_{[\sigma]}(\bar{t}^\sigma_{\text{I}},\bar{t}^{\sigma\cdot\,2}_{\text{I}})\delta_\sigma(\bar{t}^\sigma_{\text{II}},\bar{t}^\sigma_{\text{I}})}{h(\bar{t}^2,z)^{\eta_{m,2}}\prod_{\sigma=2}^{j\cdot\,2}f_{[\sigma+2]}(\bar{t}^{\sigma+2},\bar{t}^\sigma_{\text{I}})}, \qquad)6/38*$$

x i fsf $r'_m = r_m \cdot\ [j] + \eta_{m,2}$/

Ui jt fyqsfttjpo dbo cf tmjhi umz tjn qmj–fe/ Sfdbmmui bu $\hat{\delta}_i(x,y) = (\cdot\ 2)^{\eta_{m,i}}\delta_i(x,y)$/ Ui vt- di bohjoh $\delta_\sigma(\bar{t}^\sigma_{\text{II}},\bar{t}^\sigma_{\text{I}}) \to \hat{\delta}_\sigma(\bar{t}^\sigma_{\text{II}},\bar{t}^\sigma_{\text{I}})$ jo)6/38* x f pcubjo

$$\prod_{\sigma=3}^{j\cdot\,2}\delta_\sigma(\bar{t}^\sigma_{\text{II}},\bar{t}^\sigma_{\text{I}}) = (\cdot\ 2)^{([j]\cdot\ [3])r'_m}\prod_{\sigma=3}^{j\cdot\,2}\hat{\delta}_\sigma(\bar{t}^\sigma_{\text{II}},\bar{t}^\sigma_{\text{I}}). \qquad)6/39*$$

Jusfn bjot up pctfswf ui bu $[3] = \eta_{m,2}$/ Ui vt- tvctujuvujoh)6/39* jop)6/38* boe sfqmbdjoh ui f tfut $\bar{t}^k$ x jui $\bar{s}^k$ x f bssjwf bu)5/6*/ Sfdvstjpo)5/7* dbo cf pcubjofe fybdumz jo ui f tbn f x bz/

71 Rsppgpgui f tvn gpsn vmb gps ui f tdbmbs rspevdu

7/2/ I px ui f tdbmbs qspevduefqfoet po ui f wbdvvn fjhfowbmvft $\nu_i(z)$

Jo ui jt tfdujpo- x f jowftujhbuf ui f gvodujpobmefqfoefodf pg ui f tdbmbs qspevdupo ui f gvod. ujpot γ_i/ Qspqptjujpo 5/4 tubuft ui bu ui f Cfui f qbsbn fufst gspn ui f tfut $\bar{s}^i$ boe $\bar{t}^i$ dbo cf ui f bshvn fout pgui f gvodujpot γ_i pomz/ Jo pui fs x pset- ui f tdbmbs qspevduepft opuefqfoe po $\gamma_i(s^\ell_k)$ ps $\gamma_i(t^\ell_k)$ x jui $\ell \neq i$/

Xf qspwf ui jt tubufn fou wjb joevdujpo pwfs $N = m + n \cdot\ 2$/ Gps $N = 2$ jucfdpn ft pcwjpvt/ Bttvn f ui bujujt wbmje gps tpn f $N \cdot\ 2$ boe dpotjefs ui f tdbmbs qspevdupgui f wfdupst $\mathbb{C}^{m|n}(\bar{s})$ boe $\mathbb{B}^{m|n}(\bar{t})$ x jui $m + n \cdot\ 2 = N$/ P ctfswf ui bu x f beefe tvqfstdsjqut up ui f Cfui f wfdupst jo psefs up ejtujohvjti ui fn gspn ui f wfdupst dpssftqpoejoh up $\mathfrak{gl}(m \cdot\ 2|n)$ bmhfcsb/ Xf –stuqspwf ui bu ui f tdbmbs qspevduepft opuefqfoe po ui f gvodujpot $\gamma_i(s^\ell_k)$ x jui $\ell \neq i$ gps $i = 3,\dots,N$/

Tvddfttjwf bqqmjdbujpo pgui f sfdvstjpo)5/6* bmmpx t pof up fyqsftt b evbmCfui f wfdups $\mathbb{C}^{m|n}(\bar{s})$ jo ufsn t pg evbmCfui f wfdupst $\mathbb{C}^{m\cdot\ 2|n}(\bar{\tau})$/ Tdi fn bujdbmmz ui jt fyqsfttjpo dbo cf x sjuufo jo ui f gpmmpx joh gpsn

$$\mathbb{C}^{m|n}(\bar{s}) = \sum_{j_2,\dots,j_{r_2}=3}^{m+n}\ \sum_{\{\bar{\tau}^3,\dots,\bar{\tau}^N\}} \Theta^{(\bar{s})}_{j_2,\dots,j_{r_2}}(\bar{\tau})\mathbb{C}^{m\cdot\ 2|n}(\}\bar{\tau}^{\,|}\,{}^{N}_{3})\frac{T_{j_2,2}(s^2_2)\dots T_{j_{r_2},2}(s^2_{r_2})}{\nu_3(\bar{s}^2)}. \qquad)7/2*$$

I fsf $r_2 = '\,\bar{s}^2$ boe $\bar{\tau}^i \leftarrow \bar{s}^i$ gps $i = 3,\dots,N$/ Ui f tvn jt ublfo pwfs n vmuj.joefy $\{j_2,\dots,j_{r_2}\}$/ Fwfsz ufsn pg ui jt tvn dpoubjot bmtp b tvn pwfs qbsujujpot pg ui f tfut $\bar{s}^3,\dots,\bar{s}^N$ jour tvctfut

$\bar{\tau}^3, \ldots, \bar{\tau}^N$ boe ui fjs dpn qrfin foubsz tvctfut/ Ui f gbdupst $\Theta^{(\bar{s})}_{j_2,\ldots,j_{r_2}}(\bar{\tau})$ bsf tpn f ovn fsjdbmdp. fg–djfout x i ptf fyqmjdjugpsn jt opufttfoujbm/Jujt jn qpsubou- i px fwfs- up opuf ui bujo)5/6* ui fz efqfoe po $\gamma_i(s^i_k)$ x jui $i = 3, \ldots, N$ boe ep opuefqfoe po ui f gvodujpot γ_i x jui pui fs bshvn fout/

Mfu vt n vmujqmz)7/2* gspn ui f sjhi ucz b Cfui f wfdups $\mathbb{B}^{m|n}(\bar{t})$ boe bdu x jui ui f pqfsbupst $T_{j_p,2}(s^2_p)$ poup ui jt wfdups/ Evf up ui f sftvmt pg]3: ‘ ui f bdujpo pg boz pqfsbups $T_{ij}(z)$ poup ui f Cfui f wfdups $\mathbb{B}^{m|n}(\bar{t})$ hjwft b mjofbs dpn cjobujpo pg ofx Cfui f wfdupst $\mathbb{B}^{m|n}(\Omega)$- tvdi ui bu $\Omega = \{\Omega^2, \ldots, \Omega^N\}$ boe $\Omega^i \leftarrow \{\bar{t}^i \cup z\}$/ Jo ui f dbtf voefs dpotjefsbujpo fbdi pg ui f pqfsbupst $T_{j_p,2}(s^2_p)$ booji jmbuft b qbsujdmf pg dpmps 2/ I fodf- ui f upubmbdujpo pg $T_{j_2,2}(s^2_2)\ldots T_{j_{r_2},2}(s^2_{r_2})$ booji jmbuft bmm ui f qbsujdmft pg dpmps 2 jo ui f wfdups $\mathbb{B}^{m|n}(\bar{t})$/ Ui vt- bgufs ui jt bdujpo ui f Cfui f wfdups $\mathbb{B}^{m|n}(\bar{t})$ uvsot joup $\mathbb{B}^{m-2|n}(\Omega)$- x i fsf $\Omega = \{\Omega^3, \ldots, \Omega^N\}$ boe $\Omega^i \leftarrow \{\bar{t}^i \cup \bar{s}^2\}$

$$\frac{T_{j_2,2}(s^2_2)\ldots T_{j_{r_2},2}(s^2_{r_2})}{\nu_3(\bar{s}^2)}\mathbb{B}^{m|n}(\bar{t}) = \sum_{\{\Omega^3,\ldots,\Omega^N\}} \Theta^{(\bar{t})}(\Omega)\mathbb{B}^{m-2|n}(\{\Omega^k\}^N_3). \qquad)7/3*$$

I fsf ui f dpfg–djfout $\Theta^{(\bar{t})}(\Omega)$ pg ui f mjofbs dpn cjobujpo efqfoe po ui f psjhjobmtfut $\bar{t}^k$ boe tvc. tfut Ω^k/ Ui fz jowpmwf ui f gvodujpot γ_i x i ptf bshvn fout cfmpoh up ui f tfu $\{\bar{s}^2 \cup \bar{t}\}$/ Ui fsfgpsf- ui f gbdupst $\Theta^{(\bar{t})}(\Omega)$ ep opuefqfoe po $\gamma_j(s^i_k)$ x jui $i, j = 3, \ldots, N$/

Ui vt- x f pcubjo b sfdvstjpo gps ui f tdbmbs qspevdu

$$\mathbb{C}^{m|n}(\bar{s})\mathbb{B}^{m|n}(\bar{t}) = \sum_{\substack{\{\bar{\tau}^3,\ldots,\bar{\tau}^N\}\\ \{\Omega^3,\ldots,\Omega^N\}}} \Theta^{(\bar{s})}_{j_2,\ldots,j_{r_2}}(\bar{\tau})\Theta^{(\bar{t})}(\Omega)\,\mathbb{C}^{m-2|n}(\{\bar{\tau}^k\}^N_3)\mathbb{B}^{m-2|n}(\{\Omega^k\}^N_3), \qquad)7/4*$$

x i fsf $\bar{\tau}^k \leftarrow \bar{s}^k$ boe $\Omega^k \leftarrow \{\bar{s}^2 \cup \bar{t}^k\}$/ Ui f tvn jt ublfo pwfs tvctfut $\bar{\tau}^k$ boe Ω^k/

Evf up ui f joevdujpo bttvn qujpo- ui f tdbmbs qspevdu $\mathbb{C}^{m-2|n}(\{\bar{\tau}^k\}^N_3)\mathbb{B}^{m-2|n}(\{\Omega^k\}^N_3)$ efqfoet po ui f gvodujpot γ_i x jui bshvn fout τ^i_k boe Ω^i_k/ Tjodf $\tau^i_k \in \bar{s}^i$- x f dpodmvef ui bu ui f Cfui f qb. sbn fufst s^i_k gps $i = 3, \ldots, N$ dbo cf dpn f ui f bshvn fout pg ui f gvodujpot γ_i pomz/ Ui f ovn fsjdbm dpfg–djfout $\Theta^{(\bar{s})}_{j_2,\ldots,j_{r_2}}(\bar{\tau})$ boe $\Theta^{(\bar{t})}(\Omega)$ ep opucsfbl ui jt uzqf pg efqfoefodf/ Ui vt- x f qspwf ui bu jo ui f tdbmbs qspevdu $\mathbb{C}^{m|n}(\bar{s})\mathbb{B}^{m|n}(\bar{t})$ ui f Cfui f qbsbn fufst s^i_k x jui $i = 3, \ldots, N$ dbo cfdpn f ui f bshvn fout pg ui f gvodujpot γ_i pomz/

Evf up ui f tzn n fusz)5/21*- bo bobmphpvt qspqfsuz i pmet gps ui f Cfui f qbsbn fufst $\bar{t}^i$ x jui $i = 3, \ldots, N$/ Obn fmz- ui ftf qbsbn fufst dbo cf ui f bshvn fout pg ui f gvodujpot γ_i pomz/

Jusfn bjot up qspwf ui bu ui f Cfui f qbsbn fufst gspn ui f tfut $\bar{s}^2$ boe $\bar{t}^2$ dbo cf ui f bshvn fout pg ui f gvodujpo γ_2/ Gps ui jt x f vtf ui f tfdpoe sfdvstjpo gps ui f evbmCfui f wfdups)5/7* boe sfqfbu bmmui f dpotjefsbujpot bcpwf/ Ui fo x f –oe ui bu ui f Cfui f qbsbn fufst s^i_k x jui $i = 2, \ldots, N-2$ dbo cfdpn f ui f bshvn fout pg ui f gvodujpot γ_i pomz/ Ui fo- ui f vtf pg)5/21* dpn qmfuft ui f qsppg pg Qspqptjujpo 5/4/ □

7/3/ Rsppg pg ui f tvn gpsn vmb

Dpotjefs b dpn qptjuf n pefm jo x i jdi ui f n popespn z n busjy $T(u)$ jt qsftfoufe bt b qspevdu pg ux p qbsujbm n popespn z n busjdft]7-31-3: -52‘;

$$T(u) = T^{(3)}(u)T^{(2)}(u). \qquad)7/5*$$

X jui jo ui f gsbn fx psl pg ui f dpn qptjuf n pefm jujt bttvn fe ui bu ui f n busjy fmfn fout pg fwfsz $T^{(l)}(u)$ $)l = 2, 3*$ bdu jo tpn f I jmcfsu tqbdf $\mathcal{H}^{(l)}$- tvdi ui bu $\mathcal{H} = \mathcal{H}^{(2)} \otimes \mathcal{H}^{(3)}$/ Fbdi pg $T^{(l)}(u)$

tbujt–ft uif RTT.sfmbujpo)3/5* boe ibt jut pxo qtfvepwbdvvn wfdups $|1\rangle^{(l)}$ boe evbmwfdups $\langle 1|^{(l)}$- tvdi ubu $|1\rangle = |1\rangle^{(2)} \circ |1\rangle^{(3)}$ boe $\langle 1| = \langle 1|^{(2)} \circ \langle 1|^{(3)}$/ Tjodf uif pqfsbupst $T^{(3)}_{i,j}(u)$ boe $T^{(2)}_{k,l}(v)$ bdujo ejggfsfou tqbdft- uifz tvqfsdpnnvuf xjui fbdi puifs/ Xf bttvnf ubu

$$T^{(l)}_{i,i}(u)|1\rangle^{(l)} = \nu^{(l)}_i(u)|1\rangle^{(l)}, \qquad \langle 1|^{(l)}T^{(l)}_{i,i}(u) = \nu^{(l)}_i(u)\langle 1|^{(l)}, \qquad i = 2, \dots, m+n, \qquad l = 2, 3, \qquad)7/6*$$

xifsf $\nu^{(l)}_i(u)$ bsf ofx gsff gvodujpobmqbsbnfufst/ Xf bmtp jouspevdf

$$\gamma^{(l)}_k(u) = \frac{\nu^{(l)}_k(u)}{\nu^{(l)}_{k+2}(u)}, \qquad l = 2, 3, \qquad k = 2, \dots, N. \qquad)7/7*$$

Pcwjpvtmz

$$\nu_i(u) = \nu^{(2)}_i(u)\nu^{(3)}_i(u), \qquad \gamma_k(u) = \gamma^{(2)}_k(u)\gamma^{(3)}_k(u). \qquad)7/8*$$

Uif qbsujbmnpopespnz nbusjdft $T^{(l)}(u)$ ibwf uif dpssftqpoejoh Cfuif wfdupst $\mathbb{B}^{(l)}(\bar t)$ boe evbmCfuif wfdupst $\mathbb{C}^{(l)}(\bar s)$/ B Cfuif wfdups pg uif upubmnpopespnz nbusjy $T(u)$ dbo cf fyqsfttfe jo ufsnt qbsujbmCfuif wfdupst $\mathbb{B}^{(l)}(\bar t)$ wjb *dpqspevdugpsnvmb*[4]]3: -52‘

$$\mathbb{B}(\bar t) = \sum \frac{\prod_{\sigma=2}^{N} \gamma^{(3)}_\sigma(\bar t^\sigma_{\rm j})\delta_\sigma(\bar t^\sigma_{\rm jj}, \bar t^\sigma_{\rm j})}{\prod_{\sigma=2}^{N\cdot 2} f_{[\sigma+2]}(\bar t^{\sigma+2}_{\rm jj}, \bar t^\sigma_{\rm j})} \mathbb{B}^{(2)}(\bar t_{\rm j}) \circ \mathbb{B}^{(3)}(\bar t_{\rm jj}). \qquad)7/9*$$

Ifsf bmmuif tfut pg uif Cfuif qbsbnfufst $\bar t^\sigma$ bsf ejwjefe jopu uxp tvctfut $\bar t^\sigma \Rightarrow \{\bar t^\sigma_{\rm j}, \bar t^\sigma_{\rm jj}\}$- boe uif tvn jt ublfo pwfs bmmqpttjcmf qbsujujpot/

Tjnjmbs gpsnvmb fyjtut gps uif evbmCfuif wfdupst $\mathbb{C}(\bar s)$)tff Bqqfoejy B*

$$\mathbb{C}(\bar s) = \sum \frac{\prod_{\sigma=2}^{N} \gamma^{(2)}_\sigma(\bar s^\sigma_{\rm jj})\delta_\sigma(\bar s^\sigma_{\rm j}, \bar s^\sigma_{\rm jj})}{\prod_{\sigma=2}^{N\cdot 2} f_{[\sigma+2]}(\bar s^{\sigma+2}_{\rm j}, \bar s^\sigma_{\rm jj})} \mathbb{C}^{(3)}(\bar s_{\rm jj}) \circ \mathbb{C}^{(2)}(\bar s_{\rm j}), \qquad)7/:*$$

xifsf uif tvn jt pshbojfife jo uif tbnf xbz bt jo)7/9*/

Uifo uif tdbmbs qspevdu pg uif upubmCfuif wfdupst $\mathbb{C}(\bar s)$ boe $\mathbb{B}(\bar t)$ ublft uif gpsn

$$S(\bar s|\bar t) = \sum \frac{\prod_{\sigma=2}^{N} \gamma^{(2)}_\sigma(\bar s^\sigma_{\rm jj})\gamma^{(3)}_\sigma(\bar t^\sigma_{\rm j})\delta_\sigma(\bar s^\sigma_{\rm j}, \bar s^\sigma_{\rm jj})\delta_\sigma(\bar t^\sigma_{\rm jj}, \bar t^\sigma_{\rm j})}{\prod_{\sigma=2}^{N\cdot 2} f_{[\sigma+2]}(\bar s^{\sigma+2}_{\rm j}, \bar s^\sigma_{\rm jj}) f_{[\sigma+2]}(\bar t^{\sigma+2}_{\rm jj}, \bar t^\sigma_{\rm j})} S^{(2)}(\bar s_{\rm j}|\bar t_{\rm j}) S^{(3)}(\bar s_{\rm jj}|\bar t_{\rm jj}), \qquad)7/21*$$

xifsf

$$S^{(2)}(\bar s_{\rm j}|\bar t_{\rm j}) = \mathbb{C}^{(2)}(\bar s_{\rm j})\mathbb{B}^{(2)}(\bar t_{\rm j}), \qquad S^{(3)}(\bar s_{\rm jj}|\bar t_{\rm jj}) = \mathbb{C}^{(3)}(\bar s_{\rm jj})\mathbb{B}^{(3)}(\bar t_{\rm jj}). \qquad)7/22*$$

Opuf ubu jo uijt gpsnvmb $'\bar s^\sigma_{\rm j} = '\bar t^\sigma_{\rm j}$-)boe ifodf- $'\bar s^\sigma_{\rm jj} = '\bar t^\sigma_{\rm jj}$*- puifsxjtf uif tdbmbs qspevdu $S^{(2)}$ boe $S^{(3)}$ wbojti/ Mfu $'\bar s^\sigma_{\rm j} = '\bar t^\sigma_{\rm j} = k'_\sigma$- xifsf $k'_\sigma = 1, 2, \dots, r_\sigma$/ Uifo $'\bar s^\sigma_{\rm jj} = '\bar t^\sigma_{\rm jj} = r_\sigma \cdot k'_\sigma$/

Opx mfu vt uvso up fr vbujpo)5/22*/ Pvs hpbmjt up fyqsftt uif sbujpobmdpfgdjfout $W^{m|n}_{\rm qbsu}$ jo ufsnt pg uif ID/ Gps uijt xf vtf uif gbdu ubu $W^{m|n}_{\rm qbsu}$ bsf npefmjoefqfoefou/ Uifsfgpsf- xf dbo –oe uifn jo tpnf tqfdjbmnpefmxiptf npopespnz nbusjy tbujt–ft uif RTT.sfmbujpo/

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[4] Uif ufsn jopmphz *dpqspevdugpsnvmb* jt vtfe gps ijtupsjdbmsfbtpo- cfdbvtf)7/9* xbt efsjwfe gps uif –stu ujnf jo]3: ‘)tff bmtp]41‘ gps uif opo.hsbefe dbtf* bt b qspqfsuz pg uif Cfuif wfdupst joevdfe cz uif Zbohjbo dpqspevdu/

Mfuvt –y tpn f qbsujujpot pgui f Cfui f qbsbn fufst jo)5/22*; $\bar{s}^{\sigma} \Rightarrow \{\bar{s}^{\sigma}_{\rm J}, \bar{s}^{\sigma}_{\rm JJ}\}$ boe $\bar{t}^{\sigma} \Rightarrow \{\bar{t}^{\sigma}_{\rm J}, \bar{t}^{\sigma}_{\rm JJ}\}$ tvdi ui bu' $\bar{s}^{\sigma}_{\rm J} =$ ' $\bar{t}^{\sigma}_{\rm J} = k_{\sigma}$- x i fsf $k_{\sigma} = 1, 2, \ldots, r_{\sigma}$/ I fodf- ' $\bar{s}^{\sigma}_{\rm JJ} =$ ' $\bar{t}^{\sigma}_{\rm JJ} = r_{\sigma} \cdot \; k_{\sigma}$/ Dpotjefs b dpo. dsfuf n pefmjo x i jdi [5]

$$\begin{aligned} \gamma^{(2)}_{\sigma}(z) &= 1, \quad \text{jg} \quad z \in \bar{s}^{\sigma}_{\rm JJ}; \\ \gamma^{(3)}_{\sigma}(z) &= 1, \quad \text{jg} \quad z \in \bar{t}^{\sigma}_{\rm J}. \end{aligned} \qquad)7/23*$$

Evf up)7/8* ui ftf dpoejujpot jn qmz

$$\gamma_{\sigma}(z) = 1, \quad \text{jg} \quad z \in \bar{s}^{\sigma}_{\rm JJ} \cup \bar{t}^{\sigma}_{\rm J}. \qquad)7/24*$$

Ui fo ui f tdbmbs qspevdu jt qspqpsujpobmup ui f dpfg–djfou $W^{m|n}_{\rm qbsu}(\bar{s}_{\rm J}, \bar{s}_{\rm JJ}|\bar{t}_{\rm J}, \bar{t}_{\rm JJ})$- cfdbvtf bmmpui fs ufsn t jo ui f tvn pwfs qbsujujpot)5/22* wbojti evf up ui f dpoejujpo)7/24*/ Ui vt-

$$S(\bar{s}|\bar{t}) = W^{m|n}_{\rm qbsu}(\bar{s}_{\rm J}, \bar{s}_{\rm JJ}|\bar{t}_{\rm J}, \bar{t}_{\rm JJ}) \prod_{k=2}^{N} \gamma_k(\bar{s}^k_{\rm J})\gamma_k(\bar{t}^k_{\rm JJ}). \qquad)7/25*$$

Po ui f pui fs i boe-)7/23* jn qmjft ui bu b opo.fifsp dpousjcvujpo jo)7/21* pddvst jg boe pomz jg $\bar{s}^{\sigma}_{\rm jj} \leftarrow \bar{s}^{\sigma}_{\rm J}$ boe $\bar{t}^{\sigma}_{\rm j} \leftarrow \bar{t}^{\sigma}_{\rm JJ}$/ I fodf- $r_{\sigma} \cdot \; k'_{\sigma} \geq k_{\sigma}$ boe $k'_{\sigma} \geq r_{\sigma} \cdot \; k_{\sigma}$/ Cvu ui jt jt qpttjcmf jg boe pomz jg $k'_{\sigma} + k_{\sigma} = r_{\sigma}$/ Ui vt- $\bar{s}^{\sigma}_{\rm jj} = \bar{s}^{\sigma}_{\rm J}$ boe $\bar{t}^{\sigma}_{\rm j} = \bar{t}^{\sigma}_{\rm JJ}$/ Ui fo- gps ui f dpn qmfn fobsz tvctfut x f pcubjo $\bar{s}^{\sigma}_{\rm j} = \bar{s}^{\sigma}_{\rm JJ}$ boe $\bar{t}^{\sigma}_{\rm jj} = \bar{t}^{\sigma}_{\rm J}$/ Ui vt- x f bssjwf bu

$$S(\bar{s}|\bar{t}) = \frac{\prod_{\sigma=2}^{N} \gamma^{(2)}_{\sigma}(\bar{s}^{\sigma}_{\rm J})\gamma^{(3)}_{\sigma}(\bar{t}^{\sigma}_{\rm JJ})\delta_{\sigma}(\bar{s}^{\sigma}_{\rm JJ}, \bar{s}^{\sigma}_{\rm J})\delta_{\sigma}(\bar{t}^{\sigma}_{\rm J}, \bar{t}^{\sigma}_{\rm JJ})}{\prod_{\sigma=2}^{N\cdot\;2} f_{[\sigma+2]}(\bar{s}^{\sigma+2}_{\rm JJ}, \bar{s}^{\sigma}_{\rm J}) f_{[\sigma+2]}(\bar{t}^{\sigma+2}_{\rm J}, \bar{t}^{\sigma}_{\rm JJ})} S^{(2)}(\bar{s}_{\rm JJ}|\bar{t}_{\rm JJ}) S^{(3)}(\bar{s}_{\rm J}|\bar{t}_{\rm J}). \qquad)7/26*$$

Jujt fbtz up tff ui bu dbmdvmbujoh ui f tdbmbs qspevdu $S^{(2)}(\bar{s}_{\rm JJ}|\bar{t}_{\rm JJ})$ x f ti pvme ublf pomz ui f ufsn dpssftqpoejoh up ui f dpokvhbufe I D/ Joeffe- bmmpui fs ufsn t bsf qspqpsujpobmup $\gamma^{(2)}_{\sigma}(z)$ x jui $z \in \bar{s}^{\sigma}_{\rm JJ}$- ui fsfgpsf- ui fz wbojti/ I fodf

$$S^{(2)}(\bar{s}_{\rm JJ}|\bar{t}_{\rm JJ}) = \prod_{\sigma=2}^{N} \gamma^{(2)}_{\sigma}(\bar{t}^{\sigma}_{\rm JJ}) \times \overline{Z}^{m|n}(\bar{s}_{\rm JJ}|\bar{t}_{\rm JJ}). \qquad)7/27*$$

Tjn jmbsmz- dbmdvmbujoh ui f tdbmbs qspevdu $S^{(3)}(\bar{s}_{\rm J}|\bar{t}_{\rm J})$ x f ti pvme ublf pomz ui f ufsn dpssftqpoejoh up ui f I D;

$$S^{(3)}(\bar{s}_{\rm J}|\bar{t}_{\rm J}) = \prod_{\sigma=2}^{N} \gamma^{(3)}_{\sigma}(\bar{s}^{\sigma}_{\rm J}) \times Z^{m|n}(\bar{s}_{\rm J}|\bar{t}_{\rm J}). \qquad)7/28*$$

Tvctujuvujoh ui jt joup)7/26* boe vtjoh)7/8*-)7/25* x f bssjwf bu

$$W^{m|n}_{\rm qbsu}(\bar{s}_{\rm J}, \bar{s}_{\rm JJ}|\bar{t}_{\rm J}, \bar{t}_{\rm JJ}) = Z^{m|n}(\bar{s}_{\rm J}|\bar{t}_{\rm J})\; \overline{Z}^{m|n}(\bar{s}_{\rm JJ}|\bar{t}_{\rm JJ}) \frac{\prod_{k=2}^{N} \delta_k(\bar{s}^k_{\rm JJ}, \bar{s}^k_{\rm J})\delta_k(\bar{t}^k_{\rm J}, \bar{t}^k_{\rm JJ})}{\prod_{j=2}^{N\cdot\;2} f_{[j+2]}(\bar{s}^{j+2}_{\rm JJ}, \bar{s}^{j}_{\rm J}) f_{[j+2]}(\bar{t}^{j+2}_{\rm J}, \bar{t}^{j}_{\rm JJ})}. \qquad)7/29*$$

Ui jt fyqsfttjpo pcwjpvtmz dpjodjeft x jui)5/26* evf up)5/25*/

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[5] Ui jt di pjdf pgui f gvodujpot γ_k jt bmx bzt qpttjcmf- gps fybn qmf- x jui jo ui f gsbn fx psl pgjoi pn phfofpvt n pefmx jui tqjot jo i jhi fs ejn fotjpobmsfqsftfoubujpot- jo x i jdi joi pn phfofjujft dpjodjef x jui tpn f pgui f Cfui f qbsbn fufst/

81 I khi ftudpfg dkfou

9/2/ Rsppgpguif sfdvstjpo gps uif I jhiftuDpfg-djfou

Jugpmpx t gspn Qspqptjujpo 5/4 ui buu i f tdbmbs qspevdujt b tvn - jo x i jdi fwfsz ufsn jt qspqps. ujpobmp b qspevdupguif gvodujpot γ_k/ Mfuvt dbmmb ufsn *vox boufe*-jguif dpssftqpoejoh qspevdu pguif gvodujpot γ_k dpoubjot bumfbtupof $\gamma_k(t_j^k)$- x i fsf $t_j^k \in \bar{t}$/ Sftqfdujwfmz- b ufsn jt *x boufe*-jg bmmgvodujpot γ_k efqfoe po uif Cfuif qbsbn fufst s_j^k gspn uif tfu$\bar{s}$/

Cfmpx x f dpotjefs tpn f fr vbujpot n pevmpt vox boufe ufsn t/ Jo uijt dbtf x f vtf b tzn cpm$\cong$/ Ui vt-bo fr vbujpo pguif uzqf $lhs \cong rhs$ n fbot ui buu i f lhs jt fr vbmup uif rhs n pevmpt vox boufe ufsn t/

Vtjoh uif opujpo pgvox boufe ufsn t pof dbo sfef–of uif I D)5/23*bt gpmmpx t;

$$S(\bar{s}|\bar{t}) \cong \prod_{k=2}^{N} \gamma_k(\bar{s}^k) \times Z^{m|n}(\bar{s}|\bar{t}). \qquad)8/2*$$

P o uif puifs i boe- jugpmpx t gspn uif fyqmjdjujgpsn pgCfuif wfdupst]3: ‘ ui bu

$$\mathbb{B}(\bar{t}) \cong \widetilde{\mathbb{B}}(\bar{t}) = \frac{\mathbb{T}_{2,3}(\bar{t}^2)\dots\mathbb{T}_{N,N+2}(\bar{t}^N)|1\rangle}{\prod_{j=2}^{N} \nu_{j+2}(\bar{t}^j)\prod_{j=2}^{N\cdot 2} f_{[j+2]}(\bar{t}^{j+2},\bar{t}^j)}, \qquad)8/3*$$

cfdbvtf bmmpuifs ufsn t jo uif Cfuif wfdups dpoubjo gbdupst $\gamma_k(t_j^k)$- boe ui vt- uifz bsf vox boufe/ I fodf- jo psefs up –oe uif I D jujt fopvhi up dpotjefs b *sfevdfe* tdbmbs qspevdu $\tilde{S}(\bar{s}|\bar{t})$

$$S(\bar{s}|\bar{t}) \cong \tilde{S}(\bar{s}|\bar{t}) = \mathbb{C}(\bar{s})\widetilde{\mathbb{B}}(\bar{t}). \qquad)8/4*$$

Jo psefs up dbmdvmbuf uif sfevdfe tdbmbs qspevdu)8/4* x f dbo vtf uif sfdvstjpo)5/6* gps uif evbmCfuif wfdups $\mathbb{C}(\bar{s})$/ X f x sjuf jujo uif gpsn

$$\mathbb{C}(\bar{s}) = \sum_{p=3}^{N+2} \sum_{\text{qbsu}(\bar{s}^3,\dots,\bar{s}^{p\cdot 2})} \mathbb{C}(\}\bar{s}^k_{\text{JJ}}|_2^{p\cdot 2};\}\bar{s}^k|_p^N)\frac{T_{p,2}(\bar{s}^2_{\text{J}})}{\nu_3(\bar{s}^2_{\text{J}})}(\cdot\ 2)^{(r_2\cdot\ 2)\eta_{m,2}}$$
$$* \frac{\prod_{\sigma=3}^{p\cdot 2}\gamma_\sigma(\bar{s}^\sigma_{\text{J}})g_{[\sigma]}(\bar{s}^\sigma_{\text{J}},\bar{s}^{\sigma\cdot 2}_{\text{J}})\delta_\sigma(\bar{s}^\sigma_{\text{JJ}},\bar{s}^\sigma_{\text{J}})}{h(\bar{s}^2,\bar{s}^2_{\text{J}})^{\eta_{m,2}}\prod_{\sigma=2}^{p\cdot 2} f_{[\sigma+2]}(\bar{s}^{\sigma+2},\bar{s}^\sigma_{\text{J}})}. \qquad)8/5*$$

I fsf uif tvn jt ublfo pwfs qbsujujpot pguif tfut $\bar{s}^k \Rightarrow \{\bar{s}^k_{\text{J}},\bar{s}^k_{\text{JJ}}\}$ gps $k = 3,\dots,p$-tvdi ui bu’ $\bar{s}^k_{\text{J}} = 2$/ Ui f Cfuif qbsbn fufs $\bar{s}^2_{\text{J}}$ jt –yfe- boe i fodf- uif tvctfu$\bar{s}^2_{\text{JJ}}$ bmtp jt –yfe/ Ui fsf jt op uif tvn pwfs qbsujujpot pguif tfu$\bar{s}^2$ jo)8/5*/

Ui vt- x f pcubjo

$$\tilde{S}(\bar{s}|\bar{t}) = \sum_{p=3}^{N+2} \sum_{\text{qbsu}(\bar{s}^3,\dots,\bar{s}^{p\cdot 2})} (\cdot\ 2)^{(r_2\cdot\ 2)\eta_{m,2}}\mathbb{C}(\}\bar{s}^k_{\text{JJ}}|_2^{p\cdot 2},\}\bar{s}^k|_p^N)\, T_{p,2}(\bar{s}^2_{\text{J}})\widetilde{\mathbb{B}}(\bar{t})$$
$$* \frac{\prod_{\sigma=3}^{p\cdot 2}\gamma_\sigma(\bar{s}^\sigma_{\text{J}})g_{[\sigma]}(\bar{s}^\sigma_{\text{J}},\bar{s}^{\sigma\cdot 2}_{\text{J}})\delta_\sigma(\bar{s}^\sigma_{\text{JJ}},\bar{s}^\sigma_{\text{J}})}{\nu_3(\bar{s}^2_{\text{J}})h(\bar{s}^2,\bar{s}^2_{\text{J}})^{\eta_{m,2}}\prod_{\sigma=2}^{p\cdot 2} f_{[\sigma+2]}(\bar{s}^{\sigma+2},\bar{s}^\sigma_{\text{J}})}. \qquad)8/6*$$

Ui f bdujpo pg$T_{p,2}(\bar{s}^2_{\text{J}})$ poup uif wfdups $\widetilde{\mathbb{B}}(\bar{t})$ n pevmpt vox boufe ufsn t jt hjwfo cz Qspqptjujpo C/2/ Ui vt- x f pcubjo

$$\tilde{S}(\bar{s}|\bar{t}) \subseteq \gamma_2(\bar{s}_{\mathrm{J}}^2)\sum_{p=3}^{N+2}\sum_{\substack{\text{qbsu}(\bar{s}^3,\ldots,\bar{s}^{p\cdot 2})\\ \text{qbsu}(\bar{t}^2,\ldots,\bar{t}^{p\cdot 2})}} (\cdot\ 2)^{(r_2\cdot\ 2)\eta_{m,2}} \frac{g_{[3]}(\bar{t}_{\mathrm{J}}^2,\bar{s}_{\mathrm{J}}^2)\delta_2(\bar{t}_{\mathrm{J}}^2,\bar{t}_{\mathrm{JJ}}^2)f_{[2]}(\bar{t}_{\mathrm{JJ}}^2,\bar{s}_{\mathrm{J}}^2)}{f_{[p]}(\bar{s}^p,\bar{s}_{\mathrm{J}}^{p\cdot\ 2})h(\bar{s}^2,\bar{s}_{\mathrm{J}}^2)^{\eta_{m,2}}}$$

$$* \prod_{\sigma=3}^{p\cdot\ 2}\frac{\gamma_\sigma(\bar{s}_{\mathrm{J}}^\sigma)g_{[\sigma]}(\bar{s}_{\mathrm{J}}^\sigma,\bar{s}_{\mathrm{J}}^{\sigma\cdot\ 2})g_{[\sigma+2]}(\bar{t}_{\mathrm{J}}^\sigma,\bar{t}_{\mathrm{J}}^{\sigma\cdot\ 2})\delta_\sigma(\bar{s}_{\mathrm{JJ}}^\sigma,\bar{s}_{\mathrm{J}}^\sigma)\delta_\sigma(\bar{t}_{\mathrm{J}}^\sigma,\bar{t}_{\mathrm{JJ}}^\sigma)}{f_{[\sigma]}(\bar{s}^\sigma,\bar{s}_{\mathrm{J}}^{\sigma\cdot\ 2})f_{[\sigma]}(\bar{t}_{\mathrm{J}}^\sigma,\bar{t}^{\sigma\cdot\ 2})}$$

$$* \ \mathbb{C}(\}\bar{s}_{\mathrm{JJ}}^k|_2^{p\cdot\ 2},\}\bar{s}^k|_p^N)\,\widetilde{\mathbb{B}}(\}\bar{t}_{\mathrm{JJ}}^k|_2^{p\cdot\ 2};\}\bar{t}^k|_p^N). \qquad)8/7*$$

I fsf $\bar{t}^{m+n} = \bar{s}^{m+n} = \emptyset$/ Dbrdvrbujoh ui f sfevdfe tdbrbs qspevdut jo)8/7*n pevrmt vox bouf e ufsn t

$$\mathbb{C}(\}\bar{s}_{\mathrm{JJ}}^k|_2^{p\cdot\ 2},\}\bar{s}^k|_p^N)\widetilde{\mathbb{B}}(\}\bar{t}_{\mathrm{JJ}}^k|_2^{p\cdot\ 2};\}\bar{t}^k|_p^N) \subseteq \prod_{k=2}^{p\cdot\ 2}\gamma_k(\bar{s}_{\mathrm{JJ}}^k)\prod_{\ell=p}^{N}\gamma_\ell(\bar{s}^\ell)$$

$$* \ Z^{m|n}(\}\bar{s}_{\mathrm{JJ}}^k|_2^{p\cdot\ 2},\}\bar{s}^k|_p^N|\}\bar{t}_{\mathrm{JJ}}^k|_2^{p\cdot\ 2};\}\bar{t}^k|_p^N), \quad)8/8*$$

boe tvctujuvujoh ui jt joup)8/7*x f jn n fejbufm bssjwf buui f sfdvstjpo)5/28*/

Xf i bwf brtp vtfe

$$(\cdot\ 2)^{(r_2\cdot\ 2)\eta_{m,2}}\delta_2(\bar{t}_{\mathrm{J}}^2,\bar{t}_{\mathrm{JJ}}^2) = \delta_2(\bar{t}_{\mathrm{J}}^2,\bar{t}_{\mathrm{JJ}}^2), \qquad \delta_\sigma(\bar{s}_{\mathrm{JJ}}^\sigma,\bar{s}_{\mathrm{J}}^\sigma)\delta_\sigma(\bar{t}_{\mathrm{J}}^\sigma,\bar{t}_{\mathrm{JJ}}^\sigma) = \delta_\sigma(\bar{s}_{\mathrm{JJ}}^\sigma,\bar{s}_{\mathrm{J}}^\sigma)\delta_\sigma(\bar{t}_{\mathrm{J}}^\sigma,\bar{t}_{\mathrm{JJ}}^\sigma).$$

9/3/ Tfin n fusfi pgui f I jhi ftuDpf g-djf ou

Evf up jtpn psqi jtn)4/26*cfux ffo Zbohjbot $Y(\mathfrak{gl}(m|n))$ boe $Y(\mathfrak{gl}(n|m))$ pof dbo –oe b tjn qrfi sfrbujpo cfux ffo ui f I D dpssftqpoejoh up ui ftf brhfcsbt/ Jo ui jt tfdujpo x f pcubjo ui jt sfrbujpo/

Dpotjefs ui f tvn gpsn vrb)5/22*gps ui f tdbrbs qspevdupg $\mathfrak{gl}(m|n)$ Cfui f wfdupst

$$S^{m|n}(\vec{s}|\vec{t}) = \sum W_{\text{qbsu}}^{m|n}(\vec{s}_{\mathrm{J}},\vec{s}_{\mathrm{JJ}}|\vec{t}_{\mathrm{J}},\vec{t}_{\mathrm{JJ}})\prod_{k=2}^{N}\gamma_k(\bar{s}_{\mathrm{J}}^k)\gamma_k(\bar{t}_{\mathrm{JJ}}^k), \qquad)8/9*$$

x i fsf x f i bwf tusfttfe ui f psefsjoh)4/28*pgui f Cfui f qbsbn fufst/ Mfuvt bdux jui ui f n psqi jtn φ)4/26*po ui f tdbrbs qspevdu $S^{m|n}(\vec{s}|\vec{t})$/ Ui jt dbo cf epof jo ux p x bzt/ Gjstu- vtjoh)4/29*boe)4/37*x f pcubjo

$$\varphi\Big)S^{m|n}(\vec{s}|\vec{t})\Big[= \varphi\Big)\mathbb{C}^{m|n}(\vec{s})\mathbb{B}^{m|n}(\vec{t})\Big[= \frac{(\cdot\ 2)^{r_m}\mathbb{C}^{n|m}(\grave{s})\mathbb{B}^{n|m}(\grave{t})}{\prod_{k=2}^N\gamma_{N+2\cdot\ k}(\bar{s}^k)\gamma_{N+2\cdot\ k}(\bar{t}^k)}$$

$$= \frac{(\cdot\ 2)^{r_m}S^{n|m}(\grave{s}|\grave{t})}{\prod_{k=2}^N\gamma_{N+2\cdot\ k}(\bar{s}^k)\gamma_{N+2\cdot\ k}(\bar{t}^k)}. \qquad)8/:*$$

Ui f tdbrbs qspevdu $S^{n|m}(\grave{s}|\grave{t})$ i bt ui f tuboebse sfqsftfoubujpo)5/22*/ Ui vt- x f –oe

$$\varphi\Big)S^{m|n}(\vec{s}|\vec{t})\Big[= \sum_{\text{qbsu}}\frac{(\cdot\ 2)^{r_m}W_{\text{qbsu}}^{n|m}(\grave{s}_{\mathrm{J}},\grave{s}_{\mathrm{JJ}}|\grave{t}_{\mathrm{J}},\grave{t}_{\mathrm{JJ}})}{\prod_{k=2}^N\gamma_{N+2\cdot\ k}(\bar{s}^k)\gamma_{N+2\cdot\ k}(\bar{t}^k)}\prod_{k=2}^N\gamma_k(\bar{s}_{\mathrm{J}}^{N\cdot\ k+2})\gamma_k(\bar{t}_{\mathrm{JJ}}^{N\cdot\ k+2}). \qquad)8/21*$$

P o ui f pui fs i boe- bdujoh x jui φ ejsfdum po ui f tvn gpsn vrb)8/9*x f i bwf

$$\varphi\Big)S^{m|n}(\vec{s}|\vec{t})\Big[= \sum_{\text{qbsu}}W_{\text{qbsu}}^{m|n}(\vec{s}_{\mathrm{J}},\vec{s}_{\mathrm{JJ}}|\vec{t}_{\mathrm{J}},\vec{t}_{\mathrm{JJ}})\prod_{k=2}^N\Big)\gamma_{N+2\cdot\ k}(\bar{s}_{\mathrm{J}}^k)\gamma_{N+2\cdot\ k}(\bar{t}_{\mathrm{JJ}}^k)\Big[^{\cdot\ 2}. \qquad)8/22*$$

Dpn qbsjoh)8/21*boe)8/22*x f bssjwf bu

$$(\cdot\ 2)^{r_m}\sum_{\text{qbsu}} W^{n|m}_{\text{qbsu}}(\dot{s}_{\text{J}},\dot{s}_{\text{JJ}}|\dot{t}_{\text{J}},\dot{t}_{\text{JJ}})\prod_{k=2}^{N}\gamma_{N+2\cdot\ k}(\bar{s}^{k}_{\text{J}})\gamma_{N+2\cdot\ k}(\bar{t}^{k}_{\text{JJ}})$$

$$=\sum_{\text{qbsu}} W^{m|n}_{\text{qbsu}}(\vec{s}_{\text{J}},\vec{s}_{\text{JJ}}|\vec{t}_{\text{J}},\vec{t}_{\text{JJ}})\prod_{k=2}^{N}\gamma_{N+2\cdot\ k}(\bar{s}^{k}_{\text{JJ}})\gamma_{N+2\cdot\ k}(\bar{t}^{k}_{\text{J}}) \qquad)8/23*$$

Tjodf γ_i bsf gsff gvodujpobmqbsbn fufst- ui f dpfg-djfout pgui f tbn f qspevdut pg γ_i n vtucf fr vbm/ I fodf-

$$W^{m|n}_{\text{qbsu}}(\vec{s}_{\text{J}},\vec{s}_{\text{JJ}}|\vec{t}_{\text{J}},\vec{t}_{\text{JJ}})=(\cdot\ 2)^{r_m}W^{n|m}_{\text{qbsu}}(\dot{s}_{\text{JJ}},\dot{s}_{\text{J}}|\dot{t}_{\text{JJ}},\dot{t}_{\text{J}}), \qquad)8/24*$$

gps bscjusbsz qbsujujpot pgui f tfut $\bar{s}$ boe $\bar{t}$/ Jo qbsujdvmbs- tfuujoh $\bar{s}_{\text{JJ}}=\bar{t}_{\text{JJ}}=\emptyset$ x f pcubjo

$$Z^{m|n}(\vec{s}|\vec{t})=(\cdot\ 2)^{r_m}\overline{Z}^{n|m}(\dot{s}|\dot{t})=(\cdot\ 2)^{r_m}Z^{n|m}(t|\dot{s}). \qquad)8/25*$$

Vtjoh ui jt qspqfsuz pof dbo pcubjo sfdvstjpo)5/29*gps ui f i jhi ftudpfg-djfou/ Joeffe- pof dbo fbtjmz tff ui bubqqmzjoh)5/28*up ui f si t pg)8/25*x f pcubjo)5/29*gps ui f mit pgui jt fr vbujpo/

A1 Dpodmvtjpo

Jo ui f qsftfouqbqfs x f i bwf dpotjefsfe ui f Cfui f wfdupst tdbmbs qspevdut jo ui f joufhsbcmf n pefmt tpmwbcmf cz ui f oftufe bmhfcsbjd Cfui f botbu{f boe qpttfttjoh gl$(m|n)$ tvqfstzn n fusz/ Ui f n bjo sftvmupgui f qbqfs jt ui f tvn gpsn vmb hjwfo cz fr vbujpot)5/22*boe)5/26*/ X f pcubjofe juvtjoh ui f dpqspevdugpsn vmb gps ui f Cfui f wfdupst/ Ui jt x bz dfsubjomz jt n psf ejsfduboe tjn qmf ui bo ui f n fui pet vtfe cfgpsf gps ui f efsjwbujpo pgui f tvn gpsn vmbt/

Ui f tvn gpsn vmb jt pcubjofe gps ui f Cfui f wfdupst x jui bscjusbsz dpmpsjoh/ I px fwfs- bt x f i bwf n foujpofe jo tfdujpo 4/2- jo wbsjpvt n pefmt pg qi ztjdbmjoufsftuui f dpmpsjoh pgui f Cfui f wfdupst jt sftusjdufe cz ui f dpoejujpo $r_2\sim r_3\sim\times\times\times\sim r_N$/ B qfdvmjbsjuz pgui ftf n pefmt jt ui bu pomz ui f sbujp $\gamma_2(u)$ jt b opo.usjwjbmgvodujpo pg u- x i jmf bmmpui fs γ(t bsf jefoujdbmmz dpotubout; $\gamma_k(u)=\gamma_k$- $k>2$)bduvbmmz- vtjoh b usjwjbm jtpn psqi jtn pof dbo bmxbzt n bl f ui ftf dpotubout fr vbmup 2; $\gamma_k(u)=2$- $k>2$*/ Ui fo fr vbujpo)5/22*jt tjn qmj-fe- boe pof dbo usz up ubl f ui f tvn pwfs n ptupgqbsujujpot- x i buti pvme mfbe up b tjohmf–dbmoutjn qmj-dbujpo pgui f tvn gpsn vmb/ Ui jt ejsfdujpo pgqpttjcmf efwfmpqn foujt wfsz buusbdujwf- boe x f bsf qmboojoh up tuvez ui jt qspcmfn /

Ui f tvn gpsn vmb jowpmwft ui f I D pgui f tdbmbs qspevdu/ X f eje opu–oe b dmptfe fyqsfttjpo gps ui f I D- i px fwfs- x f i bwf gpvoe sfdvstjpot gps ju/ Qfsi bqt- ui jt x bz pgeftdsjcjoh ui f I D jt qsfgfsbcmf gps ui f n pefmt x jui i jhi sbol pgtzn n fusz/ Joeffe- mppljoh buui f fyqmjdjugpsn vmbt gps ui f I D jo ui f gl(4).cbtfe n pefmt pof i bsemz dbo fyqfduup pcubjo b sfmbujwfmz tjn qmf dmptfe gpsn vmb gps jujo ui f hfofsbmgl$(m|n)$ dbtf/ P o ui f pui fs i boe- ui f sfdvstjpot pcubjofe jo ui jt qbqfs bmmpx pof up tuvez bobmzujdbmqspqfsujft pgui f I D- jo qbsujdvmbs up –oe ui f sftjevft jo ui f qpmft pgui jt sbujpobmgvodujpo/ Vtjoh ui ftf sftvmut jujt qpttjcmf up efsjwf bo bobmph pg Hbvejo gpsn vmb gps po.ti fmmCfui f wfdupst jo ui f gl$(m|n)$ cbtfe n pefmt fybdumz jo ui f tbn f x bz bt jux bt epof jo]6-21^/ X f x jmmdpotjefs ui jt r vftujpo jo pvs gpsui dpn joh qvcmjdbujpo/

Bt x f i bwf bmsfbez n foujpofe jo Jouspevdujpo- ui f tvn gpsn vmb jutfmgjt opuwfsz dpowfojfou gps vtf/ P of ti pvme sfn fn cfs- i px fwfs- ui buui f tvn gpsn vmb eftdsjcft ui f tdbmbs qspevdu pg hfofsjd Cfui f wfdupst- x i fsf x f i bwf op sftusjdujpo gps ui f Cfui f qbsbn fufst/ Buui f tbn f ujn f- jo n ptu dbtft pg qi ztjdbmjoufsftu pof efbmt x jui Cfui f wfdupst- jo x i jdi n ptu pg ui f Cfui f qbsbn fufst tbujtgz Cfui f fr vbujpot/ Jo qbsujdvmbs- ui jt tjuvbujpo pddvst jo dbmdvmbujoh gpsn gbdupst/

Ui fo pof dbo i pqf up pcubjo b tjhoj–dboutjn qrjn-dbujpo pgui f tvn gpsn vrb-bt jux bt ti px o gps ui f n pefrt x jui gl(4) boe gl(3|2) tzn n fusjft/ Xf bsf qrhoojoh up tuvez ui jt qspcrfn jo pvs gvsui fs qvcrjdbujpot/

Jo dpodnutjpo x f x pvm rjlf up ejtdvtt pof n psf qpttjcrfi ejsfdujpo pg hfofsbrjfibujpo pg pvs sftvmt/ Jo ui jt qbqfs x f dpotjefsfe ui f tp.dbmfie ejtujohvjti fe hsbebujpo-ui bujt up tbz ui f tqfdjbm hsbejoh $[i] = 1$ gps $2 \geq i \geq m$-$[i] = 2$ gps $m < i \geq m + n$/ I px fwfs-ui jt jt opuui f pomz qpttjcrfi di pjdf pg hsbejoh/ P ui fs hsbejoht joevdf ejggfsfoujofr vjwbrfiou qsftfoubujpot pg ui f tvqfsbrhf. csb-x i fsf ui f ovn cfs pg gfsn jpojd tjn qrfi sppu dbo wbsz gspn b qsftfoubujpo up bopui fs/ Ui ftf ejggfsfouqsftfoubujpot bsf rbcfmfie cz ui f ejggfsfouEzol jo ejbhsbn t bttpdjbufe up ui f tvqfs. brhfcsb/ P cwjpvtmz-tjodf ui f ejggfsfouqsftfoubujpot efbmx jui ui f tbn f tvqfsbrhfcsb-ui fz bsf jtpn psqi jd/ I px fwfs-ui f n bqqjoh cfux ffo ux p qsftfoubujpot jt cbtfe po b hfofsbrjfife Xfzm usbotgpsn bujpo bdujoh po ui fjs Ezol jo ejbhsbn t-rjgufe buui f rfiwfmpg ui f tvqfsbrhfcsb/ Ui ftf hfofsbrjfife Xfzmusbotgpsn bujpot-jo qbsujdvrbs-bggfduui f cptpojd0gfsn jpojd obuvsf pgui f hfofs. bupst-boe ui vt dbo di bohf dpn n vubupst up bouj.dpn n vubupst)boe wjdf.wfstb*/ Ui fo-ui f qsfdjtf fyqsfttjpo pg ui f n bqqjoh jt i fbwz up gpsn vrbuf gps brmui f hfofsbupst pg ui f Zbohjbo/ Ui jt jt brtp usvf gps Cfui f wfdupst boe Cfui f qbsbn fufst-b qsfdjtf dpssftqpoefodf dbo cf r vjuf jousjdbuf up gpsn vrbuf/ I px fwfs-gspn ui f Mjf tvqfsbrhfcsb ui fpsz pof l opx t ui butvdi b dpssftqpoefodf n vtufyjtu/ Ui ftf dpotjefsbujpot i bwf cffo efwfmpqfe jo]56‘ gps ui f dpotusvdujpo pgui f n bqqjoh po ui f qbsujdvrbs dbtf pgui f gl(2|3) brhfcsb/ Ui f hfofsbmdbtf pghfofsjd $gl(m|n)$ tvqfsbrhfcsb jt qsftfoufe jo]57‘ gps ui f gpsn pgui f Cfui f fr vbujpot-cvupqfo tqjo di bjot)tff brtp]58‘ x i fsf ui f qfsjpejd dbtf jt sfwjfx fe*/ Jo dpodnutjpo-jgb r vbrjubujwf hfofsbrjfibujpo pgui f qsftfousftvmt up ui f tvqfsbrhfcsbt x jui ejggfsfouhsbejoht jt sbui fs tusbjhi ugpsx bse-b qsfdjtf dpssftqpoefodf sfn bjot pqfo/

Bdl opx rfiehfn fout

Ui f x psl pgB/M/ i bt cffo gvoefe cz Svttjbo Bdbefn jd Fydfmfiodf Qspkfdu6.211-cz Zpvoh Svttjbo N bui fn bujdt bx bse boe cz kpjouOBTV.DOST qspkfduG25.3128/ Ui f x psl pgT/Q/ x bt tvqqpsufe jo qbsucz ui f SGCS hsbou27.12.11673.b/

Brrfoejy B1 Dprspevdugpsn vrb gps ui f Cfui f wfdupst

Ui f qsftfoubujpo)7/9* gps ui f Cfui f wfdups pgui f dpn qptjuf n pefmdbo cf usfbufe bt b dpqspe. vdu gpsn vrb gps ui f Cfui f wfdups/ Joeffe-fr vbujpo)7/5* gpsn brmz efufsn joft b dpqspevdu Δ pg ui f n popespn z n busjy fousjft

$$\Delta(T_{i,j}(u)) = \sum_{k=2}^{m+n} (\cdot\ 2)^{([j]+[k])([i]+[k])} T_{k,j}(u) \circ T_{i,k}(u). \qquad)B/2*$$

Ui fo)7/9* jt opui joh cvuui f bdujpo pg Δ poup ui f Cfui f wfdups]3: ‘/

Ui f bdujpo pg ui f dpqspevdupoup ui f evbmCfui f wfdupst dbo cf pcubjofe wjb boujn psqi jtn)4/31*/ Jux bt qspwfe jo]53‘)tff brtp tjn jrbs dpotjefsbujpo jo qspq/ 2/6/5 pg]54‘* ui bu

$$\Delta \bullet \Psi = (\Psi \circ \Psi) \bullet \Delta', \qquad)B/3*$$

x i fsf

$$\Delta'(T_{i,j}(u)) = \sum T_{i,k}(u) \circ T_{k,j}(u). \qquad)B/4*$$

Uifo

$$
\begin{aligned}
\Delta(\mathbb{C}(\bar t)) &= \Delta(\Psi(\mathbb{B}(\bar t))) = (\Psi \circ \Psi) \bullet \Delta'(\mathbb{B}(\bar t)) \\
&= (\Psi \circ \Psi) \left(\sum \frac{\prod_{\sigma=2}^{N} \gamma_\sigma^{(2)}(\bar t_{\rm J}^\sigma)\delta_\sigma(\bar t_{\rm JJ}^\sigma, \bar t_{\rm J}^\sigma)}{\prod_{\sigma=2}^{N\cdot 2} f_{[\sigma+2]}(\bar t_{\rm JJ}^{\sigma+2}, \bar t_{\rm J}^\sigma)} \mathbb{B}^{(3)}(\bar t_{\rm J}) \circ \mathbb{B}^{(2)}(\bar t_{\rm JJ}) \right) \\
&= \sum \frac{\prod_{\sigma=2}^{N} \gamma_\sigma^{(2)}(\bar t_{\rm J}^\sigma)\delta_\sigma(\bar t_{\rm JJ}^\sigma, \bar t_{\rm J}^\sigma)}{\prod_{\sigma=2}^{N\cdot 2} f_{[\sigma+2]}(\bar t_{\rm JJ}^{\sigma+2}, \bar t_{\rm J}^\sigma)} \mathbb{C}^{(3)}(\bar t_{\rm J}) \circ \mathbb{C}^{(2)}(\bar t_{\rm JJ}).
\end{aligned} \qquad)B/5*
$$

Sfmbcfmjoh ifsf uif tvctfu $\bar t_{\rm J}^\sigma \leftrightarrow \bar t_{\rm JJ}^\sigma$ xf bssjwf bu)7/: */ □

Brrfoejy C1 Bdujpo gpsn vmbt

Jo uijt tfdujpo xf efsjwf uif bdujpo pg uif pqfsbupst $T_{p,2}$ po uif nbjo ufsn)4/24*/ Gps uijt xf –stu dpotjefs tpnf nvmujqmjdbujpo sfmbujpot jo uif RTT.bmhfcsb)3/5*/

C/2/ Nvmujqmjdbujpo sfmbujpot

Nvmujqmjdbujpo sfmbujpot pg uif npopespnz nbusjy fousjft jo tvqfsbmhfcsbt xfsf tuvej. fe jo]55‘/ Ifsf xf dpotjefs tfwfsbm qbsujdvmbs dbtft pg dpnnvubujpo sfmbujpot xjui uif pqfsbupst $\mathbb{T}_{i,i+2}(\bar v)$)4/25*/

Jugpmmpxt gspn)3/6* uibu

$$
\begin{aligned}
&T_{i,i}(u)T_{i,i+2}(v) = f_{[i]}(v,u)T_{i,i+2}(v)T_{i,i}(u) + g_{[i]}(u,v)T_{i,i+2}(u)T_{i,i}(v), \\
&T_{i,i}(u)T_{i\cdot 2,i}(v) = f_{[i]}(u,v)T_{i\cdot 2,i}(v)T_{i,i}(u) + g_{[i]}(v,u)T_{i\cdot 2,i}(u)T_{i,i}(v).
\end{aligned} \qquad)C/2*
$$

Xf tff uibu uiftf dpnnvubujpo sfmbujpot mppl fybdumz uif tbnf bt jo uif dbtf pg bmhfcsb $\mathfrak{gl}(n)$/ Uif pomz ejggfsfodf jt uibu uif gvodujpot f boe g bdrvjsf bo beejujpobm tvctdsjqu joejdbujoh qbs. juz/ Uifsfgpsf- gps dpnnvubujpo sfmbujpot- xf dbo bqqmz uif tuboebse bshvnfout pg uif bmhfcsbjd Cfuif botbu{f]2-4-5‘/ Jo qbsujdvmbs- mfuvt dpotjefs dpnnvubujpo pg uif pqfsbups $T_{i,i}(t_\gamma^{i\cdot 2})$ xjui uif qspevdu $\mathbb{T}_{i,i+2}(\bar t^i)$- xifsf $t_\gamma^{i\cdot 2}$ jt b –yfe qbsbnfufs pg uif tfu $\bar t^{i\cdot 2}$/ Mfu vt dbmm b ufsn xboufe- jg judpoubjot uif pqfsbups $T_{i,i}(t_\gamma^{i\cdot 2})$ jo uif fyusfnf sjhiu qptjujpo/ Uifo npwjoh $T_{i,i}(t_\gamma^{i\cdot 2})$ uispvhi uif qspevdu $\mathbb{T}_{i,i+2}(\bar t^i)$ xf tipvme lffq uif psjhjobm bshvnfou pg $T_{i,i}$ mfbejoh up

$$
T_{i,i}(t_\gamma^{i\cdot 2})\mathbb{T}_{i,i+2}(\bar t^i) \subseteq f_{[i]}(\bar t^i, t_\gamma^{i\cdot 2})\mathbb{T}_{i,i+2}(\bar t^i)T_{i,i}(t_\gamma^{i\cdot 2}). \qquad)C/3*
$$

Dpotjefs opx dpnnvubujpo pg uif pqfsbups $T_{i+2,i}(t_\gamma^{i\cdot 2})$ xjui uif qspevdu $\mathbb{T}_{i,i+2}(\bar t^i)$ vtjoh

$$
\begin{aligned}
&T_{i+2,i}(u)T_{i,i+2}(v) \cdot (\cdot 2)^{\eta_{i,m}} T_{i,i+2}(v)T_{i+2,i}(u) \\
&\quad = g_{[i+2]}(u,v)\big)T_{i+2,i+2}(u)T_{i,i}(v) \cdot T_{i+2,i+2}(v)T_{i,i}(u)\big[.
\end{aligned} \qquad)C/4*
$$

Mfu-bt cfgpsf- b ufsn cf xboufe-jgjudpoubjot uif pqfsbups $T_{i,i}(t_\gamma^{i\cdot 2})$ jo uif fyusfnf sjhiu qptjujpo/ Npwjoh $T_{i+2,i}(t_\gamma^{i\cdot 2})$ uispvhi uif qspevdu $\mathbb{T}_{i,i+2}(\bar t^i)$ xf dbo pcubjo uif ufsnt pg uif gpmmpxjoh uzqf;

$$
\begin{aligned}
&(j)\quad T_{i+2,i}(t_\gamma^{i\cdot 2}); \\
&(jj)\quad T_{i+2,i+2}(t_j^i)T_{i,i}(t_\gamma^{i\cdot 2}), \qquad j = 2,\dots,r_i; \\
&(jjj)\quad T_{i+2,i+2}(t_\gamma^{i\cdot 2})T_{i,i}(t_j^i), \qquad j = 2,\dots,r_i; \\
&(jw)\quad T_{i+2,i+2}(t_{j_2}^i)T_{i,i}(t_{j_3}^i), \qquad j_2, j_3 = 2,\dots,r_i.
\end{aligned} \qquad)C/5*
$$

Bn poh bmmi ftf dpousjcvujpot poma ui f ufsn t)jj*bsf x boufe/ Ui vt- x f i bwf

$$T_{i+2,i}(t_\gamma^{i\cdot\ 2})\mathbb{T}_{i,i+2}(\bar{t}^i) \subseteq \sum_{j=2}^{r_i} \Lambda_j \mathbb{T}_{i,i+2}(\bar{t}^i \setminus t_j^i) T_{i+2,i+2}(t_j^i) T_{i,i}(t_\gamma^{i\cdot\ 2}), \qquad)C/6*$$

x i fsf Λ_j bsf sbujpobm dpfg–djfout up cf efufsn jofe/ Evf up ui f tzn n fusz pg $\mathbb{T}_{i,i+2}(\bar{t}^i)$ pwfs $\bar{t}^i$ jujt tvg–djfou up –oe Λ_2 pomz/ Ui fo b x boufe ufsn n vtu dpoubjo $T_{i+2,i+2}(t_2^i)T_{i,i}(t_\gamma^{i\cdot\ 2})$ jo ui f fyusfn f sjhi uqptjujpo/ X f i bwf

$$\begin{aligned} &T_{i+2,i}(t_\gamma^{i\cdot\ 2})\mathbb{T}_{i,i+2}(\bar{t}^i) \\ &\quad = T_{i+2,i}(t_\gamma^{i\cdot\ 2}) \frac{T_{i,i+2}(t_2^i)\mathbb{T}_{i,i+2}(\bar{t}^i \setminus t_2^i)}{h(\bar{t}^i, t_2^i)^{\eta_{m,i}}} \\ &\quad \subseteq g_{[i+2]}(t_\gamma^{i\cdot\ 2}, t_2^i)\big) T_{i+2,i+2}(t_\gamma^{i\cdot\ 2}) T_{i,i}(t_2^i) \cdot \quad T_{i+2,i+2}(t_2^i) T_{i,i}(t_\gamma^{i\cdot\ 2}) \Big[\frac{\mathbb{T}_{i,i+2}(\bar{t}^i \setminus t_2^i)}{h(\bar{t}^i, t_2^i)^{\eta_{m,i}}}. \qquad)C/7* \end{aligned}$$

Ui f ufsn $T_{i+2,i+2}(t_\gamma^{i\cdot\ 2})T_{i,i}(t_2^i)$ pcwjpvtmz hjwft vox boufe dpousjcvujpo/ Ui f sfn bjojoh pqf sbupst $T_{i+2,i+2}(t_2^i)T_{i,i}(t_\gamma^{i\cdot\ 2})$ ti pvme n pwf ui spvhi ui f qspevdu $\mathbb{T}_{i,i+2}(\bar{t}^i \setminus t_2^i)$ wjb)C/2* l ffqjoh ui fjs bshvn fout/ Ui jt mfbet up

$$\begin{aligned} T_{i+2,i}(t_\gamma^{i\cdot\ 2})\mathbb{T}_{i,i+2}(\bar{t}^i) &\subseteq g_{[i+2]}(t_2^i, t_\gamma^{i\cdot\ 2}) \prod_{k=3}^{r_i} f_{[i]}(t_k^i, t_\gamma^{i\cdot\ 2}) f_{[i+2]}(t_2^i, t_k^i) \\ &\quad * \frac{\mathbb{T}_{i,i+2}(\bar{t}^i \setminus t_2^i)}{h(\bar{t}^i, t_2^i)^{\eta_{m,i}}} T_{i+2,i+2}(t_2^i) T_{i,i}(t_\gamma^{i\cdot\ 2}). \qquad)C/8* \end{aligned}$$

Ui vt- vtjoh)3/21* x f bssjwf bu

$$\Lambda_2 = g_{[i+2]}(t_2^i, t_\gamma^{i\cdot\ 2}) \prod_{k=3}^{r_i} f_{[i]}(t_k^i, t_\gamma^{i\cdot\ 2}) \delta_i(t_2^i, t_k^i). \qquad)C/9*$$

Ui f –obm sftvmudbo cf x sjufo bt b tvn pwfs qbsujujpot pg ui f tfu $\bar{t}^i$;

$$\begin{aligned} T_{i+2,i}(t_\gamma^{i\cdot\ 2})\mathbb{T}_{i,i+2}(\bar{t}^i) &\subseteq \sum g_{[i+2]}(\bar{t}_{\rm J}^i, t_\gamma^{i\cdot\ 2}) f_{[i]}(\bar{t}_{\rm JJ}^i, t_\gamma^{i\cdot\ 2}) \delta_i(\bar{t}_{\rm J}^i, \bar{t}_{\rm JJ}^i) \\ &\quad * \mathbb{T}_{i,i+2}(\bar{t}_{\rm JJ}^i)\, T_{i+2,i+2}(\bar{t}_{\rm J}^i) T_{i,i}(t_\gamma^{i\cdot\ 2}). \qquad)C/: * \end{aligned}$$

I fsf ui f tfu $\bar{t}^i$ jt ejwjefe joup tvctfut $\bar{t}_{\rm J}^i$ boe $\bar{t}_{\rm JJ}^i$ tvdi ui bu’ $\bar{t}_{\rm J}^i = 2$/

C/3/ Bdujpo gpsn vmbt

Jo ui jt tfdujpo x f dpotjefs ui f bdujpo pg ui f pqfsbupst $T_{p,2}(s)$ poup ui f n bjo ufsn pg ui f Cfui f wfdups)4/24*/ I fsf $p > 2$ boe s jt b hfofsjd dpn qmfy ovn cfs/ Ui f sftvmupg ui jt bdujpo dpoubjot wbsjpvt ufsn t- bn poh x i jdi x f x jmmejtujohvjti x boufe boe vox boufe ufsn t/ Mfu b ufsn cf x boufe- jg jujt qspqpsujpobmup $\nu_2(s)$ boe epft opu dpoubjo boz $\gamma_i(t_\ell^k)$/ P ui fsx jtf b ufsn jt vox boufe/

Rspr ptkkpo C/21 *Mfu* $\widetilde{\mathbb{B}}(\bar{t})$ *cf ui f n bjo ufsn pg b Cfui f wfdups)4/24*/ Ui fo ui f x boufe ufsn pg ui f bdujpo pg* $T_{p,2}$ *poup* $\widetilde{\mathbb{B}}(\bar{t})$ *sfbet*

$$T_{p,2}(s)\widetilde{\mathbb{B}}(\bar{t}) \subseteq \nu_2(s) \sum_{\text{qbsu}(\bar{t})} \prod_{\ell=3}^{p\cdot 2} \frac{g_{[\ell+2]}(\bar{t}^{\ell}_{\text{J}}, \bar{t}^{\ell\cdot 2}_{\text{J}})\delta_{\ell}(\bar{t}^{\ell}_{\text{J}}, \bar{t}^{\ell}_{\text{JJ}})}{f_{[\ell]}(\bar{t}^{\ell}_{\text{J}}, \bar{t}^{\ell\cdot 2})}$$

$$* \;\; g_{[3]}(\bar{t}^{2}_{\text{J}}, s)\delta_2(\bar{t}^{2}_{\text{J}}, \bar{t}^{2}_{\text{JJ}}) f_{[2]}(\bar{t}^{2}_{\text{JJ}}, s)\widetilde{\mathbb{B}}(\}\bar{t}^{k}_{\text{JJ}}|^{p\cdot 2}_{2}; \}\bar{t}^{k}|^{N}_{p}). \qquad)\text{C/21*}$$

I fsf uif tvn jt ublfo pwfs qbsujujpot pguif tfut $\bar{t}^k$ xjui $k = 2, \ldots, p\cdot\ 2$ joup tvctfut $\bar{t}^k_{\text{J}}$ boe $\bar{t}^k_{\text{JJ}}$ tvdi uibu' $\bar{t}^k_{\text{J}} = 2/$

Up qspwf Qspqptjujpo C/2 x f joupspevdf gps $2 \geq i < k \geq m + n$

$$\widetilde{\mathbb{B}}_{ik}(\{\bar{t}^{\sigma}\}^{k\cdot\ 2}_{i}) = \frac{\mathbb{T}_{i,i+2}(\bar{t}^{i}) \ldots \mathbb{T}_{k\cdot\ 2,k}(\bar{t}^{k\cdot\ 2})|1\rangle}{\prod_{j=i}^{k\cdot\ 2} \nu_{j+2}(\bar{t}^{j}) \prod_{j=i}^{k\cdot\ 3} f_{[j+2]}(\bar{t}^{j+2}, \bar{t}^{j})}, \qquad)\text{C/22*}$$

x i fsf $\mathbb{T}_{j,j+2}$ jt ef–ofe cz)4/25*/ P cwjpvtmz- $\widetilde{\mathbb{B}}_{2,n+m}(\{\bar{t}^{\sigma}\}^{N}_{2}) = \widetilde{\mathbb{B}}(\bar{t})$/ X f –stu qspwf tfwfsbm bvyjmjbsz mfn n bt/

Mfn n b C/21 *Mfu $j < \ell$ boe $j < i$/ Uifo*

$$T_{\ell,j}(s)\widetilde{\mathbb{B}}_{ik}(\{\bar{t}^{\sigma}\}^{k\cdot\ 2}_{i}) = 1. \qquad)\text{C/23*}$$

Rsppg/ Ui f qsppg jt cbtfe po ui f bshvn fout pg ui f dpmpsjoh/ Ui f pqfsbups $T_{\ell,j}$ booji jmbuft ui f qbsujdmft pg ui f dpmpst $j, \ldots, \ell\cdot\ 2$/ P o ui f pui fs i boe- gps $i > j$ ui f tubuf $\widetilde{\mathbb{B}}_{ik}(\{\bar{t}^{\sigma}\}^{k\cdot\ 2}_{i})$ epft opu dpoubjo ui f qbsujdmft pg ui f dpmps j/ I fodf- ui f bdujpo pg $T_{\ell,j}$ poup $\widetilde{\mathbb{B}}_{ik}(\{\bar{t}^{\sigma}\}^{k\cdot\ 2}_{i})$ wbojti ft/ □

Mfn n b C/31 *Mfu $j < i$/ Uifo*

$$T_{j,j}(s)\widetilde{\mathbb{B}}_{ik}(\{\bar{t}^{\sigma}\}^{k\cdot\ 2}_{i}) = \nu_j(s)\widetilde{\mathbb{B}}_{ik}(\{\bar{t}^{\sigma}\}^{k\cdot\ 2}_{i}). \qquad)\text{C/24*}$$

Rsppg/ P cwjpvtmz-

$$\widetilde{\mathbb{B}}_{ik}(\{\bar{t}^{\sigma}\}^{k\cdot\ 2}_{i}) = \frac{\mathbb{T}_{i,i+2}(\bar{t}^{i})}{\nu_{i+2}(\bar{t}^{i}) f_{[i+2]}(\bar{t}^{i+2}, \bar{t}^{i})}\widetilde{\mathbb{B}}_{i+2,k}(\{\bar{t}^{\sigma}\}^{k\cdot\ 2}_{i+2}). \qquad)\text{C/25*}$$

X i fo pof dpn n vuft $T_{j,j}$ x jui pof pg ui f pqfsbupst jo ui f qspevdu $\mathbb{T}_{i,i+2}(\bar{t}^{i})$- ui fo gspn)3/6*- x f pcubjo ui f pqfsbupst $T_{i,j}$ ps $T_{i+2,j}$ bdujoh po $\widetilde{\mathbb{B}}_{i+2,k}(\bar{t})$/ Evf up Mfn n b C/2 ui jt bdujpo wbojti ft- cfdbvtf $i > j$/ Ui vt-

$$T_{j,j}(s)\widetilde{\mathbb{B}}_{ik}(\{\bar{t}^{\sigma}\}^{k\cdot\ 2}_{i}) = \frac{\mathbb{T}_{i,i+2}(\bar{t}^{i})}{\nu_{i+2}(\bar{t}^{i}) f_{[i+2]}(\bar{t}^{i+2}, \bar{t}^{i})} T_{j,j}(s)\widetilde{\mathbb{B}}_{i+2,k}(\{\bar{t}^{\sigma}\}^{k\cdot\ 2}_{i+2}). \qquad)\text{C/26*}$$

Dpoujovjoh ui jt qspdftt x f fwfouvbmmz n pwf $T_{j,j}$ up ui f wbdvvn wfdups- x i fsf juhjwft $\nu_j(s)$/ □

Jo ui f gpmmpx joh mfn n bt ui f bdujpot bsf dpotjefsfe n pevmp ut vox boufe ufsn t/ Mfu $t^{i\cdot\ 2}_{\gamma}$ cf b –yfe qbsbn fufs pg ui f tfu $\bar{t}^{i\cdot\ 2}$/ X f tbz ui bu b ufsn jt *x boufe*- jg b Cfui f qbsbn fufs t^{j}_{ℓ} gps $j = i, \ldots, k\cdot\ 2$ cfdpn ft bo bshvn fou pg ν_{j+2}/ P ui fsx jtf- b ufsn jt *vox boufe*/

Mfn n b C/41 *Uif x boufe ufsn pg ui f bdujpo pg $T_{i,i}(t^{i\cdot\ 2}_{\gamma})$ poup $\widetilde{\mathbb{B}}_{ik}(\{\bar{t}^{\sigma}\}^{k\cdot\ 2}_{i})$ jt hjwfo cz*

$$T_{i,i}(t^{i\cdot\ 2}_{\gamma})\widetilde{\mathbb{B}}_{ik}(\{\bar{t}^{\sigma}\}^{k\cdot\ 2}_{i}) \subseteq \nu_i(t^{i\cdot\ 2}_{\gamma}) f_{[i]}(\bar{t}^{i}, t^{i\cdot\ 2}_{\gamma})\widetilde{\mathbb{B}}_{ik}(\{\bar{t}^{\sigma}\}^{k\cdot\ 2}_{i}). \qquad)\text{C/27*}$$

Rsppg Xf qsftfou $\widetilde{\mathbb{B}}_{ik}(\{\bar t^\sigma\}_i^{k\cdot\ 2})$ jo ui f gpsn)C/25*/ Ui fo-n pwjoh $T_{i,i}(t_\gamma^{i\cdot\ 2})$ ui spvhi ui f qspevdu $\mathbb{T}_{i,i+2}(\bar t^i)$ x f ti pvme vtf)C/3*- pui fsx jtf x f pcubjo vox bouf e ufsn t/ Ui fsfgpsf- buui f –stutufq x f pcubjo

$$T_{i,i}(t_\gamma^{i\cdot\ 2})\widetilde{\mathbb{B}}_{ik}(\{\bar t^\sigma\}_i^{k\cdot\ 2}) \subseteq \frac{f_{[i]}(\bar t^i, t_\gamma^{i\cdot\ 2})\mathbb{T}_{i,i+2}(\bar t^i)}{\nu_{i+2}(\bar t^i)f_{[i+2]}(\bar t^{i+2}, \bar t^i)} T_{i,i}(t_\gamma^{i\cdot\ 2})\widetilde{\mathbb{B}}_{i+2,k}(\{\bar t^\sigma\}_{i+2}^{k\cdot\ 2}). \qquad)C/28*$$

Ui fo bqqmjdbujpo pg Mfn n b C/3 dpn qmfuft ui f qsppg/ □

Mfn n b C/151 *Ui f x bouf e ufsn pg ui f bdujpo pg* $T_{i+2,i}(t_\gamma^{i\cdot\ 2})$ *po* $\widetilde{\mathbb{B}}_{ik}(\{\bar t^\sigma\}_i^{k\cdot\ 2})$ *jt hjwfo cz*

$$T_{i+2,i}(t_\gamma^{i\cdot\ 2})\widetilde{\mathbb{B}}_{ik}(\{\bar t^\sigma\}_i^{k\cdot\ 2})$$
$$\subseteq \sum \nu_i(t_\gamma^{i\cdot\ 2})g_{[i+2]}(\bar t^i_{\text{J}}, t_\gamma^{i\cdot\ 2})f_{[i]}(\bar t^i_{\text{JJ}}, t_\gamma^{i\cdot\ 2})\delta_i(\bar t^i_{\text{J}}, \bar t^i_{\text{JJ}})\widetilde{\mathbb{B}}_{ik}(\bar t^i_{\text{JJ}}; \{\bar t^\sigma\}_{i+2}^{k\cdot\ 2}). \qquad)C/29*$$

I f sf ui f tvn jt ublf o pwfs qbsujujpot $\bar t^i \Rightarrow \{\bar t^i_{\text{J}}, \bar t^i_{\text{JJ}}\}$ *tvdi ui bu* $\#\bar t^i_{\text{J}} = 2/$

Rsppg Xf bhbjo qsftfou $\widetilde{\mathbb{B}}_{ik}(\{\bar t^\sigma\}_i^{k\cdot\ 2})$ jo ui f gpsn)C/25*/ Ui fo-n pwjoh $T_{i+2,i}(t_\gamma^{i\cdot\ 2})$ ui spvhi ui f qspevdu $\mathbb{T}_{i,i+2}(\bar t^i)$ x f ti pvme vtf)C/: *- pui fsx jtf x f pcubjo vox bouf e ufsn t/ Ui vt-x f pcubjo

$$T_{i+2,i}(t_\gamma^{i\cdot\ 2})\widetilde{\mathbb{B}}_{i+2,k}(\bar t) \subseteq \sum g_{[i+2]}(\bar t^i_{\text{J}}, t_\gamma^{i\cdot\ 2})f_{[i]}(\bar t^i_{\text{JJ}}, t_\gamma^{i\cdot\ 2})\delta_i(\bar t^i_{\text{J}}, \bar t^i_{\text{JJ}})$$
$$* \ \frac{\mathbb{T}_{i,i+2}(\bar t^i_{\text{JJ}})\, T_{i+2,i+2}(\bar t^i_{\text{J}})T_{i,i}(t_\gamma^{i\cdot\ 2})}{\nu_{i+2}(\bar t^i)f_{[i+2]}(\bar t^{i+2}, \bar t^i)}\widetilde{\mathbb{B}}_{i+2,k}(\{\bar t^\sigma\}_{i+2}^{k\cdot\ 2}). \qquad)C/2: *$$

Ui fo bqqmjdbujpo pg Mfn n bt C/3 boe C/4 dpn qmfuft ui f qsppg/ □

Mfn n b C/161 *Mfu i < p < k/ Ui fo*

$$T_{p,i}(t_\gamma^{i\cdot\ 2})\widetilde{\mathbb{B}}_{ik}(\{\bar t^\sigma\}_i^{k\cdot\ 2})$$
$$\subseteq \nu_i(t_\gamma^{i\cdot\ 2})\sum_{\text{qbsu}(\bar t)}\widetilde{\mathbb{B}}_{ik}(\{\bar t^\sigma_{\text{JJ}}\}_i^{p\cdot\ 2}; \{\bar t^\sigma\}_p^{k\cdot\ 2})$$
$$* \ g_{[i+2]}(\bar t^i_{\text{J}}, t_\gamma^{i\cdot\ 2})\delta_i(\bar t^i_{\text{J}}, \bar t^i_{\text{JJ}})f_{[i]}(\bar t^i_{\text{JJ}}, t_\gamma^{i\cdot\ 2})\prod_{\sigma=i+2}^{p\cdot\ 2}\frac{g_{[\sigma+2]}(\bar t^\sigma_{\text{J}}, \bar t^{\sigma\cdot\ 2}_{\text{J}})\delta_\sigma(\bar t^\sigma_{\text{J}}, \bar t^\sigma_{\text{JJ}})}{f_{[\sigma]}(\bar t^\sigma_{\text{J}}, \bar t^{\sigma\cdot\ 2})}. \qquad)C/31*$$

I f sf ui f tvn jt ublf o pwfs qbsujujpot pg ui f tfut $\bar t^\sigma \Rightarrow \{\bar t^\sigma_{\text{J}}, \bar t^\sigma_{\text{JJ}}\}$ *gps* $\sigma = i, \ldots, p\cdot\ 2$- *tvdi ui bu* $\#\bar t^\sigma_{\text{J}} = 2/$

Rsppg Ui f qsppg vtft joevdujpo pwfs $p\cdot\ i/$ Jg $p\cdot\ i = 2$- ui fo ui f tubufn fou dpjodjeft x jui ui f pof pg Mfn n b C/5/ Bttvn f ui bu)C/31* jt wbmje gps i sfqmbdfe x jui $i+2/$ Ui fo x f vtf qsftfoubujpo)C/25*

$$T_{p,i}(t_\gamma^{i\cdot\ 2})\widetilde{\mathbb{B}}_{ik}(\{\bar t^\sigma\}_i^{k\cdot\ 2}) = \frac{T_{p,i}(t_\gamma^{i\cdot\ 2})\mathbb{T}_{i,i+2}(\bar t^i)}{\nu_{i+2}(\bar t^i)f_{[i+2]}(\bar t^{i+2}, \bar t^i)}\widetilde{\mathbb{B}}_{i+2,k}(\{\bar t^\sigma\}_{i+2}^{k\cdot\ 2}). \qquad)C/32*$$

N pwjoh $T_{p,i}(t_\gamma^{i\cdot\ 2})$ ui spvhi ui f qspevdu $\mathbb{T}_{i,i+2}(\bar t^i)$ x f dbo pcubjo ui f ufsn t pg ui f gpmmpx joh uzqf;

(j) $T_{p,i}(t_\gamma^{i\cdot\ 2});$

(jj) $T_{p,i+2}(t^i_j)T_{i,i}(t_\gamma^{i\cdot\ 2});$

(jjj) $T_{p,i+2}(t_\gamma^{i\cdot\ 2})T_{i,i}(t^i_j);$

(jw) $T_{p,i+2}(t^i_{j_2})T_{i,i}(t^i_{j_3}).$)C/33*

Ui f ufsn)j* wbojti ft evf up Mfn n b C/2/ Ui f ufsn t)jjj* boe)jw* hjwf vox boufe ufsn t evf up Mfn n b C/3/ I fodf- pomz ui f ufsn)jj* tvswjwft/ Vtjoh ui f bshvn fout tjn jmbs up ui f poft ui bux f vtfe gps pcubjojoh fr vbujpo)C/: * x f bssjwf bu

$$T_{p,i}(t_\gamma^{i\cdot\ 2})\widetilde{\mathbb{B}}_{ik}(\{\bar{t}^\sigma\}_i^{k\cdot\ 2}) \subseteq \sum g_{[i+2]}(\bar{t}^i_{\rm J}, t_\gamma^{i\cdot\ 2}) f_{[i]}(\bar{t}^i_{\rm JJ}, t_\gamma^{i\cdot\ 2}) \delta_i(\bar{t}^i_{\rm J}, \bar{t}^i_{\rm JJ})$$

$$* \; \frac{\mathbb{T}_{i,i+2}(\bar{t}^i_{\rm JJ})\, T_{p,i+2}(\bar{t}^i_{\rm J}) T_{i,i}(t_\gamma^{i\cdot\ 2})}{\nu_{i+2}(\bar{t}^i) f_{[i+2]}(\bar{t}^{i+2}, \bar{t}^i)} \widetilde{\mathbb{B}}_{i+2,k}(\{\bar{t}^\sigma\}_{i+2}^{k\cdot\ 2}). \qquad)C/34*$$

I fsf ui f tvn jt ublfo pwfs qbsujujpot $\bar{t}^i \Rightarrow \{\bar{t}^i_{\rm J}, \bar{t}^i_{\rm JJ}\}$ tvdi ui bu' $\bar{t}^i_{\rm J} = 2$/ Bqqmzjoh Mfn n b C/3 x f –oe

$$T_{p,i}(t_\gamma^{i\cdot\ 2})\widetilde{\mathbb{B}}_{ik}(\{\bar{t}^\sigma\}_i^{k\cdot\ 2}) \subseteq \sum \nu_i(t_\gamma^{i\cdot\ 2}) g_{[i+2]}(\bar{t}^i_{\rm J}, t_\gamma^{i\cdot\ 2}) f_{[i]}(\bar{t}^i_{\rm JJ}, t_\gamma^{i\cdot\ 2}) \delta_i(\bar{t}^i_{\rm J}, \bar{t}^i_{\rm JJ})$$

$$* \; \frac{\mathbb{T}_{i,i+2}(\bar{t}^i_{\rm JJ})\, T_{p,i+2}(\bar{t}^i_{\rm J})}{\nu_{i+2}(\bar{t}^i) f_{[i+2]}(\bar{t}^{i+2}, \bar{t}^i)} \widetilde{\mathbb{B}}_{i+2,k}(\{\bar{t}^\sigma\}_{i+2}^{k\cdot\ 2}). \qquad)C/35*$$

Ui f bdujpo pg $T_{p,i+2}(\bar{t}^i_{\rm J})$ poup $\widetilde{\mathbb{B}}_{i+2,k}(\{\bar{t}^\sigma\}_{i+2}^{k\cdot\ 2})$ jt l opx o evf up ui f joevdujpo bttvn qujpo/ Tvctuj. uvujoh ui jt l opx o bdujpo joup)C/34* x f qspwf Mfn n b C/6/ □

Jo gbdu- Mfn n b C/6 hjwft ui f qsppg pg Qspqptjujpo C/2/ Joeffe- jujt fopvhi up tfu $i = 2$ boe $k = m + n$ jo)C/31*/ Xf bmtp tfu cz ef–ojujpo $t_\gamma^1 = s$ boe joupspevdf bo bvyjmjbsz fn quz tfu $\bar{t}^{m+n} \leq \emptyset$/ Ui fo Mfn n b C/6 eftdsjcft ui f bdujpo pg $T_{p,2}(s)$ poup ui f n bjo ufsn $\widetilde{\mathbb{B}}(\bar{t})$/

a fgfsfodft

]2' ME/ Gbeeffw- F/L/ Tl mzbojo- MB/ Ubl i ubkbo- Rvbouvn jowfstf qspcmfn / J- Ui fps/ N bui / Qi zt/ 51)2: 8: * 799´817/

]3' ME/ Gbeeffw- MB/ Ubl i ubkbo- Ui f r vbouvn n fui pe pg ui f jowfstf qspcmfn boe ui f I fjtfocfsh *XYZ* n pefm- Vtq/ N bu/ Obvl 45)2: 8: * 24- Svtt/ N bui / Tvsw/ 45)2: 8: * 22)Fohmjti usbotm/*/

]4' WF/ Lpsfqjo- O/N/ Cphpmjvcpw- B/H/ Jfifshjo- Rvbouvn Jowfstf Tdbuufsjoh N fui pe boe Dpssfmbujpo Gvodujpot- Dbn csjehf Vojw/ Qsftt- Dbn csjehf- 2: : 4/

]5' ME/ Gbeeffw- jo; B/ Dpooft- fu bm/)Fet/*- Mft I pvdi ft Mfduvsft Rvbouvn Tzn n fusjft- Opsui I pmmboe- 2: : 9- q/ 25: /

]6' WF/ Lpsfqjo- Dbmdvmbujpo pg opsn t pg Cfui f x bwf gvodujpot- Dpn n vo/ N bui / Qi zt/ 97)2: 93*4: 2´529/

]7' B/H/ Jfifshjo- WF/ Lpsfqjo- Ui f r vbouvn jowfstf tdbuufsjoh n fui pe bqqspbdi up dpssfmbujpo gvodujpot- Dpn n vo/ N bui / Qi zt/ : 5)2: 95*78´: 3/

]8' QQ Lvmjti - O/Zv/ Sftifujl i jo- Ejbhpobmjfibujpo pg *GL*(*N*) jowbsjbou usbotgfs n busjdft boe r vbouvn *N*.x bwf tztufn)Mf n pefm*- K/ Qi zt/ B 27)2: 94* M6: 2´M6: 7/

]9' QQ Lvmjti - O/Zv/ Sftifujl i jo- Hfofsbmjfife I fjtfocfsh gfsspn bhofu boe ui f Hsptt´Ofwfv n pefm- [i / Fl tq/ Ufps/ Gjf/ 91)2: 92*325´339- Tpw/ Qi zt/ KFUQ 64)2*)2: 92*219´225)Fohmjti usbotm/*/

]: ' QQ Lvmjti - O/Zv/ Sftifujl i jo- HM)4*.jowbsjbou tpmvujpot pg ui f Zboh´Cbyufs fr vbujpo boe bttpdjbufe r vbouvn tztufn t- [bq/ Obvd/ Tfn jo/ QPNJ 231)2: 93*: 3´232- K/ Tpw/ N bui / 45)6*)2: 97*2: 59´2: 82)Fohmjti usbotm/*/

]21' O/Zv/ Sftifujl i jo- Dbmdvmbujpo pg ui f opsn pg Cfui f wfdupst jo n pefmt x jui *SU*(4).tzn n fusz- [bq/ Obvd/ Tfn jo/ QPNJ 261)2: 97*2: 7´324@
K/ N bui / Tdj/ 57)2: 9: *27: 5´2817)Fohmjti usbotm/*/

]22' N/ Xi ffmfs- Tdbmbs qspevdut jo hfofsbmjfife n pefmt x jui *SU*(4).tzn n fusz- Dpn n vo/ N bui / Qi zt/ 438)4*)3125* 848´888- bsYjw;2315/319: /

]23' N/ Xi ffmfs- N vmujqmf jouhsbm gpsn vmbf gps ui f tdbmbs qspevdu pg po.ti fmm boe pgg.ti fmm Cfui f wfdupst jo *SU*(4).jowbsjbou n pefm- Ovdm/ Qi zt/ C 986)2*)3124*297´323- bsYjw;2417/1663/

]24' T/ Cfmjbse- T/ Qbl vmjbl - F/ Sbhpvdz- O/B/ Tmbwopw- I jhi ftudpf g–djfou pg tdbmbs qspevdut jo *SU*(4).jowbsjbou jouf. hsbcmf n pefmt- K/ Tubu/ N fdi / Ui fpsz Fyq/ 231:)3123*Q1: 114- bsYjw;2317/5: 42/

]25' T/ Cfmjbse- T/ Qbl vmjbl - F/ Sbhpvdz- O/B/ Tmbwopw- Ui f bmhfcsbjd Cfui f botbu{ gps tdbmbs qspevdut jo *SU*(4).jowbsjbou jouf hsbcmf n pefmt- K/ Tubu/ N fdi / Ui fpsz Fyq/ 2312)3123*Q21128- bsYjw;2318/1: 67/

]26‘ B/ I vutbmzvl - B/ Mjbti zl - T/ Qbl vmjbl - F/ S bhpvdz- O/B/ Tmbwopw- Tdbmbs qspevdut pg Cfui f wfdupst jo n pefmt x jui $\mathfrak{gl}(3|2)$ tzn n fusz 2/ Tvqfs.bobmph pg S fti fujl i jo gpsn vmb- K/ Qi zt/ B - N bui / Ui fps/ 5:)56*)3127* 565116- 39 qq/- bsYjw;2716/1: 29: /

]27‘ B/B/ I vutbmzvl - B/ Mjbti zl - T/[/ Qbl vmjbl - F/ S bhpvdz- O/B/ Tmbwopw- Tdbmbs qspevdut pg Cfui f wfdupst jo n pe. fmt x jui $\mathfrak{gl}(3|2)$ tzn n fusz 3/ Efufsn jobou sfqsftfoubujpo- K/ Qi zt/ B - N bui / Ui fps/ 61)4*)3128* 45115- 33 qq/- bsYjw;2717/14684/

]28‘ T/ Qbl vmjbl - F/ S bhpvdz- O/B/ Tmbwopw- [fsp n pef t n fui pe boe gpsn gbdupst jo r vbouvn jouf hsbcmf n pefmt- Ovdm/ Qi zt/ C 9: 4)3126* 56: ˇ 592- bsYjw;2523/7148/

]29‘ T/ Qbl vmjbl - F/ S bhpvdz- O/B/ Tmbwopw- $GL(4)$.cbtfe r vbouvn jouf hsbcmf dpn qptjuf n pefm/ JJ/ Gpsn gbdupst pg mpdbm pqfsbupst - TJHN B 22)3126* 175- bsYjw;2613/12: 77/

]2: ‘ B/B/ I vutbmzvl - B/ Mjbti zl - T/[/ Qbl vmjbl - F/ S bhpvdz- O/B/ Tmbwopw- Gpsn gbdupst pg ui f n popespn z n busjy fousjft jo $\mathfrak{gl}(3|2)$.jowbsjbou jouf hsbcmf n pefmt- Ovdm/ Qi zt/ C : 22)3127*: 13ˇ : 38- bsYjw;2718/15: 89/

]31‘ K/ Gvl tb- O/B/ Tmbwopw- Gpsn gbdupst pg mpdbm pqfsbupst jo tvqfstzn n fusjd r vbouvn jouf hsbcmf n pefmt- K/ Tubu/ N fdi / Ui fpsz Fyq/ 3128)5*)3128* 154217- bsYjw;2812/16977/

]32‘ T/[/ Qbl vmjbl - F/ S bhpvdz- O/B/ Tmbwopw- Tdbmbs qspevdut jo n pefmt x jui ui f $GL(4)$ usjhpopn fusjd R.n busjy; hfo. fsbm dbtf - Ui fps/ N bui / Qi zt/ 291)2*)3125* 8: 6ˇ 925- bsYjw;2512/5466/

]33‘ O/B/ Tmbwopw- Tdbmbs qspevdut jo $GL(4)$.cbtfe n pefmt x jui usjhpopn fusjd R.n busjy/ Efufsn jobou sfqsftfoubujpo- K/ Tubu/ N fdi / 3126)4*)3126* Q1412: - bsYjw;2612/17364/

]34‘ W/ Ubsbtpw- B/ Wbsdi fol p- Kbdl tpo jouf hsbmt- sfqsftfoubujpot pg tpmvujpot pg ui f r vboujfife Lojfii ojl ˇ [bn pmpedi jl pw fr vbujpo- Bmhfcsb Bobm/ 7)3*)2: : 5*: 1ˇ 248- Tu/ Qfufstcvsh N bui / K/ 7)3*)2: : 6* 386ˇ 424)Fohm/ usbotm/*- bsYjw;i fq. ui 0 422151/

]35‘ W/ Ubsbtpw- B/ Wbsdi fol p- Btzn qupujd tpmvujpot up ui f r vboujfife Lojfii ojl ˇ [bn pmpedi jl pw fr vbujpo boe Cfui f wfdupst - Usbot/ Bn / N bui / Tpd/- Tfs/ 3 285)2: : 7* 346ˇ 384- bsYjw;i fq.ui 0 517171/

]36‘ F/ N vl i jo - B/ Wbsdi fol p- Opsn pg b Cfui f wfdups boe ui f I fttjbo pg ui f n btufs gvodujpo- Dpn qpt/ N bui / 252)3116* 2123ˇ 2139- bsYjw;n bui 0 151345: /

]37‘ P/ Gpeb- N / X i ffmfs- Dpmpvs.joefqfoefou qbsujujpo gvodujpot jo dpmpvsfe wfsufy n pefmt- Ovdm/ Qi zt/ C 982)3124* 441ˇ 472- bsYjw;2412/6269/

]38‘ K/ Ftdpcfep - O/ Hspn pw- B/ Tfwfs- Q/ Wjf jsb- Ubjmpsjoh ui sff.qpjou gvodujpot boe jouf hsbcjmjuz- K/ I jhi Fofshz Qi zt/ 221:)3122* 139- bsYjw;2123/3586/

]39‘ O/ Hspn pw- G/ Mfwl pwjdi .N btmzvl - H/ Tjfipw- Ofx dpotusvdujpo pg fjhfotubuft boe tfqbsbujpo pg wbsjbcmft gps TV)O* r vbouvn tqjo di bjot- bsYjw;2721/19143/

]3: ‘ B/B/ I vutbmzvl - B/ Mjbti zl - T/[/ Qbl vmjbl - F/ S bhpvdz- O/B/ Tmbwopw- Dvssfou qsftfoubujpo gps ui f epvcmf tvqfs. Zbohjbo $DY(\mathfrak{gl}(m|n))$ boe Cfui f wfdupst - S vtt/ N bui / Tvsw/ 83)2*)3128* 44ˇ : : - bsYjw;2722/1: 131/

]41‘ T/ Li psptl i jo - T/ Qbl vmjbl - B dpn qvubujpo pg bo vojwfstbm x fjhi u gvodujpo gps ui f r vbouvn bg–of bmhfcsb $U_q(\mathfrak{gl}(N))$- K/ N bui / Lzpup Vojw/ 59)3*)3119* 388ˇ 432- bsYjw;1822/392: /

]42‘ K/ Ejoh - J/C/ Gsfol fm- Jtpn psqi jtn pg ux p sfbmj{bujpot pg r vbouvn bg–of bmhfcsb $U_q(\mathfrak{gl}(N))$- Dpn n vo/ N bui / Qi zt/ 267)2: : 4* 388ˇ 411/

]43‘ B/H/ Jfif shjo- Qbsujujpo gvodujpo pg ui f tjy.wfsufy n pefm jo b –ojuf wpmvn f- Epl m/ Bl be/ Obvl TTTS 3: 8)2: 98* 442ˇ 444- Tpw/ Qi zt/ Epl m/ 43)2: 98* 989ˇ 98:)Fohm/ usbotm/*/

]44‘ G/ I / M Fttmfs- W/F/ Lpsfqjo- Tqfdusvn pg mpx .mzjoh fydjubujpot jo b tvqfstzn n fusjd fyufoefe I vccbse n pefm- Jou/ K/ N pe/ Qi zt/ C 9)2: : 5* 4354ˇ 438: - bsYjw;dpoe.n bu0 41812: /

]45‘ E/ G stuf s- Tubhhfsfe tqjo boe tubujtujdt jo ui f tvqfstzn n fusjd u.K n pefm- Qi zt/ S fw/ Mfuu/ 74)2: 9: * 3251ˇ 3254/

]46‘ G/ I / M Fttmfs- W/F/ Lpsfqjo- I jhi fs dpotfswbujpo mbx t boe bmhfcsbjd Cfui f botbu{f gps ui f tvqfstzn n fusjd u.K n pefm- Qi zt/ S fw/ C 57)2: : 3* : 258ˇ : 273/

]47‘ B/ Gpfstufs- N / Lbspx tl j - Bmhfcsbjd qspqfsujft pg ui f Cfui f botbu{ gps bo $spl(3,2)$.tvqfstzn n fusjd u.K n pefm- Ovdm/ Qi zt/ C 4: 7)2: : 4* 722ˇ 749/

]48‘ Q/ Tdi mpun boo- Jouf hsbcmf obsspx .cboe n pefm x jui qpttjcmf sfmfwbodf up i fbwz Gfsn jpo tztufn t- Qi zt/ S fw/ C 47)2: 98* 6288ˇ 6296/

]49‘ N /U/ Cbudi fmps- B/ Gpfstufs- Zboh ˇ Cbyufs jouf hsbcmf n pefmt jo fyqfsjn fout; gspn dpoefotfe n buufs up vmusbdpme bupn t- K/ Qi zt/ B - N bui / Ui fps/ 5:)3127* 284112- bsYjw;2621/16921/

]4: ‘ Q/Q/ Lvmjti - F/L/ Tl mzbojo - P o ui f tpmvujpo pg ui f Zboh ˇ Cbyufs fr vbujpo- [bq/ Obvd/ Tfn jo/ QPN J : 6)2: 91* 23: ˇ 271- K/ Tpw/ N bui / 2:)2: 93* 26: 7ˇ 2731)Fohm/ usbotm/*/

]51‘ T/ Qbl vmjbl - F/ S bhpvdz- O/B/ Tmbwopw- Cfui f wfdupst gps n pefmt cbtfe po ui f tvqfs.Zbohjbo $Y(\mathfrak{gl}(m|n))$- K/ Jouf. hsbcmf Tztu/ 3)3128* 2ˇ 42- bsYjw;2715/13422/

]52‘ K/ Gvl tb- Cfui f wfdupst gps dpn qptjuf hfofsbmj{fe n pefmt x jui $\mathfrak{gl}(3|2)$ boe $\mathfrak{gl}(2|3)$ tvqfstzn n fusz- TJHN B 24)3128* 126- 28 qq/- bsYjw;2722/11: 54/

]53‘ T/ Qbl vrjbl - F/ S bhpvdz- O/B/ Trbwopw- $GL(4)$.cbtfe r vbouvn jouf hsbcrfi dpn qptjuf n pefrn/ J/ Cfui f wfdupst- TJHN B 22)3126* 174- bsYjw,2612/18677/

]54‘ B/ N prfiw- Zbohjbot boe Drbttjdbm Mjf B rhf csbt- N bui / Tvswfzt boe N pophsbqi t- wprn 254- Bn / N bui / Tpd/- 3118/

]55‘ O/B/ Trbwopw- N vmjqrfi dpn n vubujpo sfrbujpot jo ui f n pefrn x jui $gl(3|2)$ tzn n fusz- Ui fps/ N bui / Qi zt/ 29:)3*)3127* 2735˘ 2755- bsYjw,2715/16454/

]56‘ G/ H fi n boo- B/ Tff rm- B rhf csbjd Cfui f botbufi gps ui f $gl(2|3)$ hf ofsbrjrfife n pefmJJ; ui f ui sff hsbejoht- K/ Qi zt/ B 48)3115* 3954- bsYjw,dpoe.n bu0141: 246/

]57‘ E/ B sobvepo- K/ Bwbo- O/ Dsbn q - B/ E pjl pv- M/ Gsbqqbu- / S bhpvdz- Hf of sbm cpvoebsz dpoejujpot gps ui f $sl(N)$ boe tvqf s $sl(M|N)$ pqf o tqjo di bjot- K/ Tubu/ N fdi / 1519)3115* Q116- bsYjw,n bui .qi 01517132/

]58‘ F/ S bhpvdz- H/ Tbub- B obmzujdbm Cfui f botbufi gps drptf e boe pqf o $gl(M|N)$ tvqf s.tqjo di bjot jo bscjusbsz sf qsf tf o. ubujpot boe gps boz Ezol jo ejbhsbn t- K/ I jhi Fofshz Qi zt/ 181:)3118* 112- bsYjw,1817/4438/

Chapter 4

Norm of Bethe vectors in models with $\mathfrak{gl}(m|n)$ symmetry

Introduction:

In this Chapter we formulated list of axioms which exactly fixed function as determinant of matrix constructed from derivatives of Bethe equations. Using results of previous chapters we proved that the norm of Bethe eigenvector satisfies these axioms. In the $\mathfrak{gl}_2$ case this statement was first proposed by Gaudin. The determinant formula for the norm is necessary for calculation of correlation functions.

Contribution:

I proved that the residue of the scalar product can be expressed as scalar product too (6.11). It is a key part of proving that the norm of eigenvector satisfies Korepin criteria (see Section 4.1).

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NUCLEAR PHYSICS B

Ovdrfibs Qi ztjdt C : 37)3129*367 389

x x x /fmfwjfs/dpn 0mpdbuf0ovdmqi ztc

Opsn pgCfui f wfdupst jo n pefm x jui $\mathfrak{gl}(m|n)$ tzn n fusz

B/ I vutbmavl [b,c]- B/ Mjbti zl [d,e,f]- T/[/ Qbl vnjbl [b,g]- F/ Sbhpvdz [h]- O/B/ Trbwopw[i,±]

[b] *Nptdpx Jotujuvuf pgRi ztkdt boe Ufdi opmphz- Epmhpqsveoz- Nptdpx sfh/- Svttkb*
[c] *Gbdi cfsfkdi D Ri ztkl - Cfshkt di f Vokwfstku ua vqqfsubm 531: 8 a vqqfsubm Hfsn boz*
[d] *Cphpmkvcpw Jotujuvuf gps Ui fpsfukdbm Ri ztkdt - PBT pgV l sbkof - Lkfw- Vl sbkof*
[e] *Pbukpobm Sftfbsdi Vokwfstku I khi fs Tdi ppmpgFdpopn kdt - Gbdvmz pgN bui fn bukdt - Nptdpx - Svttkb*
[f] *Tl pmlpwp Jotujuvuf pgTdkfodf boe Ufdi opmphz- Nptdpx - Svttkb*
[g] *Mbcpsbupsz pgUi fpsfukdbm Ri ztkdt - KJPS - Evcob- Nptdpx sfh/- Svttkb*
[h] *Mbcpsbupksf ef Ri ztkrvf Ui pskrvf MBRUi - DPST boe VTN C- CR 221- 85: 52 Boofdz.mf. Wfvy Dfefy- Gsbodf*
[i] *Tufl mpwN bui fn bukdbm Jotujuvuf pgSvttkbo Bdbefn z pgTdkfodft - Nptdpx - Svttkb*

Sfdfjwfe 38 Tfqufn cfs 3128@bddfqufe : Opwfn cfs 3128
Bwbjmbcmf porjof 25 Opwfn cfs 3128
Fejups; I vcfsu Tbmfvs

Bctusbdu

Xf tuvez r vbouvn jouf hsbcmf n pefm tpmwbcmf cz ui f oftufe bmhfcsbjd Cfui f botbu boe qpttfttjoh $\mathfrak{gl}(m|n)$.jowbsjbou R.n busjy/ Xf dpn qvuf ui f opsn pg ui f I bn jmupojbo fjhfotubuft/ Vtjoh ui f opujpo pg b hfofsbmjfe n pefm x f ti px ui bu ui f tr vbsf pg ui f opsn pcfzt b ovn cfs pgqspqfsujft ui bu vojr vfmz flyju/ Xf bmtp ti px ui bu b Kbdpcjbo pg ui f tztufn pgCfui f fr vbujpot pcfzt ui f tbn f qspqfsujft/ Jo ui jt x bz x f qspwf b hfofsbmjfe Hbvejo i zqpui ftjt gps ui f opsn pg ui f I bn jmupojbo fjhfotubuft/

⁎ Dpssftqpoejoh bvui ps/
F.n bjmbeesfttft; Ai vutbmavl A hn bjm/dpn)B/ I vutbmavl *- b/mjbti zl A hn bjm/dpn)B/ Mjbti zl *- tubojtmbw/qbl vnjbl A kjos/sv)T/[/ Qbl vnjbl *- f sjd/sbhpvdz A mbqui /dost/gs)F/ Sbhpvdz*- ot mbwopw A n j/sbt/sv)O/B/ Trbwopw*/

i uuqt;00epj/psh021/21270k/ovdmqi ztc/3128/22/117
1661.43240

21 Jouspevdujpo

Jo 2: 83 N/ Hbvejo gpsn vmbufe b i zqpui ftjt bcpvu ui f opsn pg ui f I bn jmupojbo fjhfogvodujpo pg ui f r vbouvn opomjofbs Tdi s ejohfs fr vbujpo]2')tff bmtp]3'*/ Bddpsejoh up ui jt i zqpui ftjt- ui f tr vbsf pg ui f fjhfogvodujpo opsn jt qspqpsujpobmup b Kbdpcjbo efufsn jobou sfmbufe up ui f Cfui f fr vbujpot/ Jo 2: 93 W Lpsfqjo qspwfe ui f Hbvejo i zqpui ftjt gps b x jef dmbtt pg r vbouvn jo. ufhsbcmf n pefmt]4'/ Jo ui bu x psl ui f Rvbouvn Jowfstf Tdbuufsjoh N fui pe)RJTN*]5 8' x bt vtfe/ Bo bewboubhf pg ui jt n fui pe jt ui bu ju bmmpx t pof up dpotjefs r vbouvn n pefmt pg ejggfs. fouqi ztjdbmpsjhjo jo b dpn n po gsbn fx psl / Ui f x psl]4' efbmu x jui ui f n pefmt eftdsjcfe cz $gl(3)$.jowbsjbou R.n busjy boe ju q.efgpsn bujpo/ Vtjoh ui f tbn f bqqspbdi O/ Sftifujl i jo hfo. fsbmjjfe ui jt sftvmu up ui f n pefmt x jui $gl(4)$.jowbsjbou R.n busjy]9'/ Sfdfoumz- ui f opsn t pg ui f I bn jmupojbo fjhfogvodujpot jo ui f n pefmt x jui $gl(4)$ usjhpopn fusjd R.n busjy x fsf dbmdvmbufe jo]: '/

B ofx bqqspbdi up ui f qspcmfn cbtfe po ui f r vboujjfe Lojfii ojl [bn pmpedi jl pwfr vbujpo x bt efwfmpqfe jo b tfsjft pg qbqfst]21 23'/ Ui fsf ui f opsn t pg ui f fjhfotubuft jo $gl(N)$ cbtfe n pefmt x fsf dbmdvmbufe/ Ju x bt ti px o ui bu ui ftf sftvmut bsf fr vjwbmfou up ui f Hbvejo i zqpui ftjt/ Dpodfsojoh n pefmt eftdsjcfe cz tvqfsbmhfcsbt ju jt x psui n foujpojoh ui f x psl]24'- x i fsf bo bobmph pg ui f Hbvejo gpsn vmb x bt dpokfduvsfe gps I vccbse n pefm/ Sfdfoumz- ui f Hbvejo opsn pg ui f gvmmz pu$(3, 3|5)$ tqjo di bjo x bt tuvejfe jo]25'/

Jo bmm ui f dbtft mjtufe bcpwf ui f psjhjobmi zqpui ftjt x bt dpogjsn fe/ Tdi fn bujdbmmz ju dbo cf gpsn vmbufe bt gpmmpx t/ Mfu| ⟩ cf b I bn jmupojbo fjhfotubuf/ Gps r vbouvn jouhfhsbcmf n pefmt ju dbo cf qbsbn fufsjjfe cz b tfu pg qbsbn fufst | ⟩ = | $(t_2, \ldots, t_L)$⟩ tbujtgzjoh b tztufn pg fr vbujpot)Cfui f fr vbujpot*

$$F_i(t_2, \ldots, t_L) = 2, \qquad i = 2, \ldots, L, \qquad)2/2*$$

x i fsf F_i bsf tpn f gvodujpot efqfoejoh po ui f n pefm/ Ui fo ui f tr vbsf pg ui f opsn pg | ⟩ jt qspqpsujpobmup ui f gpmmpx joh Kbdpcjbo

$$\langle \ | \ \rangle \quad \text{efu} \frac{\partial \text{mph} F_i}{\partial t_j}. \qquad)2/3*$$

Jo ui f qsftfou qbqfs x f qspwf ui f Hbvejo i zqpui ftjt gps jouhfhsbcmf n pefmt x jui $gl(m|n)$ tzn . n fusz eftdsjcfe cz ui f tvqfs.Zbohjbo $Y\big)gl(m|n)\big\{$/ Pvs bqqspbdi jt wfsz dmptf up ui f pof pg ui f x psl]4'/ Ju jt cbtfe po ui f oftufe bmhfcsbjd Cfui f botbu{]26 28' boe ui f opujpo pg b hfofsbmjjfe n pefm]4-29-2: ')tff bmtp]7'*/ Xf cfhjo x jui b *tvn gpsn vmb* gps ui f tdbmbs qspevdu pg hfofsjd Cfui f wfdupst pcubjofe jo]31'/ Vtjoh ui jt gpsn vmb x f gjoe b sfdvstjpo gps ui f tdbmbs qspevdu boe ui fo tqfdjgz ju up ui f dbtf pg ui f opsn / Jo ui jt x bz x f qspwf ui bu ui f opsn boe ui f Hbvejo ef. ufsn jobou tbujtgz ui f tbn f sfdvstjpo/ Ubl joh jouP bddpvou ui f dpjodjefodf pg ui f jojujbmebub- x f ui fsfcz qspwf ui f Hbvejo i zqpui ftjt gps ui f n pefmt eftdsjcfe cz ui f tvqfs.Zbohjbo $Y\big)gl(m|n)\big\{$/

Ui f qbqfs jt pshbojjfe bt gpmmpx t/ Jo tfdujpo 3 x f csjf−z sfdbmmcbtjd opujpot pg RJTN tqfd. jgzjoh ui fn up ui f n pefmt cbtfe po ui f tvqfs.Zbohjbo $Y\big)gl(m|n)\big\{$/ Jo tfdujpo 4 x f eftdsjcf ui f Cfui f wfdupst pg ui f n pefmt x jui $gl(m|n)$.jowbsjbou R.n busjy boe dpotjefs ui fjs tdbmbs qspevdut/ Tfdujpo 5 jt efwpufe up ui f qspqfsujft pg ui f Hbvejo n busjy/ I fsf x f gpsn vmbuf ui f n bjo sftvmu pg ui f qbqfs/ Jo tfdujpo 6 x f jouspevdf ui f opujpo pg b hfofsbmjjfe n pefm ui bu tfswft bt b n bjo uppmpg pvs bqqspbdi / Jo tfdujpo 7 x f gjoe b sfdvstjpo gps ui f tdbmbs qspevdu pg Cfui f wfdupst/ Xf tqfdjgz ui jt sfdvstjpo up ui f dbtf pg ui f opsn jo tfdujpo 8 boe ti px ui bu ju dpjodjeft x jui ui f sfdvstjpo gps ui f Hbvejo efufsn jobou/ Jo ui jt x bz x f qspwf ui f hfofsbmjjfe Hbvejo i zqpui ftjt gps ui f n pefmt x jui $gl(m|n)$.jowbsjbou R.n busjy/ Tfwfsbmbvyjmjbsz tubufn fout bsf hbui fsfe jo bq. qfoejdft/ Jo Bqqfoejy B x f fyqmbjo i px up dpotusvdutpn f sfqsftfoubujwft pg ui f hfofsbmjfife n pefmjo ui f gsbn fx psl pg fwbmvbujpo sfqsftfoubujpo/ Bqqfoejy C dpoubjot sfdvstjpot gps ui f i jhi ftudpf gfldjfout pg ui f tdbmbs qspevdut/ Gjobmmz- jo Bqqfoejy D x f floe sftjevft jo ui f qpmft pg ui f i jhi ftudpf gfldjfout/

31 Cbtkd opubpot

Jo ui jt tfdujpo x f csjf–z sfdbmmcbtjd opujpot pg r vbouvn jouf hsbcmfi hsbefe n pefmt/ B n psf efubjmfe qsftfoubujpo dbo cf gpvoe jo]32‘/

Ui f $\mathbb{Z}_3$.hsbefe wfdups tqbdf $\mathbf{E}^{m|n}$ x jui ui f hsbejoh $[i] = 1$ gps $2 \sim i \sim m$- $[i] = 2$ gps $m < i \sim m+n$ jt b ejsfdutvn pg tqbdft; $\mathbf{E}^{m|n} = \mathbf{E}^m \otimes \mathbf{E}^n$/ Wfdupst cfmpohjoh up $\mathbf{E}^m$ bsf dbmmfe fwfo- wfdupst cfmpohjoh up $\mathbf{E}^n$ bsf dbmmfe pee/ N busjdft bdujoh jo $\mathbf{E}^{m|n}$ bsf hsbefe bt $[E_{ij}] = [i] + [j] \in \mathbb{Z}_3$- x i fsf E_{ij} bsf fmfn fobsz vojut; $(E_{ij})_{ab} = \lambda_{ia}\lambda_{jb}$/

Ui f R.n busjy pg $gl(m|n)$.jowbsjbou n pefmt i bt ui f gpsn

$$R(u,v) = \mathbb{I} + g(u,v)P, \qquad g(u,v) = \frac{c}{u \cdot\ v}. \qquad)3/2*$$

I fsf c jt b dpotubou- $\mathbb{I}$ boe P sftqfdujwfmz bsf ui f jefoujuz n busjy boe ui f hsbefe qfsn vubujpo pqf sbups]32‘;

$$\mathbb{I} = \mathbf{2} \leq \mathbf{2} = \sum_{i,j=2}^{n+m} E_{ii} \leq E_{jj}, \qquad P = \sum_{i,j=2}^{n+m} (\cdot\ 2)^{[j]} E_{ij} \leq E_{ji}. \qquad)3/3*$$

Jo)3/3* x f efbmx jui ui f n busjdft bdujoh jo ui f ufotps qspevdu $\mathbf{E}^{m|n} \leq \mathbf{E}^{m|n}$/ Jo jut uvso- ui f ufotps qspevdu pg $\mathbf{E}^{m|n}$ tqbdft jt hsbefe bt gpmmpx t;

$$(\mathbf{2} \leq E_{ij})\ \times(E_{kl} \leq \mathbf{2}) = (\cdot\ 2)^{([i]+[j])([k]+[l])}\, E_{kl} \leq E_{ij}. \qquad)3/4*$$

B cbtjd sfmbujpo pg ui f RJTN jt bo RTT.sfmbujpo[2]

$$R(u,v)\big)T(u) \leq \mathbf{2}\big\{\big)\mathbf{2} \leq T(v)\big\{ = \big)\mathbf{2} \leq T(v)\big\{\big)T(u) \leq \mathbf{2}\big\{R(u,v). \qquad)3/5*$$

I fsf $T(u)$ jt b n popespn z n busjy- x i ptf n busjy fmfn fout bsf r vbouvn pqf sbupst bdujoh jo b I jmcfsutqbdf $\mathcal{H}$/ Ui jt I jmcfsutqbdf dpjodjeft x jui ui f tqbdf pg tubuft pg ui f I bn jmupojbo voefs dpotjefsbujpo/ Ui f n busjy fmfn fout $T_{i,j}(u)$ bsf hsbefe jo ui f tbn f x bz bt ui f n busjdft $[E_{ij}]$; $[T_{i,j}(u)] = [i] + [j] \in \mathbb{Z}_3$/ Fr vbujpo)3/5* i pmet jo ui f ufotps qspevdu $\mathbf{E}^{m|n} \leq \mathbf{E}^{m|n} \leq \mathcal{H}$/ Bmmui f ufotps qspevdut bsf hsbefe/

Gps ui f hjwfo R.n busjy)3/2* ui f RTT.sfmbujpo)3/5* jn qmjft b tfu pg dpn n vubujpo sfmbujpot gps ui f n popespn z n busjy fousjft

$$\begin{aligned}[T_{i,j}(u), T_{k,l}(v)\} &= (\cdot\ 2)^{[i]([k]+[l])+[k][l]} g(u,v)\Big)T_{k,j}(v)T_{i,l}(u) \cdot\ \ T_{k,j}(u)T_{i,l}(v)\Big[\\ &= (\cdot\ 2)^{[l]([i]+[j])+[i][j]} g(u,v)\Big)T_{i,l}(u)T_{k,j}(v) \cdot\ \ T_{i,l}(v)T_{k,j}(u)\Big[,\end{aligned} \qquad)3/6*$$

x i fsf x f jouspevdfe ui f hsbefe dpn n vubups

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2 Tusjdumz tqfbl joh- jo sfmbujpo)3/5*- x f ti pvme vtf $R(u,v) \leq 1_{\mathcal{H}}$ jotufbe pg $R(u,v)$- x i fsf $1_{\mathcal{H}}$ jt ui f vojubdujoh po $\mathcal{H}$/ Ui jt n bl ft bmmsfmbujpot wfsz i fbwz- boe x f x sjuf mptfmz $R(u,v)$/ Ui jt x jmmcf ui f dbtf ui spvhi pvuui f qbqfs- cvux f n bl f ui jt ejtujodujpo jo Bqqfoejy B up dmbsjgz ui f dpotusvdujpo pg ui f fwbmvbujpo n bq/

$$[T_{i,j}(u), T_{k,l}(v)\} = T_{i,j}(u)T_{k,l}(v) \cdot \ (\cdot\ 2)^{([i]+[j])([k]+[l])} T_{k,l}(v)T_{i,j}(u). \qquad)3/7*$$

Ui f I bn jmpojbo boe pui fs jouhsbmt pgn pujpo pgb r vbouvn jouhsbcmf tztufn dbo cf pcubjofe gspn b hsbefe usbotgfs n busjy/ Jujt efflofe bt ui f tvqfsusbdf pgui f n popespn z n busjy

$$\mathcal{T}(u) = \text{tus}\, T(u) = \sum_{j=2}^{m+n} (\cdot\ 2)^{[j]}\, T_{j,j}(u). \qquad)3/8*$$

Pof dbo fbtjmz difdl]32' ui bu $[\mathcal{T}(u), \mathcal{T}(v)] = 1$/ Fjhfotubuft pg ui f hsbefe usbotgfs n busjy bsf fjhfotubuft pg ui f r vbouvn I bn jmpojbo/ Bt vtvbm ui fz bsf efflofe vq up b opsn bmjfibujpo gbdups/ Ui f n bjo hpbm pg ui jt qbqfs jt up floe opsn bmjfibujpo gbdupst tvdi ui bu ui f opsn t pg ui f dpssftqpoejoh fjhfotubuft bsf fr vbm up 2/

41 Cfui f wfdupst boe ui fjs tdbmbs rspevdut

Xf ep oputqfdjgz b I jmcfsutqbdf $\mathcal{H}$ x i fsf ui f n popespn z n busjy fousjft bdu- i px fwfs- x f bttvn f ui bujudpoubjot b *qtfvepwbdvvn wfdups* $|1\rangle$- tvdi ui bu

$$\begin{aligned} &T_{i,i}(u)|1\rangle = \mu_i(u)|1\rangle, && i = 2, \ldots, m+n, \\ &T_{i,j}(u)|1\rangle = 1, && i > j, \end{aligned} \qquad)4/2*$$

x i fsf $\mu_i(u)$ bsf tpn f tdbmbs gvodujpot/ Cfmpx ju x jmm cf dpowfojfou up efbm x jui sbujpt pg ui ftf gvodujpot

$$\gamma_i(u) = \frac{\mu_i(u)}{\mu_{i+2}(u)}, \qquad i = 2, \ldots, m+n \cdot\ 2. \qquad)4/3*$$

Jo ui f gsbn fx psl pg ui f hfofsbmjfife n pefm dpotjefsfe jo ui jt qbqfs- ui fz sfn bjo gsff gvodujpobm qbsbn fufst/ Xf ejtdvtt tpn f qspqfsujft pg ui f hfofsbmjfife n pefm jo tfdujpo 6/

Xf bmtp bttvn f ui bu ui f n popespn z n busjy fousjft bdujo b evbm tqbdf $\mathcal{H}^{\pm}$ x jui b evbm qtfv. epwbdvvn $\langle 1|$ tvdi ui bu

$$\begin{aligned} &\langle 1|T_{i,i}(u) = \mu_i(u)\langle 1|, && i = 2, \ldots, m+n, \\ &\langle 1|T_{i,j}(u) = 1, && i < j. \end{aligned} \qquad)4/4*$$

I fsf ui f gvodujpot $\mu_i(u)$ bsf ui f tbn f bt jo)4/2*/

Jo ui f gsbn fx psl pg ui f bmhfcsbjd Cfui f botbu{- ju jt bttvn fe ui bu ui f tqbdf pg tubuft $\mathcal{H}$ jt hfofsbufe cz ui f bdujpo pg ui f vqqfs usjbohvmbs fmfn fout pg ui f n popespn z n busjy $T_{i,j}(u)$ x jui $i < j$ pou ui f wfdups $|1\rangle$/ Jo qi ztjdbm n pefmt- wfdupst pg ui f tqbdf $\mathcal{H}$ eftdsjcf tubuft x jui r vbtjqbsujdmft pg ejggfsfou uzqft)dpmpst*/ Jo $\mathfrak{gl}(m|n)$.jowbsjbou n pefmt r vbtjqbsujdmft n bz i bwf $N = m+n \cdot\ 2$ dpmpst/ Mfu $\{r_2, \ldots, r_N\}$ cf b tfu pg opo.ofhbujwf jouhfst/ Xf tbz ui bu b tubuf i bt dpmpsjoh $\{r_2, \ldots, r_N\}$- jg ju dpoubjot r_i r vbtjqbsujdmft pg ui f dpmps i- x i fsf $i = 2, \ldots, N$/ Ui f bd. ujpo pg $T_{i,j}(u)$ pou b tubuf pg b flyfe dpmpsjoh dsfbuft $j \cdot\ i$ r vbtjqbsujdmft pg ui f dpmpst $i, \ldots, j \cdot\ 2$/ N psf efubjmt po dpmpsjoh dbo cf gpvoe jo]31'/

B Cfui f wfdups jt b qpmzopn jbm jo ui f dsfbujpo pqfsbupst $T_{i,j}$ x jui $i < j$ bqqmjfe up ui f wfdups $|1\rangle$/ Bmm ui f ufsn t pg ui jt qpmzopn jbm i bwf ui f tbn f dpmpsjoh/ Jo ui jt qbqfs x f ep opu vtf bo fyqmjdju gpsn pg ui f Cfui f wfdupst- i px fwfs- ui f sfbefs dbo floe ju jo]33'/ B hfofsjd Cfui f wfdups pg $\mathfrak{gl}(m|n)$.jowbsjbou n pefm efqfoet po $N = m+n \cdot\ 2$ tfut pg wbsjbcmft $\bar{t}^2, \bar{t}^3, \ldots, \bar{t}^N$ dbmmfe Cfui f qbsbn fufst/ Xf efopuf Cfui f wfdupst cz $\mathbb{B}(\bar{t})$- x i fsf

$$\bar{t} = \{t_2^2, \ldots, t_{r_2}^2; t_2^3, \ldots, t_{r_3}^3; \ldots; t_2^N, \ldots, t_{r_N}^N\}, \qquad)4/5*$$

boe ui f dbsejobmjujft r_i pgui f tfu $\bar{t}^i = \{t^i_2, \ldots, t^i_{r_i}\}$ dpjodjef xjui ui f dpmpsjoh/ Ui vt-fbdi Cfui f qbsbn fufs t^i_k dbo cf bttpdjbufe xjui b rvbtjqbsujdmf pgui f dpmps i/ Xf bmtp jouspevdf ui f upubm ovn cfs pgui f Cfui f qbsbn fufst

$$\mathbf{s} = '\,\bar{t} = \sum_{i=2}^{N} r_i. \qquad)4/6*$$

Cfui f wfdupst bsf tzn n fusjd pwfs qfsn vubujpot pgui f qbsbn fufst t^i_k xjui jo ui f tfu $\bar{t}^i$-i px fwfs-ui fz bsf opu tzn n fusjd pwfs qfsn vubujpot pwfs qbsbn fufst cfmpohjoh up ejggfsfou tfut $\bar{t}^i$ boe $\bar{t}^j$/ Gps hfofsjd Cfui f wfdupst ui f Cfui f qbsbn fufst t^i_k bsf hfofsjd dpn qmfy ovn cfst/ Jg ui ftf qbsbn fufst tbujtgz b tqfdjbm tztufn pgfr vbujpot)Cfui f fr vbujpot*-ui fo ui f dpssftqpoejoh wfdups cfdpn ft bo fjhfowfdups pgui f usbotgfs n busjy)3/8*/ Jo ui jt dbtf jujt dbmmfe *po.ti fmmCfui f wfdups*/ Xf hjwf fyqmjdjumz ui f tztufn pgCfui f fr vbujpot)4/22* b cjumbufs- bgufs jouspevdujpo b ofdfttbsz opubujpo/

EvbmCfui f wfdupst cfmpoh up ui f evbm tqbdf $\mathcal{H}^{\pm}$/ Ui fz dbo cf pcubjofe bt b hsbefe usbotqptj.ujpo pgui f Cfui f wfdupst)tff f/h/]31-33-34‘*/ Xf efopuf evbmCfui f wfdupst cz $\mathbb{C}(\bar{t})$-xi fsf $\bar{t}$ bsf ui f Cfui f qbsbn fufst)4/5*/ EvbmCfui f wfdupst cfdpn f po.ti fmmjg ui f tfu $\bar{t}$ tbujtgz ui f tztufn)4/22*/

4/2/ Ppubujpo

Jo ui jt qbqfs xf vtf opubujpo boe dpowfoujpot pgui f xpsl]31‘/ Cftjeft ui f gvodujpo $g(u,v)$ xf vtf pof n psf sbujpobmgvodujpo

$$f(u,v) = 2 + g(u,v) = \frac{u \cdot\ v + c}{u \cdot\ v}. \qquad)4/7*$$

Jo psefs up n blf gpsn vmbt vojgpsn xf bmtp jouspevdf b ahsbefe(dpotubou $c_{[i]} = (\cdot\ 2)^{[i]} c$/ Sf.tqfdujwfmz-xf vtf ahsbefe(sbujpobmgvodujpot $g_{[i]}(u,v)$ boe $f_{[i]}(u,v)$;

$$g_{[i]}(u,v) = \frac{c_{[i]}}{u \cdot\ v}, \qquad f_{[i]}(u,v) = 2 + g_{[i]}(u,v) = \frac{u \cdot\ v + c_{[i]}}{u \cdot\ v}. \qquad)4/8*$$

Gjobmmz-xf efflof $\delta_i(u,v)$ bt

$$\delta_i(u,v) = \begin{cases} f_{[i]}(u,v), & i \neq m, \\ g_{[i]}(u,v), & i = m. \end{cases} \qquad)4/9*$$

Pctfswf ui buui f gvodujpo δ_i ubl ft ui sff wbmvft-obn fmz-$\delta_i(u,v) = f(u,v)$ gps $i < m$-$\delta_i(u,v) = g(u,v)$ gps $i = m$-boe $\delta_i(u,v) = f(v,u)$ gps $i > m$/

Mfuvt gpsn vmbuf opx b dpowfoujpo po ui f opubujpo/ Xf vtf b cbs up efopuf tfut pgwbsjbcmft/ Ui f tfupgui f Cfui f qbsbn fufst jt efopufe cz $\bar{t}$)mjl f jo)4/5** ps $\bar{s}$/ Ui f mbuufs opubujpo n ptumz jt vtfe gps ui f Cfui f qbsbn fufst pgevbmCfui f wfdupst/ Gspn opx po joejwjevbmCfui f qbsbn fufst bsf mbcfmfe xjui b Hsffl tvqfstdsjqu boe b Mbujo tvctdsjqu-j/f/ t^{ν}_j- t^{ξ}_k-boe tp po/ Ui f tvqfstdsjqu sfgfst up ui f dpmps-xi jmf ui f tvctdsjqu dpvout ui f ovn cfs pgui f Cfui f qbsbn fufst pgui f flyfe dpmps/ Ui vt- $\bar{t} = \{\bar{t}^2, \ldots, \bar{t}^N\}$-xi fsf $\bar{t}^{\nu} = \{t^{\nu}_2, \ldots, t^{\nu}_{r_\nu}\}$/ Ui f jouhfst r_ν efopuf ui f dbsejobmjujft $r_\nu = '\,\bar{t}^{\nu}$-boe ui f upubmdbsejobmjuz s jt hjwfo cz)4/6*/ Tjn jmbs opubujpo jt vtfe gps ui f tfu $\bar{s}$/

Cfmpx xf dpotjefs qbsujujpot pgui f Cfui f qbsbn fufst jop ejtkpjou tvctfut/ Ui f tvctfut bsf efopufe cz Spn bo ovn cfst-j/f/ $\bar{t}^{\nu}_{\mathrm{J}}$-$\bar{s}^{\xi}_{\mathrm{JJ}}$-boe tp po/ B tqfdjbm opubujpo $\bar{t}^{\nu}_j$)sftq/ $\bar{s}^{\nu}_j$*jt vtfe gps ui f tvctfupg $\bar{t}^{\nu}$)sftq/ $\bar{s}^{\nu}$ * dpn qmfin foubsz up ui f qbsbn fufs t_j^{ν})sftq/ s_j^{ν} *- j/f/ $\bar{t}_j^{\nu} = \bar{t}^{\nu} \setminus \{t_j^{\nu}\}$)sftq/ $\bar{s}_j^{\nu} = \bar{s}^{\nu} \setminus \{s_j^{\nu}\}$*/

Xf vtf b ti psui boe opubujpo gps qspevdut pg ui f gvodujpot)4/3*-)4/8*- boe)4/9*/ Obn fmz- jg tpn f pg ui ftf gvodujpot efqfoe po b tfu pg wbsjbcmft)ps ux p tfut pg wbsjbcmft*- ui jt n fbot ui bu pof ti pvme ubl f ui f qspevdu pwfs ui f dpssftqpoejoh tfu)ps epvcmf qspevdu pwfs ux p tfut*/ Gps fybn qmf-

$$\gamma_{\xi}(\bar{t}^{\xi}) = \prod_{t_j^{\xi} \in \bar{t}^{\xi}} \gamma_{\xi}(t_j^{\xi}), \qquad f_{[\nu]}(t_k^{\nu}, \bar{t}_k^{\nu}) = \prod_{\substack{t_\ell^{\nu} \in \bar{t}^{\nu} \\ \ell \neq k}} f_{[\nu]}(t_k^{\nu}, t_\ell^{\nu}),$$

$$\delta_{\xi}(\bar{s}_{\mathrm{J}}^{\xi}, \bar{s}_{\mathrm{JJ}}^{\xi}) = \prod_{s_j^{\xi} \in \bar{s}_{\mathrm{J}}^{\xi}} \prod_{s_k^{\xi} \in \bar{s}_{\mathrm{JJ}}^{\xi}} \delta_{\xi}(s_j^{\xi}, s_k^{\xi}). \qquad)4/:*$$

Cz efgjojujpo- boz qspevdu pwfs ui f fn quz tfu jt frvbm up 2/ B epvcmf qspevdu jt frvbm up 2 jg bu mfbtu pof pg ui f tfut jt fn quz/

Up jmmvtusbuf ui f vtf pg ui f ti psui boe opubujpo)4/: * x f hjwf i fsf b tztufn pg Cfui f frvbujpot/ Sfdbmm ui bu jg ui f Cfui f qbsbn fufst $\bar{t}$ tbujtgz ui f tztufn pg Cfui f frvbujpot- ui fo ui f dpssftqpoejoh)evbm* Cfui f wfdups jt po.ti fmm/ Cfjoh x sjuufo jo b tuboebse opubujpo ui jt tztufn i bt ui f gpmmpx joh gpsn ;

$$\gamma_{\xi}(t_j^{\xi}) = (\cdot\ 2)^{\lambda_{\xi,m}(r_m \cdot\ 2)} \left(\prod_{\substack{k=2 \\ k \neq j}}^{r_{\xi}} \frac{\delta_{\xi}(t_j^{\xi}, t_k^{\xi})}{\delta_{\xi}(t_k^{\xi}, t_j^{\xi})} \right) \frac{\prod_{k=2}^{r_{\xi+2}} f_{[\xi+2]}(t_k^{\xi+2}, t_j^{\xi})}{\prod_{k=2}^{r_{\xi \cdot 2}} f_{[\xi]}(t_j^{\xi}, t_k^{\xi \cdot\ 2})}, \qquad \begin{array}{l} \xi = 2, \ldots, N, \\ j = 2, \ldots, r_{\xi}. \end{array} \qquad)4/21*$$

Ui f vtf pg ui f ti psui boe opubujpo bmmpx t pof up sfx sjuf ui jt tztufn bt

$$\gamma_{\xi}(t_j^{\xi}) = (\cdot\ 2)^{\lambda_{\xi,m}(r_m \cdot\ 2)} \frac{\delta_{\xi}(t_j^{\xi}, \bar{t}_j^{\xi}) f_{[\xi+2]}(\bar{t}^{\xi+2}, t_j^{\xi})}{\delta_{\xi}(\bar{t}_j^{\xi}, t_j^{\xi}) f_{[\xi]}(t_j^{\xi}, \bar{t}^{\xi \cdot\ 2})}, \qquad \begin{array}{l} \xi = 2, \ldots, N, \\ j = 2, \ldots, r_{\xi}. \end{array} \qquad)4/22*$$

4/3/ Jojujbm opsn bmjfibujpo pg Cfui f wfdupst

Bmui pvhi x f ep opu vtf fyqmjdju gpsn vmbt gps ui f Cfui f wfdupst- x f ti pvme gjy ui fjs jojujbm opsn bmj{bujpo/ Xf vtf ui f tbn f opsn bmj{bujpo bt jo]31‘/

Ju x bt bmsfbez n foujpofe ui bu b hfofsjd Cfui f wfdups i bt ui f gpsn pg b qpmzopn jbm jo $T_{i,j}$ x jui $i < j$ bqqmjfe up ui f qtfvepwbdvvn $|1\rangle$/ Bn poh bmm ui f ufsn t pg ui jt qpmzopn jbm ui fsf jt pof n popn jbm ui bu dpoubjot ui f pqfsbupst $T_{i,j}$ x jui $j \cdot\ i = 2$ pomz/ Xf dbmm ui jt n popn jbm ui f *n bjo ufsn* boe gjy ui f opsn bmj{bujpo pg ui f Cfui f wfdupst cz gjyjoh b ovn fsjd dpfggjdjfou pg ui f n bjo ufsn

$$\mathbb{B}(\bar{t}) = \frac{\mathbb{T}_{2,3}(\bar{t}^2) \ldots \mathbb{T}_{N,N+2}(\bar{t}^N)|1\rangle}{\prod_{i=2}^{N} \mu_{i+2}(\bar{t}^i) \prod_{i=2}^{N \cdot\ 2} f_{[i+2]}(\bar{t}^{i+2}, \bar{t}^i)} + \ldots, \qquad)4/23*$$

x i fsf fmmjqtjt n fbot bmm ui f ufsn t dpoubjojoh bu mfbtu pof pqfsbups $T_{i,j}$ x jui $j \cdot\ i > 2$/ Xf bmtp jouspevdf tzn n fusjd pqfsbupst qspevdut jo)4/23*;

$$\mathbb{T}_{i,i+2}(\bar{t}^i) = \frac{T_{i,i+2}(t_2^i) \ldots T_{i,i+2}(t_{r_i}^i)}{\left) \prod_{2 \sim j < k \sim r_i} h(t_k^i, t_j^i) \right[^{\lambda_{i,m}}}. \qquad)4/24*$$

Pof dbo fbtjmz di fdl ui buevf up ui f dpn n vubujpo sfmbujpot)3/6* ui f pqfsbups qspevdut $\mathbb{T}_{i,i+2}(\bar{t}^i)$ ep bsf tzn n fusjd pwfs $\bar{t}^i$ gps bmm $i = 2,\ldots,m+n\cdot\ 2$/

Sfdbmmui bu x f vtf i fsf ui f ti psu boe opubujpo gps ui f qspevdut pg ui f gvodujpot μ_{j+2} boe $f_{[j+2]}$/ Ui f opsn bmjfibujpo jo)4/23* jt ejggfsfou gspn ui f pof vtfe jo]33‘ cz ui f qspevdu $\prod_{j=2}^{N}\mu_{j+2}(\bar{t}^j)$/ Ui jt beejujpobmopsn bmjfibujpo gbdups jt dpowfojfou cf dbvtf jo ui jt dbtf ui f tdbmbs qspevdut pg ui f Cfui f wfdupst efqfoe po ui f sbujpt γ_i)4/3* pomz/

Tjodf ui f pqfsbupst $T_{i,i+2}$ boe $T_{j,j+2}$ ep opu dpn n vuf gps $i \neq j$- ui f n bjo ufsn dbo cf x sjuufo jo tfwfsbmgpsn t dpssftqpoejoh up ejggfsfou psefsjoh pg ui f n popespn z n busjy fousjft/ Ui f psefs. joh jo)4/23* obuvsbmmz bsjtft jg x f dpotusvdu Cfui f wfdupst wjb ui f fn cfeejoh pg $Y\big)\mathfrak{gl}(m\cdot\ 2|n)\big\{$ joup $Y\big)\mathfrak{gl}(m|n)\big\{$/

4/4/ Tdbmbs qspevdu pg Cfui f wfdupst

Ui f tdbmbs qspevdu pg Cfui f wfdupst jt efflofe bt

$$S(\bar{s}|\bar{t}) = \mathbb{C}(\bar{s})\mathbb{B}(\bar{t}). \qquad)4/25*$$

I fsf $\bar{s}$ boe $\bar{t}$ bsf tfut pg hfofsjd dpn qmfy ovn cfst pg ui f tbn f dbsejobmjuz ’ $\bar{s}$ = ’ $\bar{t}$/ Pof dbo ti px ui bu ui f tdbmbs qspevdu pg Cfui f wfdupst pg ejggfsfou dpmpsjoh wbojti ft]31‘- ui fsfgpsf- cfmpx x f dpotjefs pomz ui f dbtf ’ $\bar{s}^\xi$ = ’ $\bar{t}^\xi = r_\xi$- $\xi = 2,\ldots,N$)sfdbmmui bu $N = m+n\cdot\ 2$*/

Jo]31‘ x f gpvoe b tvn gpsn vmb gps ui jt tdbmbs qspevdu

$$S(\bar{s}|\bar{t}) = \sum \frac{\prod_{\xi=2}^{N}\gamma_\xi(\bar{s}^\xi_{\rm J})\gamma_\xi(\bar{t}^\xi_{\rm JJ})\delta_\xi(\bar{s}^\xi_{\rm JJ},\bar{s}^\xi_{\rm J})\delta_\xi(\bar{t}^\xi_{\rm J},\bar{t}^\xi_{\rm JJ})}{\prod_{\xi=2}^{N\cdot\ 2} f_{[\xi+2]}(\bar{s}^{\xi+2}_{\rm JJ},\bar{s}^\xi_{\rm J})f_{[\xi+2]}(\bar{t}^{\xi+2}_{\rm J},\bar{t}^\xi_{\rm JJ})} Z^{m|n}(\bar{s}_{\rm J}|\bar{t}_{\rm J})\ Z^{m|n}(\bar{t}_{\rm JJ}|\bar{s}_{\rm JJ}). \qquad)4/26*$$

I fsf bmmui f tfut pg ui f Cfui f qbsbn fufst $\bar{t}^\xi$ boe $\bar{s}^\xi$ bsf ejwjefe joup ux p tvctfut $\bar{t}^\xi \Rightarrow \{\bar{t}^\xi_{\rm J},\bar{t}^\xi_{\rm JJ}\}$ boe $\bar{s}^\xi \Rightarrow \{\bar{s}^\xi_{\rm J},\bar{s}^\xi_{\rm JJ}\}$- tvdi ui bu ’ $\bar{t}^\xi_{\rm J}$ = ’ $\bar{s}^\xi_{\rm J}$/ Ui f tvn jt ublfo pwfs bmmqpttjcmf qbsujujpot pg ui jt uzqf/

Ui f gvodujpo $Z^{m|n}(\bar{s}|\bar{t})$ jt ui f i jhi ftu dpfgfldjfou)I D*/ Ui jt jt b sbujpobmgvodujpo pg ui f Cfui f qbsbn fufst/ Ju dbo cf dpotusvdufe sfdvstjwfmz tubsujoh x jui I D jo $\mathfrak{gl}(2|2)$ tvqfsbmhfcsb)tff bmtp]36‘ gps bo fyqmjdju efufsn jobou sfqsftfoubujpo pg I D jo $\mathfrak{gl}(3|2)$ tvqfsbmhfcsb*

$$Z^{2|2}(\bar{s}|\bar{t}) = g(\bar{s},\bar{t}). \qquad)4/27*$$

Ui f sfdvstjpot gps I D bsf hjwfo jo Bqqfoejy C/

Ui f n ptu jn qpsubou qspqfsuz pg I D jt ui bu ui jt gvodujpo i bt tjn qmf qpmft bu $s^\nu_j = t^\nu_j$- $\nu = 2,\ldots,N$- $j = 2,\ldots,r_\nu$/

Rsprptkukpo 4121 *Ui f sftkevft pg I D ko ui f qpmft bu $s^\nu_j = t^\nu_j$- $\nu = 2,\ldots,N$- $j = 2,\ldots,r_\nu$ bsf qspqpsukpobmup $Z^{m|n}(\bar{s}\setminus\{s^\nu_j\}|\bar{t}\setminus\{t^\nu_j\})$A*

$$Z^{m|n}(\bar{s}|\bar{t})\Big|_{s^\nu_j\to t^\nu_j} = g_{[\nu+2]}(t^\nu_j,s^\nu_j)\,\frac{\delta_\nu(\bar{t}^\nu_j,t^\nu_j)\delta_\nu(s^\nu_j,\bar{s}^\nu_j)\ Z^{m|n}(\bar{s}\setminus\{s^\nu_j\}|\bar{t}\setminus\{t^\nu_j\})}{f_{[\nu+2]}(\bar{t}^{\nu+2},t^\nu_j)f_{[\nu]}(s^\nu_j,\bar{s}^{\nu\cdot\ 2})} + reg, \qquad)4/28*$$

x i fsf reg nfbot sfhvmbs ufsn t/

Xf qspwf ui jt qspqptjujpo jo Bqqfoejy D/

Ui f tr vbsf pg ui f opsn pg ui f Cfui f wfdups ubsbejujpobmmz jt efflofe bt

$$S(\bar{t}|\bar{t}) = \mathbb{C}(\bar{t})\mathbb{B}(\bar{t}), \qquad)4/29*$$

ui bujt- ui jt jt ui f tdbmbs qspevdu bu $\bar{s} = \bar{t}$/ Fr vbujpo)4/26*tujmi pmt jo ui jt dbt f- i px fwfs- tfqbsbuf ufsn t pgui f tvn pwfs qbsujujpot n bz i bwf tjohvmbsjujft evf up ui f qpmft pgI D/ Ui vt- jo psefs up bqqspbdi ui f dbtf pgui f opsn pof ti pvme ublf b mjn ju $\bar{s} \to \bar{t}$ jo)4/26*/ Ui f mjn ju $\bar{s} \to \bar{t}$ n fbot ui bu $s_j^{\nu} \to t_j^{\nu}$ gps bmm $\nu = 2, \dots, N$ boe $j = 2, \dots, r_\nu$/

Gjobmmz- up pcubjo ui f opsn pg po.ti fmmCfui f wfdupst- pof ti pvme jn qptf Cfui f fr vbujpot)4/22*/ Bddpsejoh up ui f hfofsbmjmfe Hbvejo i zqpui ftjt- ui f tr vbsf pgui f opsn pgpo.ti fmmCfui f wfdupst jo $\mathfrak{gl}(m|n)$.jowbsjbou n pefmt jt qspqpsujpobmup b tqfdjbmKbdpcjbo/ X f eftdsjcf ui jt Kbdp. cjbo jo ui f ofyu tfdujpo/

51 I bvejo n busjy

Ui f Hbvejo n busjy G gps $\mathfrak{gl}(m|n)$.jowbsjbou n pefmt jt bo $N * N$ cmpdl .n busjy/ Ui f tjmf pgui f cmpdl $G^{(\nu,\xi)}$ jt $r_\nu * r_\xi$/ Up eftdsjcf ui f fousjft $G^{(\nu,\xi)}_{jk}$ x f jouspevdf b gvodujpo

$$\Phi_j^{(\nu)} = (\cdot\ 2)^{\lambda_{\nu,m}(r_m \cdot\ 2)} \gamma_\nu(t_j^\nu) \frac{\delta_\nu(\bar{t}_j^\nu, t_j^\nu)}{\delta_\nu(t_j^\nu, \bar{t}_j^\nu)} \frac{f_{[\nu]}(t_j^\nu, \bar{t}^{\nu \cdot\ 2})}{f_{[\nu+2]}(\bar{t}^{\nu+2}, t_j^\nu)}. \qquad)5/2*$$

Jujt fbtz up tff ui bu Cfui f fr vbujpot)4/22* dbo cf x sjuufo jo ufsn t pg $\Phi_j^{(\nu)}$ bt

$$\Phi_j^{(\nu)} = 2, \qquad \nu = 2, \dots, N, \quad j = 2, \dots, r_\nu. \qquad)5/3*$$

Ui f fousjft pgui f Hbvejo n busjy bsf efmofe bt

$$G^{(\nu,\xi)}_{jk} = \cdot\ c_{[\nu+2]} \frac{\partial \mathrm{mph}\, \Phi_j^{(\nu)}}{\partial t_k^\xi}. \qquad)5/4*$$

X f bsf opx jo qptjujpo up tubuf ui f n bjo sftvmupgui jt qbqfs;

Zi fpsfn 5/2/ *Ui f tr vbsf pgui f opsn pgui f po.ti fmmCfui f wfdupst sfbet*

$$\mathbb{C}(\bar{t})\mathbb{B}(\bar{t}) = \prod_{\xi=2}^{N} \prod_{\substack{p,q=2\\ p\neq q}}^{r_\xi} \delta_\xi(t_p^\xi, t_q^\xi) \left(\prod_{\xi=2}^{N\cdot\ 2} f_{[\xi+2]}(\bar{t}^{\xi+2}, \bar{t}^\xi) \right)^{\cdot\ 2} \mathrm{efu} G, \qquad)5/5*$$

x i fsf ui f n busjy G jt hjwfo cz)5/4/*

X f qspwf ui jt gpsn vmb jo ui f sftu pgui f qbqfs/

5/2/ Rspqfsujft pgui f Hbvejo n busjy

Gjstu pgbmmmfu vt hjwf fyqmjdju fyqsfttjpot gps ui f n busjy fmfn fout pgui f Hbvejo n busjy)5/4*/ X f i bwf gps ui f fmfn fout jo ui f ejbhpobmcmpdl t $G^{(\nu,\nu)}$;

$$G^{(\nu,\nu)}_{jk} = \lambda_{jk}\Big[X_j^\nu \cdot \sum_{\ell=2}^{r_\nu} \mathcal{K}_\nu(t_j^\nu, t_\ell^\nu) + (\cdot\ 2)^{\lambda_{\nu,m}} \sum_{q=2}^{r_{\nu\cdot 2}} \mathcal{J}_{[\nu]}(t_j^\nu, t_q^{\nu\cdot 2})$$
$$+ \sum_{p=2}^{r_{\nu+2}} \mathcal{J}_{[\nu+2]}(t_p^{\nu+2}, t_j^\nu) \Big] + \mathcal{K}_\nu(t_j^\nu, t_k^\nu). \qquad)5/6*$$

I fsf

$$X_j^{\nu} = \cdot\ c_{[\nu+2]}\frac{d}{dz}\,\text{mph}\,\gamma_{\nu}(z)\Big|_{z=t_j^{\nu}},\qquad)5/7*$$

boe

$$\mathcal{K}_{\nu}(x,y) = \frac{3c^3(2\cdot\ \lambda_{\nu,m})}{(x\cdot\ y)^3\cdot\ c^3},\qquad \mathcal{J}_{[\nu]}(x,y) = \frac{c^3}{(x\cdot\ y)(x\cdot\ y+c_{[\nu]})}.\qquad)5/8*$$

Ui f ofbs.ejbhpobmcmpdl t bsf

$$G_{jk}^{(\nu,\nu\cdot\ 2)} = (\cdot\ 2)^{\lambda_{\nu,m}+2}\mathcal{J}_{[\nu]}(t_j^{\nu}, t_k^{\nu\cdot\ 2}),\qquad G_{jk}^{(\nu,\nu+2)} = \cdot\ \mathcal{J}_{[\nu+2]}(t_k^{\nu+2}, t_j^{\nu}).\qquad)5/9*$$

Jg $|\nu\cdot\ \xi| > 2$- ui fo $G_{jk}^{(\nu,\xi)} = 1$/

Dpotjefs opx tpn f qspqfsujft pgui f Hbvejo n busjy efufsn jobou/ Mfu

$$\mathbf{G}^{(\mathbf{s})}(\bar{X};\bar{t}) = \text{efu}G.\qquad)5/:*$$

I fsf x f i bwf tusfttfe ui buui f gvodujpo $\mathbf{G}^{(\mathbf{s})}(\bar{X};\bar{t})$ efqfoet po ux p tfut pgwbsjbcmft/ P of pgui ftf tfut dpotjtut pgui f Cfui f qbsbn fufst $\bar{t}$)4/5*/ Bopui fs tfujt

$$\bar{X} = \{X_2^2,\ldots,X_{r_2}^2; X_2^3,\ldots,X_{r_3}^3;\ldots;X_2^N,\ldots,X_{r_N}^N\}.\qquad)5/21*$$

Ui f tvqfstdsjqu $\mathbf{s}$ ti px t ui f upubmovn cfs pg Cfui f qbsbn fufst ps- x i bujt ui f tbn f- ui f upubm ovn cfs pgqbsbn fufst X_j^{ν}; $\mathbf{s} = '\ \bar{t} = '\ \bar{X}$/

Jo tqfdjfld n pefmui f wbsjbcmft X_j^{ν} bsf gvodujpot pgui f Cfui f qbsbn fufst)tff)5/7**/ I fsf x f dpotjefs b n psf hfofsbmdbtf- x i fsf ui f tfut $\bar{X}$ boe $\bar{t}$ bsf joefqfoefou/ Jo pui fs x pset- x f tuvez efu$G$ x jui ui f n busjy fmfn fout)5/6*-)5/9*- cvux f ep opujn qptf)5/7*/

Lpsfqko dskufskb Ui f gvodujpo $\mathbf{G}^{(\mathbf{s})}(\bar{X};\bar{t})$ pcfzt tpn f di bsbdufsjtujdqspqfsujft/ Ui ftf qspqfsujft mjtufe cfmpx bsf rvjuf bobmphpvt up ui f qspqfsujft pgui f Hbvejo efufsn joboujo ui f $\mathfrak{gl}(3)$ dbtf/ Evf up ui f qbsbmfmp ui f psjhjobmqbqfs]4' x f dbmmui fn *Lpsfqko dskufskb*/

)j* Ui f gvodujpo $\mathbf{G}^{(\mathbf{s})}(\bar{X};\bar{t})$ jt tzn n fusjd pwfs ui f sfqmbdfn fou pg ui f qbjst $(X_j^{\nu}, t_j^{\nu}) \leftrightarrow (X_k^{\nu}, t_k^{\nu})$/

)jj* Jujt b mjofbs gvodujpo pgfbdi X_j^{ν}/

)jjj* $\mathbf{G}^{(2)}(X_2^2; t_2^2) = X_2^2$ gps $'\ \bar{t} = \mathbf{s} = 2$/

)jw* Ui f dpfgfldjfoupg X_j^{ν} jt hjwfo cz b gvodujpo $\mathbf{G}^{(\mathbf{s}\cdot\ 2)}$ x jui n pejflfe qbsbn fufst X_k^{ξ}

$$\frac{\partial\mathbf{G}^{(\mathbf{s})}(\bar{X};\bar{t})}{\partial X_j^{\nu}} = \mathbf{G}^{(\mathbf{s}\cdot\ 2)}(\{\bar{X}^{\text{n pe}}\setminus X_j^{\text{n pe};\nu}\};\{\bar{t}\setminus t_j^{\nu}\}),\qquad)5/22*$$

x i fsf ui f psjhjobmwbsjbcmft X_k^{ξ} ti pvme cf sfqmbdfe cz $X_k^{\text{n pe};\xi}$;

$$\begin{aligned}
X_k^{\text{n pe};\nu} &= X_k^{\nu}\cdot\ \mathcal{K}_{\nu}(t_j^{\nu}, t_k^{\nu}),\\
X_k^{\text{n pe};\nu+2} &= X_k^{\nu+2} + (\cdot\ 2)^{\lambda_{m,\nu+2}}\mathcal{J}_{[\nu+2]}(t_k^{\nu+2}, t_j^{\nu}),\\
X_k^{\text{n pe};\nu\cdot\ 2} &= X_k^{\nu\cdot\ 2} + \mathcal{J}_{[\nu]}(t_j^{\nu}, t_k^{\nu\cdot\ 2}),\\
X_k^{\text{n pe};\xi} &= X_k^{\xi},\qquad |\xi\cdot\ \nu| > 2.
\end{aligned}\qquad)5/23*$$

)w* $\mathbf{G}^{(\mathbf{s})}(\bar{X};\bar{t}) = 1$- jg bmm $X_j^{\xi} = 1$/

Ui f qspqfsujft)j*)jw* bsf rvjuf pcwjpvt/ Jo psefs up di fdl ui f qspqfsuz)w* pof ti pvme ubl f ui f tvn pgbmmdpmvn ot)ps spx t* pgui f n busjy G

$$\sum_{\xi=2}^{N}\sum_{k=2}^{r_\xi} G_{jk}^{(\nu,\xi)} = X_j^{\nu}.$$)5/24*

I fodf-jgbmm $X_j^\nu = 1$- ui fo ui jt mjofbs dpn cjobujpo wbojti ft- boe ui vt- efu $G = 1$/

Rspr ptkukpo 5121 *Ui f Lpsfqko dskufskb –yft ui f gvodukpo* $\mathbf{G}^{(\mathbf{s})}(\bar X;\bar t)$ *vokrvfm/*

Rsppg Ui f qsppgjt fybdumz ui f tbn f bt jo ui f gl(3) dbtf]4'/ Gps dpn qmfufoftt-x f sfqfbujui fsf/ Mfugvodujpot $\mathbf{G}_2^{(\mathbf{s})}(\bar X;\bar t)$ boe $\mathbf{G}_3^{(\mathbf{s})}(\bar X;\bar t)$ tbujtgz Lpsfqjo dsjufsjb/ Ui fo gps $\mathbf{s} = '\ \bar t = 2$ x f i bwf $\mathbf{G}_2^{(2)}(X_2^2;t_2^2) = \mathbf{G}_3^{(2)}(X_2^2;t_2^2)$/ Bttvn f ui bu $\mathbf{G}_2^{(\mathbf{s}\cdot 2)}(\bar X;\bar t) = \mathbf{G}_3^{(\mathbf{s}\cdot 2)}(\bar X;\bar t)$/ Ui fo gps $'\ \bar t = \mathbf{s}$ x f i bwf

$$\frac{\partial}{\partial X_j^\nu})\mathbf{G}_2^{(\mathbf{s})}(\bar X;\bar t)\cdot \mathbf{G}_3^{(\mathbf{s})}(\bar X;\bar t)) = 1,$$)5/25*

evf up ui f qspqfsuz)jw* boe ui f joevdujpo bttvn qujpo- boe

$$(\mathbf{G}_2^{(\mathbf{s})}(\bar X;\bar t)\cdot \mathbf{G}_3^{(\mathbf{s})}(\bar X;\bar t))\Big|_{\bar X=1} = 1,$$)5/26*

evf up ui f qspqfsuz)w*/ Tjodf ui f gvodujpo $\mathbf{G}_2^{(\mathbf{s})}(\bar X;\bar t)\cdot \mathbf{G}_3^{(\mathbf{s})}(\bar X;\bar t)$ jt mjofbs pwfs fbdi X_j^ν- frvb. ujpot)5/25* boe)5/26* zjfme $\mathbf{G}_2^{(\mathbf{s})}(\bar X;\bar t)\cdot \mathbf{G}_3^{(\mathbf{s})}(\bar X;\bar t) = 1$ gps $'\ \bar t = \mathbf{s}$/ □

Ui vt- jo psefs up qspwf)5/5* jujt fopvhi up ti px ui buui f qspqfsmz opsn bmjfife tdbmbs qspevdu pgpo.ti fmmCfui f wfdupst $\mathbb{C}(\bar t)\mathbb{B}(\bar t)$ pcfzt Lpsfqjo dsjufsjb/

61 I fofsbmkfife n pefm

Ui f opujpo pgui f hfofsbmjfife n pefmx bt jouspevdfe jo]4' gps gl(3) cbtfe n pefmt)tff bmtp]7-9-29-2: '*/ Ui jt n pefmbmtp dbo cf dpotjefsfe jo ui f dbtf pgui f tvqfs.Zbohjbo $Y\big)\mathfrak{gl}(m|n)\big\{$/ Jo gbdu- ui f hfofsbmjfife n pefmjt b dmbtt pg n pefmt/ Fbdi sfqsftfoubujwf pg ui jt dmbtt i bt b n popespn z n busjy tbujtgzjoh ui f RTT.sfmbujpo)3/5* x ju ui f R.n busjy)3/2*- boe qpttfttft qtfvepwbdvvn wfdupst x ju ui f qspqfsujft)4/2*-)4/4*/ B sfqsftfoubujwf pgui f hfofsbmjfife n pefm dbo cf di bsbdufsjfife cz b tfupgui f gvodujpobmqbsbn fufst $\gamma_\nu(u)$)4/3*/ Ejggfsfousfqsftfoubujwft bsf ejtujohvjti fe cz ejggfsfoutfut pgui f sbujpt $\gamma_\nu(u)$/

Ui f tvn gpsn vmb)4/26* gps ui f tdbmbs qspevdujt wbmje gps boz sfqsftfoubujwf pgui f hfofsbm jfife n pefm/ Ui fo x f dbo dpotjefs ui f tdbmbs qspevdubt b gvodujpo efqfoejoh po ux p uzqft pg wbsjbcmft; ui f Cfui f qbsbn fufst $\bar s$ boe $\bar t$ po ui f pof i boe- boe ui f gvodujpobmqbsbn fufst γ_ν po ui f pui fs i boe/ Joeffe-fwfo jgtpn f t_j^ν)sftq/ s_j^ν *jt flyfe- ui fo ui f gvodujpo $\gamma_\nu(t_j^\nu)$)sftq/ $\gamma_\nu(s_j^\nu)$* di bohft gsffmz x i fo svoojoh ui spvhi ui f dmbtt pgui f hfofsbmjfife n pefmt/ Jo qbsujdvmbs- vtjoh pomz joi pn phfofpvt n pefmt x ju tqjot jo i jhi fs ejn fotjpobmsfqsftfoubujpot pof dbo fbtjmz dpotusvdu sfqsftfoubujwft pgui f hfofsbmjfife n pefmt)tff Bqqfoejy B*- gps x i jdi

$$\gamma_\nu(u) = \prod_{j=2}^{L^{(\nu)}} f_{[\nu]}(u,\pi_j^{(\nu)}).$$)6/2*

I fsf joi pn phfofjujft $\pi_j^{(\nu)}$ bsf bscjusbsz dpn qmfy ovn cfst- boe $L^{(\nu)}$ bsf bscjusbsz qptjujwf jo. ufhfst/ Ju jt dmfbs ui bu fwfo cfjoh sftusjdufe up ui jt dmbtt pg gvodujpot γ_ν x f dbo bqqspbdi boz qsfefflofe wbmvf pg $\gamma_\nu(u)$ bu u flyfe/

Ui f n fbojoh pg Cfui f fr vbujpot)4/22* bmtp di bohft jo ui f hfofsbmjif e n pefm/ Gps b hjwfo sfqsftfoubujwf ui jt jt b tfu pg fr vbujpot gps ui f Cfui f qbsbn fufst/ Jo ui f hfofsbmjif e n pefm ui jt jt b tfu pg dpotusbjout cfux ffo ux p hspvqt pg joefqfoefou wbsjbcmft t_j^ν boe $\gamma_\nu(t_j^\nu)$/ Joeffe- pof dbo fly bo bscjusbsz tfu pg ui f Cfui f qbsbn fufst $\bar t$ boe ui fo floe b tfu pg gvodujpot γ_ν tvdi ui bu ui f tztufn)4/22* jt gvmflmmfe/ Gps fybn qmf- pof dbo mppl gps ui f gvodujpot γ_ν jo ui f gpsn)6/2*/ Ui fo Cfui f fr vbujpot cfdpn f b tfu pg dpotusbjout gps joi pn phfofjujft $\pi_j^{(\nu)}$/ Tjodf ui f ovn cfs pg joi pn phfofjujft jt opu sftusjdufe- pof dbo bmxbzt qspwjef tpmwbcjmjuz pg ui f tztufn)4/22*/

Xf x jmm tff jo tfdujpo 7 ui bu jg $t_j^\nu = s_j^\nu$ gps tpn f ν boe j- ui fo ui f tdbmbs qspevdu efqfoet bmtp po ui f efsjwbujwft $\gamma'_\nu(t_j^\nu)$ pg ui f gvodujpobm qbsbn fufst γ_ν/ Ui fz bsjtf evf up ui f qsftfodf pg qpmft jo ui f I D $Z^{m|n}(\bar s_{\rm I}|\bar t_{\rm I})$ boe $Z^{m|n}(\bar t_{\rm II}|\bar s_{\rm II})$/ Ui f efsjwbujwft $\gamma'_\nu(t_j^\nu)$ bmtp dbo cf usfbufe bt joefqfoefou gvodujpobm qbsbn fufst- cfdbvtf hfofsjdbmmz ui f wbmvft pg b gvodujpo boe jut efsjwbujwf jo b flyfe qpjou bsf opu sfmbufe up fbdi pui fs/ Jo qbsujdvmbs- ui f tr vbsf pg ui f opsn pg b Cfui f wfdups efqfoet po ui sff uzqf pg wbsjbcmft; ui f Cfui f qbsbn fufst- ui f wbmvft pg ui f gvodujpot γ_ν jo ui f qpjout t_j^ν- boe ui f wbmvft pg ui f efsjwbujwft γ'_ν jo ui f tbn f qpjout/ Jg ui f Cfui f wfdups jt po.ti fmm- ui fo x f dbo fyqsftt $\gamma_\nu(t_j^\nu)$ jo ufsn t pg ui f Cfui f qbsbn fufst evf up)4/22*/ I px fwfs- ui f efsjwbujwft $\gamma'_\nu(t_j^\nu)$ tujmm sfn bjo gsff/ Jo qbsujdvmbs- ui f wbsjbcmft X_j^ν)5/7* boe ui f Cfui f qbsbn fufst $\bar t$ dbo cf dpotjefsfe bt joefqfoefou wbsjbcmft jo ui f gsbn fx psl pg ui f hfofsbmjif e n pefm/

Up jmmvtusbuf bo bewboubhf pg ui f hfofsbmjif e n pefm x f qspwf i fsf bo jefoujuz ui bu x jmm cf vtfe cfmpx/

Rspr ptkkpo 6/2/ *Gps bscjusbsz dpn qmfy* $\bar t$ *boe* $\bar s$ *tvdi ui bu* $\#\bar s = \#\bar t > 1$

$$\sum \frac{\prod_{\xi=2}^{N} \delta_\xi(\bar s^\xi_{\rm II}, \bar s^\xi_{\rm I})\delta_\xi(\bar t^\xi_{\rm I}, \bar t^\xi_{\rm II})}{\prod_{\xi=2}^{N-2} f_{[\xi+2]}(\bar s^{\xi+2}_{\rm II}, \bar s^\xi_{\rm I}) f_{[\xi+2]}(\bar t^{\xi+2}_{\rm I}, \bar t^\xi_{\rm II})} Z^{m|n}(\bar s_{\rm I}|\bar t_{\rm I})\ Z^{m|n}(\bar t_{\rm II}|\bar s_{\rm II}) = 1. \qquad)6/3*$$

Rsppg/ Pctfswf ui bu ui f mit pg)6/3* jt b qbsujdvmbs dbtf pg ui f tdbmbs qspevdu gspn vmb)4/26* bu $\gamma_\xi(u) = 2$ gps $\xi = 2, \dots, N$/

Sfdbmm ui bu ui f tvn gspn vmb)4/26* i pmet gps bo bscjusbsz sfqsftfoubujwf pg ui f hfofsbmjif e n pefm/ Bn poh ui ftf sfqsftfoubujwft ui fsf fyjtut b n pefm tvdi ui bu $T(u) = \mathbf{2}$/ Joeffe- ui jt n po. pespn z n busjy pcwjpvtmz tbujtflft ui f RTT.sfmbujpo)3/5*/ Pof dbo qptuvmbuf ui bu ui f n busjy fmfn fout $T_{i,j}(u)$ bdu jo tpn f I jmcfsu tqbdf $\mathcal H$- gps fybn qmf- $\mathcal H = \mathbf{E}$ x jui b qtfvepwbdvvn $|1\rangle = 2$/ Ui f evbm tqbdf $\mathcal H^\pm$ ui fo dpjodjeft x jui $\mathcal H$- boe $\langle 1| = 2$/ Ui f dpoejujpot)4/2*-)4/4* pcwjpvtmz bsf gvmflmmfe- boe $\gamma_\xi(u) = 2$ gps $\xi = 2, \dots, N$/ Ui vt- ui f mit pg)6/3* jt fr vbm up ui f tdbmbs qspevdu pg Cfui f wfdupst jo ui f n pefm x jui $T(u) = \mathbf{2}$/ Cvu ui f mbuufs wbojti ft- cfdbvtf $T_{i,j} = 1$ gps $i \ne j$- boe i fodf- $\mathbb{B}(\bar t) = 1$- $\mathbb{C}(\bar s) = 1$ gps $\#\bar t = \#\bar s > 1$/ □

71 Tfdvstkpo gps ui f tdbmbs rspevdu

Mfu vt uvso cbdl up ui f tdbmbs qspevdu jo ui f gpsn)4/26*/ Tvqqptf ui bu $s_j^\nu = t_j^\nu$ gps tpn f j boe ν/ Ui f upubm tdbmbs qspevdu jt opu tjohvmbs- cfdbvtf ui f RTT.dpn n vubujpo sfmbujpot bsf opu tjohvmbs/ I px fwfs- ui f i jhi ftu dpfgfldjfout jo)4/26* n jhi u i bwf qpmft/ Ui f qpmft pddvs jg fjui fs $s_j^\nu \in \bar s_{\rm I}$ boe $t_j^\nu \in \bar t_{\rm I}$ ps $s_j^\nu \in \bar s_{\rm II}$ boe $t_j^\nu \in \bar t_{\rm II}$/ Sftpmwjoh ui ftf tjohvmbsjujft bu $s_j^\nu = t_j^\nu$ x f pcubjo efsjwbujwft pg ui f gvodujpot $\gamma_\nu(z)$/ Pvs hpbm jt up floe- i px ui f tdbmbs qspevdu efqfoet po ui ftf efsjwbujwft/

Gps ui jt jujt dpowfojfou up jouspevdf

$$\hat\gamma_\xi(t^\xi_j)=(\cdot\ 2)^{\lambda_{\xi,m}(r_m\cdot\ 2)}\gamma_\xi(t^\xi_j)\frac{\delta_\xi(\bar t^\xi_j,t^\xi_j)f_{[\xi]}(t^\xi_j,\bar t^{\xi\cdot\ 2})}{\delta_\xi(t^\xi_j,\bar t^\xi_j)f_{[\xi+2]}(\bar t^{\xi+2},t^\xi_j)},\qquad \xi=2,\dots,N,$$

$$\hat\gamma_\xi(s^\xi_j)=(\cdot\ 2)^{\lambda_{\xi,m}(r_m\cdot\ 2)}\gamma_\xi(s^\xi_j)\frac{\delta_\xi(\bar s^\xi_j,s^\xi_j)f_{[\xi]}(s^\xi_j,\bar s^{\xi\cdot\ 2})}{\delta_\xi(s^\xi_j,\bar s^\xi_j)f_{[\xi+2]}(\bar s^{\xi+2},s^\xi_j)},\qquad j=2,\dots,r_\xi, \qquad)7/2*$$

x i fsf)i fsf boe cfmpx * $\bar t^1=\bar s^1=\bar t^{m+n}=\bar s^{m+n}=\emptyset$/ Ui jt jn qmjft jo qbsujdvmbs ui bu ui f qspevdut jowpmwjoh fmfn fout gspn ui ftf fn quz tfut bsf frvbm up 2/

Ui fo- sfqmbdjoh γ_ξ x ju $\hat\gamma_\xi$ jo ui f tdbmbs qspevdu)4/26* x f bssjwf bu

$$S(\bar s|\bar t)=\sum\frac{\prod_{\xi=2}^{N}\hat\gamma_\xi(\bar s^\xi_{\rm J})\hat\gamma_\xi(\bar t^\xi_{\rm JJ})\delta_\xi(\bar s^\xi_{\rm J},\bar s^\xi_{\rm JJ})\delta_\xi(\bar t^\xi_{\rm JJ},\bar t^\xi_{\rm J})}{\prod_{\xi=2}^{N\cdot\ 2}f_{[\xi+2]}(\bar s^{\xi+2}_{\rm J},\bar s^\xi_{\rm JJ})f_{[\xi+2]}(\bar t^{\xi+2}_{\rm JJ},\bar t^\xi_{\rm J})}Z^{m|n}(\bar s_{\rm J}|\bar t_{\rm J})\,Z^{m|n}(\bar t_{\rm JJ}|\bar s_{\rm JJ}). \qquad)7/3*$$

Opuf ui bu ui f qspevdu pg ui f tjho gbdupst $(\cdot\ 2)^{\lambda_{\xi,m}(r_m\cdot\ 2)}$ hjwft 2- cfdbvtf ’$\bar s^m_{\rm J}$ + ’$\bar t^m_{\rm JJ}=r_m$/

Mfu $s^\nu_j\in\bar s_{\rm J}$ boe $t^\nu_j\in\bar t_{\rm J}$/ Xf efopuf ui f dpssftqpoejoh dpousjcvujpo up ui f tdbmbs qspevdu cz $S^{(2)}(\bar s|\bar t)$/ Jg $s^\nu_j\to t^\nu_j$- ui fo evf up)4/28* ui f I D $Z^{m|n}(\bar s_{\rm J}|\bar t_{\rm J})$ i bt b qpmf/ Mfu $\bar s^\nu_{\rm J}=\{s^\nu_j,\bar s^\nu_{\rm J'}\}$- $\bar t^\nu_{\rm J}=\{t^\nu_j,\bar t^\nu_{\rm J'}\}$- boe $\bar s^\xi_{\rm J}=\bar s^\xi_{\rm J'}$- $\bar t^\xi_{\rm J}=\bar t^\xi_{\rm J'}$ gps $\xi\neq\nu$/ Ui fo vtjoh)4/28* x f pcubjo

$$Z^{m|n}(\bar s_{\rm J}|\bar t_{\rm J})\Big|_{s^\nu_j\to t^\nu_j}=g_{[\nu+2]}(t^\nu_j,s^\nu_j)\frac{\delta_\nu(\bar t^\nu_{\rm J'},t^\nu_j)\delta_\nu(s^\nu_j,\bar s^\nu_{\rm J'})}{f_{[\nu+2]}(\bar t^{\nu+2}_{\rm J},t^\nu_j)f_{[\nu]}(s^\nu_j,\bar s^{\nu\cdot\ 2}_{\rm J})}Z^{m|n}(\bar s_{\rm J'}\mid\bar t_{\rm J'})+reg, \qquad)7/4*$$

x i fsf reg n fbot sfhvmbs qbsu/

Ui f qspevdu pg ui f f.gvodujpot boe δ.gvodujpot jo)7/3* usbotgpsn t bt gpmmpx t;

$$\frac{\prod_{\xi=2}^{N}\delta_\xi(\bar s^\xi_{\rm J},\bar s^\xi_{\rm JJ})\delta_\xi(\bar t^\xi_{\rm JJ},\bar t^\xi_{\rm J})}{\prod_{\xi=2}^{N\cdot\ 2}f_{[\xi+2]}(\bar s^{\xi+2}_{\rm J},\bar s^\xi_{\rm JJ})f_{[\xi+2]}(\bar t^{\xi+2}_{\rm JJ},\bar t^\xi_{\rm J})}=\frac{\delta_\nu(s^\nu_j,\bar s^\nu_{\rm JJ})\delta_\nu(\bar t^\nu_{\rm JJ},t^\nu_j)}{f_{[\nu]}(s^\nu_j,\bar s^{\nu\cdot\ 2}_{\rm JJ})f_{[\nu+2]}(\bar t^{\nu+2}_{\rm JJ},t^\nu_j)}$$
$$*\ \frac{\prod_{\xi=2}^{N}\delta_\xi(\bar s^\xi_{\rm J'},\bar s^\xi_{\rm JJ})\delta_\xi(\bar t^\xi_{\rm JJ},\bar t^\xi_{\rm J'})}{\prod_{\xi=2}^{N\cdot\ 2}f_{[\xi+2]}(\bar s^{\xi+2}_{\rm J'},\bar s^\xi_{\rm JJ})f_{[\xi+2]}(\bar t^{\xi+2}_{\rm JJ},\bar t^\xi_{\rm J'})}. \qquad)7/5*$$

Dpn cjojoh)7/4* boe)7/5* x f pcubjo gps ui f dpousjcvujpo $S^{(2)}(\bar s|\bar t)$

$$S^{(2)}(\bar s|\bar t)\Big|_{s^\nu_j\to t^\nu_j}=\hat\gamma_\nu(s^\nu_j)g_{[\nu+2]}(t^\nu_j,s^\nu_j)\frac{\delta_\nu(\bar t^\nu_j,t^\nu_j)\delta_\nu(s^\nu_j,\bar s^\nu_j)}{f_{[\nu]}(s^\nu_j,\bar s^{\nu\cdot\ 2})f_{[\nu+2]}(\bar t^{\nu+2},t^\nu_j)}$$
$$*\ \sum\frac{\prod_{\xi=2}^{N}\hat\gamma_\xi(\bar s^\xi_{\rm J'})\hat\gamma_\xi(\bar t^\xi_{\rm JJ})\delta_\xi(\bar s^\xi_{\rm J'},\bar s^\xi_{\rm JJ})\delta_\xi(\bar t^\xi_{\rm JJ},\bar t^\xi_{\rm J'})}{\prod_{\xi=2}^{N\cdot\ 2}f_{[\xi+2]}(\bar s^{\xi+2}_{\rm J'},\bar s^\xi_{\rm JJ})f_{[\xi+2]}(\bar t^{\xi+2}_{\rm JJ},\bar t^\xi_{\rm J'})}Z^{m|n}(\bar s_{\rm J'}|\bar t_{\rm J'})\,Z^{m|n}(\bar t_{\rm JJ}|\bar s_{\rm JJ})+reg, \qquad)7/6*$$

x i fsf opx ui f tvn jt ublfo pwfs qbsujujpot pg ui f tfut $\bar t\setminus\{t^\nu_j\}$ boe $\bar s\setminus\{s^\nu_j\}$ sftqfdujwfmz jop tvctfut $\{\bar s_{\rm J'},\bar s_{\rm JJ}\}$ boe $\{\bar t_{\rm J'},\bar t_{\rm JJ}\}$/ Sfdbmm bmtp ui bu $\bar s^\nu_j=\bar s^\nu\setminus\{s^\nu_j\}$ boe $\bar t^\nu_j=\bar t^\nu\setminus\{t^\nu_j\}$/

Tjn jmbsmz pof dbo dpotjefs ui f dbtf $s^\nu_j\in\bar s_{\rm JJ}$ boe $t^\nu_j\in\bar t_{\rm JJ}$/ Efopujoh ui f dpssftqpoejoh dpousj. cvujpo cz $S^{(3)}(\bar s|\bar t)$ x f floe

$$S^{(3)}(\bar s|\bar t)\Big|_{s_j^\nu\to t_j^\nu} = \hat\gamma_\nu(t_j^\nu)g_{[\nu+2]}(s_j^\nu,t_j^\nu)\frac{\delta_\nu(\bar s_j^\nu,s_j^\nu)\delta_\nu(t_j^\nu,\bar t_j^\nu)}{f_{[\nu]}(t_j^\nu,\bar t^{\nu\cdot 2})f_{[\nu+2]}(\bar s^{\nu+2},s_j^\nu)}$$
$$* \sum\frac{\prod_{\xi=2}^{N}\hat\gamma_\xi(\bar s_{\rm J}^\xi)\hat\gamma_\xi(\bar t_{\rm JJ'}^\xi)\delta_\xi(\bar s_{\rm J}^\xi,\bar s_{\rm JJ'}^\xi)\delta_\xi(\bar t_{\rm JJ'}^\xi,\bar t_{\rm J}^\xi)}{\prod_{\xi=2}^{N\cdot 2}f_{[\xi+2]}(\bar s_{\rm J}^{\xi+2},\bar s_{\rm JJ'}^\xi)f_{[\xi+2]}(\bar t_{\rm JJ'}^{\xi+2},\bar t_{\rm J}^\xi)}Z^{m|n}(\bar s_{\rm J}|\bar t_{\rm J})\,Z^{m|n}(\bar t_{\rm JJ'}|\bar s_{\rm JJ'}) + reg. \qquad)7/7*$$

I fsf ui f tvn jt ubl fo pwfs qbsujujpot pg ui f tfut $\bar t\setminus\{t_j^\nu\}$ boe $\bar s\setminus\{s_j^\nu\}$ sftqfdujwfmz joup tvctfut $\{\bar s_{\rm J},\bar s_{\rm JJ'}\}$ boe $\{\bar t_{\rm J},\bar t_{\rm JJ'}\}$/

Opx x f dpn cjof)7/6* boe)7/7*/ Sfmbcfmjoh ui f tvctdsjqut pg tvctfut $\rm J'\to J$- $\rm JJ'\to JJ$ boe tvctujuvujoh $\hat\gamma(s_j^\nu)$ boe $\hat\gamma(t_j^\nu)$ sftqfdujwfmz jo ufsn t pg $\gamma(s_j^\nu)$ boe $\gamma(t_j^\nu)$ x f bssjwf bu

$$S(\bar s|\bar t)\Big|_{s_j^\nu\to t_j^\nu} = g_{[\nu+2]}(t_j^\nu,s_j^\nu)\Big)\gamma_\nu(s_j^\nu)\cdot\ \gamma_\nu(t_j^\nu)\Big[\frac{(\cdot\ 2)^{\lambda_{\nu,m}(r_m\cdot\ 2)}\delta_\nu(\bar s_j^\nu,s_j^\nu)\delta_\nu(\bar t_j^\nu,t_j^\nu)}{f_{[\nu+2]}(\bar s^{\nu+2},s_j^\nu)f_{[\nu+2]}(\bar t^{\nu+2},t_j^\nu)}$$
$$* \sum\frac{\prod_{\xi=2}^{N}\hat\gamma_\xi(\bar s_{\rm J}^\xi)\hat\gamma_\xi(\bar t_{\rm JJ}^\xi)\delta_\xi(\bar s_{\rm J}^\xi,\bar s_{\rm JJ}^\xi)\delta_\xi(\bar t_{\rm JJ}^\xi,\bar t_{\rm J}^\xi)}{\prod_{\xi=2}^{N\cdot 2}f_{[\xi+2]}(\bar s_{\rm J}^{\xi+2},\bar s_{\rm JJ}^\xi)f_{[\xi+2]}(\bar t_{\rm JJ}^{\xi+2},\bar t_{\rm J}^\xi)}Z^{m|n}(\bar s_{\rm J}|\bar t_{\rm J})\,Z^{m|n}(\bar t_{\rm JJ}|\bar s_{\rm JJ}) + \tilde S. \qquad)7/8*$$

I fsf $\tilde S$ efopuft ui f ufsn t ui buefqfoe po ui f gvodujpo $\gamma_\nu(t_j^\nu)$ cvuopupo jut efsjwbujwf/ Ui f tvn jt ubl fo pwfs qbsujujpot pg ui f tfut $\bar t\setminus\{t_j^\nu\}$ boe $\bar s\setminus\{s_j^\nu\}$ sftqfdujwfmz joup tvctfut $\{\bar s_{\rm J},\bar s_{\rm JJ}\}$ boe $\{\bar t_{\rm J},\bar t_{\rm JJ}\}$/

Ui fo qfsgpsn joh ui f mjn ju $s_j^\nu\to t_j^\nu$ jo)7/8* x f pcubjo

$$S(\bar s|\bar t)\Big|_{s_j^\nu=t_j^\nu} = (\cdot\ 2)^{\lambda_{\nu,m}(r_m\cdot\ 2)}\frac{X_j^\nu\gamma_\nu(t_j^\nu)\delta_\nu(\bar s_j^\nu,t_j^\nu)\delta_\nu(\bar t_j^\nu,t_j^\nu)}{f_{[\nu+2]}(\bar s^{\nu+2},t_j^\nu)f_{[\nu+2]}(\bar t^{\nu+2},t_j^\nu)}$$
$$* \sum\frac{\prod_{\xi=2}^{N}\hat\gamma_\xi(\bar s_{\rm J}^\xi)\hat\gamma_\xi(\bar t_{\rm JJ}^\xi)\delta_\xi(\bar s_{\rm J}^\xi,\bar s_{\rm JJ}^\xi)\delta_\xi(\bar t_{\rm JJ}^\xi,\bar t_{\rm J}^\xi)}{\prod_{\xi=2}^{N\cdot 2}f_{[\xi+2]}(\bar s_{\rm J}^{\xi+2},\bar s_{\rm JJ}^\xi)f_{[\xi+2]}(\bar t_{\rm JJ}^{\xi+2},\bar t_{\rm J}^\xi)}Z^{m|n}(\bar s_{\rm J}|\bar t_{\rm J})\,Z^{m|n}(\bar t_{\rm JJ}|\bar s_{\rm JJ}) + \tilde S, \qquad)7/9*$$

x i fsf X_j^ν jt efflofe cz)5/7* boe $\tilde S$ epft opuefqfoe po X_j^ν/

Pof n jhi ui bwf ui f jn qsfttjpo ui buui f tvn pwfs qbsujujpot jo ui f tfdpoe mjof pg)7/9* hjwft ui f tdbmbs qspevdu $S(\bar s\setminus\{s_j^\nu\}\,|\,\bar t\setminus\{t_j^\nu\})$/ Ui jt jt opufybdumz tp-cfdbvtf ui f gvodujpot $\hat\gamma_\nu$ boe $\hat\gamma_{\nu\oplus 2}$ tujmm efqfoe po t_j^ν)tff)7/2**/ I px fwfs- x f dbo hfusje pg ui jt efqfoefodf jg x f joupspevdf n pejflfe gvodujpobmqbsbn fufst $\gamma_\xi^{(\rm n\,pe)}$/ Obn fmz- gps ν flyfe x f tfu $\gamma_\xi^{(\rm n\,pe)}(z) = \gamma_\xi(z)$- jg $|\xi\cdot\ \nu|>2$- boe

$$\gamma_\nu^{(\rm n\,pe)}(z) = (\cdot\ 2)^{\lambda_{\nu,m}}\gamma_\nu(z)\frac{\delta_\nu(t_j^\nu,z)}{\delta_\nu(z,t_j^\nu)},$$
$$\gamma_{\nu+2}^{(\rm n\,pe)}(z) = \gamma_{\nu+2}(z)f_{[\nu+2]}(z,t_j^\nu), \qquad)7/:*$$
$$\gamma_{\nu\cdot\ 2}^{(\rm n\,pe)}(z) = \frac{\gamma_{\nu\cdot\ 2}(z)}{f_{[\nu]}(t_j^\nu,z)}.$$

Ui fo- tvctujuvujoh $\hat\gamma_\xi$ jo)7/9* jo ufsn t pg $\gamma_\xi^{(\rm n\,pe)}$ x f pcubjo

$$S(\bar s|\bar t)\Big|_{s_j^\nu=t_j^\nu} = (\cdot\ 2)^{\lambda_{\nu,m}(r_m\cdot\ 2)}\frac{X_j^\nu\gamma_\nu(t_j^\nu)\delta_\nu(\bar s_j^\nu,t_j^\nu)\delta_\nu(\bar t_j^\nu,t_j^\nu)}{f_{[\nu+2]}(\bar s^{\nu+2},t_j^\nu)f_{[\nu+2]}(\bar t^{\nu+2},t_j^\nu)}$$
$$* \sum\frac{\prod_{\xi=2}^{N}\gamma_\xi^{(\rm n\,pe)}(\bar s_{\rm J}^\xi)\gamma_\xi^{(\rm n\,pe)}(\bar t_{\rm JJ}^\xi)\delta_\xi(\bar s_{\rm JJ}^\xi,\bar s_{\rm J}^\xi)\delta_\xi(\bar t_{\rm J}^\xi,\bar t_{\rm JJ}^\xi)}{\prod_{j=2}^{N\cdot 2}f_{[j+2]}(\bar s_{\rm JJ}^{j+2},\bar s_{\rm J}^j)f_{[j+2]}(\bar t_{\rm J}^{j+2},\bar t_{\rm JJ}^j)}Z^{m|n}(\bar s_{\rm J}|\bar t_{\rm J})\ \ Z^{m|n}(\bar t_{\rm JJ}|\bar s_{\rm JJ}) + \tilde S.$$

)7/21*

Ui f tvn pwfs qbsujujpot jo)7/21* hjwft ui f tdbmbs qspevdu $\mathbb{C}(\bar{s}\setminus\{s_j^\nu\})\mathbb{B}(\bar{t}\setminus\{t_j^\nu\})$ jo b ofx sfq. sftfoubujwf pg ui f hfofsbmjfife n pefm-jo x i jdi ui f γ.gvodujpot bsf n pejflfe bddpsejoh up)7/: */ Ui vt-x f bssjwf bu

$$S(\bar{s}|\bar{t})\Big|_{s_j^\nu=t_j^\nu} = (\cdot\ 2)^{\lambda_{\nu,m}(r_m\cdot\ 2)}\frac{X_j^\nu\gamma_\nu(t_j^\nu)\delta_\nu(\bar{s}_j^\nu,t_j^\nu)\delta_\nu(\bar{t}_j^\nu,t_j^\nu)}{f_{[\nu+2]}(\bar{s}^{\nu+2},t_j^\nu)f_{[\nu+2]}(\bar{t}^{\nu+2},t_j^\nu)}S^{(\text{n pe})}(\bar{s}\setminus\{s_j^\nu\}\mid\bar{t}\setminus\{t_j^\nu\}) + \tilde{S}, \qquad)7/22*$$

x i fsf ui f n pejflfdbujpo pg ui f tdbmbs qspevdu n fbot ui bu opx x f ti pvme vtf ui f n pejflfe γ.gvodujpot)7/: */

Ui vt- x f dpodmvef ui bu jg $s_j^\nu = t_j^\nu$- ui fo ui f tdbmbs qspevdu mjofbsmz efqfoet po ui f mphb. sjui n jd efsjwbujwf X_j^ν/ Ui f dpfgfldjfou pg X_j^ν jt qspqpsujpobm up ui f n pejflfe tdbmbs qspevdu $\mathbb{C}(\bar{s}\setminus\{s_j^\nu\})\mathbb{B}(\bar{t}\setminus\{t_j^\nu\})$ jo b ofx sfqsftfoubujwf pg ui f hfofsbmjfife n pefm/

81 Ppsn pg po.ti fmmCfui f wfdupst

Ju x bt bmsfbez ejtdvttfe ui bu gps $\bar{t}=\bar{s}$ ui f tdbmbs qspevdu efqfoet po ui f Cfui f qbsbn fufst t_j^ξ- ui f gvodujpobm qbsbn fufst $\gamma_\xi(t_j^\xi)$- boe ui f mphbsjui n jd efsjwbujwft X_j^ξ)5/7*/ Jo ui f dbtf pg ui f opsn pg po.ti fmmCfui f wfdupst ui f gvodujpot γ_ξ bsf sfmbufe up ui f qbsbn fufst $\bar{t}$ wjb Cfui f fr vbujpot)4/22*/ Ui fsfgpsf- ui f opsn pg bo po.ti fmmCfui f wfdups jt b gvodujpo pg ui f Cfui f qbsbn fufst t_j^ξ boe ui f qbsbn fufst X_j^ξ/

Mfu

$$\mathbf{P}^{(s)}(\bar{X};\bar{t}) = \prod_{\xi=2}^{N}\prod_{\substack{p,q=2\\p\neq q}}^{r_\xi}\delta_\xi(t_p^\xi,t_q^\xi)\Bigg)^{\cdot\ 2}\prod_{\xi=2}^{N\cdot\ 2}f_{[\xi+2]}(\bar{t}^{\xi+2},\bar{t}^\xi)\lim_{\bar{s}\to\bar{t}}\mathbb{C}(\bar{s})\mathbb{B}(\bar{t}), \qquad)8/2*$$

x i fsf $\mathbb{B}(\bar{t})$ jt po.ti fmm/

Nfn n b 8/2/ *Ui f gvodujpo* $\mathbf{P}^{(s)}(\bar{X};\bar{t})$ *gvmflmt ui f Lpsfqjo dsjufsjb/*

Rsppg/ Qspqfsujft)j*)jj* bsf r vjuf pcwjpvt/ Qspqfsuz)jjj* gpmmpx t gspn b ejsfdu dbmdvmbujpo/ Jg pomz pof Cfui f qbsbn fufs pg ui f dpmps 2 jt jowpmwfe- ui fo ui f Cfui f wfdups boe ui f evbm Cfui f wfdups i bwf sftqfdujwfmz ui f gpmmpx joh gpsn)tff]33‘*

$$\mathbb{B}(t_2^2) = \frac{T_{2,3}(t_2^2)}{\mu_3(t_2^2)}|1\rangle; \qquad \mathbb{C}(t_2^2) = \langle 1|\frac{T_{3,2}(t_2^2)}{\mu_3(t_2^2)}. \qquad)8/3*$$

Vtjoh dpn n vubujpo sfmbujpot)3/6* x f jn n fejbufmz pcubjo

$$\mathbb{C}(s)\mathbb{B}(t) = \frac{\langle 1|T_{3,2}(s)T_{2,3}(t)|1\rangle}{\mu_3(s)\mu_3(t)} = (\cdot\ 2)^{[3]}g(s,t)\big)\gamma_2(t)\cdot\ \gamma_2(s)\big\{. \qquad)8/4*$$

Tfujoh i fsf $s=t=t_2^2$ x f floe

$$\mathbb{C}(t_2^2)\mathbb{B}(t_2^2) = \gamma_2(t_2^2)X_2^2, \qquad)8/5*$$

boe flobmmz- vtjoh ui f Cfui f fr vbujpo $\gamma_2(t_2^2)=2$ x f bssjwf bu qspqfsuz)jjj*/

Ui f sfdvstjpo)5/22* boe ui f n pejflfdbujpo)5/23* gpmmpx gspn ui f dpotjefsbujpot pg ui f qsfwj. pvt tfdujpo/ Joeffe- ubljoh ui f mjn ju $\bar{s}\to\bar{t}$ jo)7/22* x f floe

$$\frac{\partial}{\partial X_j^\nu}\underset{\bar s\to\bar t}{\text{rjn}}\, S(\bar s|\bar t) = (\cdot\ 2)^{\lambda_{\nu,m}(r_m\cdot\ 2)}\gamma_\nu(t_j^\nu)\ \left(\frac{\delta_\nu(\bar t_j^\nu, t_j^\nu)}{f_{[\nu+2]}(\bar t^{\nu+2}, t_j^\nu)}\right)^3 \underset{\bar s\to\bar t}{\text{rjn}}\, S^{(\text{n pe})}(\bar s\setminus\{s_j^\nu\}\,|\,\bar t\setminus\{t_j^\nu\}).$$
)8/6*

Tvctujuvujoh i fsf $\gamma_\nu(t_j^\nu)$ gspn ui f Cfui f fr vbujpot)4/22*x f i bwf

$$\frac{\partial}{\partial X_j^\nu}\underset{\bar s\to\bar t}{\text{rjn}}\, S(\bar s|\bar t) = \frac{\delta_\nu(\bar t_j^\nu, t_j^\nu)\delta_\nu(t_j^\nu, \bar t_j^\nu)}{f_{[\nu+2]}(\bar t^{\nu+2}, t_j^\nu) f_{[\nu]}(t_j^\nu, \bar t^{\nu\cdot\ 2})}\underset{\bar s\to\bar t}{\text{rjn}}\, S^{(\text{n pe})}(\bar s\setminus\{s_j^\nu\}\,|\,\bar t\setminus\{t_j^\nu\}). \qquad)8/7*$$

Ui vt-ui f dpfgfldjfoupg $\partial S/\partial X_j^\nu$ jt qspqpsujpobmup ui f opsn pgui f Cfui f wfdups pgb ofx sfqsf. tfoubujwf pgui f hfofsbrjfife n pefm/Jo ui jt sfqsftfoubujwf ui f gvodujpobmqbsbn fufst γ_ξ ti pvme cf n pejflfe bddpsejoh up)7/: */ Pcwjpvtm- ui jt n pejfldbujpo jn qrjft ui f n pejfldbujpo)5/23*pgui f qbsbn fufst X_k^ν/

Sfn bsl bcm- ui f ofx wfdups jt tujmpo.ti fm/Joeffe- jujt fbtz up tff ui buui f gvodujpobmqb. sbn fufst $\gamma_\xi^{(\text{n pe})}$ dbo cf fyqsfttfe jo ufsn t pgui f Cfui f qbsbn fufst $\bar t\setminus\{t_j^\nu\}$ wjb Cfui f fr vbujpot/ Jo qbsujdvrbs-

$$\gamma_\nu^{(\text{n pe})}(t_k^\nu) = (\cdot\ 2)^{\lambda_{\nu,m}(r_m\cdot\ 3)}\frac{\delta_\nu(t_k^\nu, \bar t_{k,j}^\nu) f_{[\nu+2]}(\bar t^{\nu+2}, t_k^\nu)}{\delta_\nu(\bar t_{k,j}^\nu, t_k^\nu) f_{[\nu]}(t_k^\nu, \bar t^{\nu\cdot\ 2})}, \qquad)8/8*$$

x i fsf x f jouspevdfe $\bar t_{k,j}^\nu = \bar t^\nu\setminus\{t_j^\nu, t_k^\nu\}$/ Pctfswf ui bujg $\nu = m$- ui fo $'\bar t_j^\nu = '\bar s_j^\nu = r_m\cdot\ 2$- ui fsfgpsf ui f tjho gbdups jo)8/8*di bohft/ Xf bmtp i bwf

$$\gamma_{\nu+2}^{(\text{n pe})}(t_k^{\nu+2}) = (\cdot\ 2)^{\lambda_{\nu+2,m}(r_m\cdot\ 2)}\frac{\delta_{\nu+2}(t_k^{\nu+2}, \bar t_k^{\nu+2}) f_{[\nu+3]}(\bar t^{\nu+3}, t_k^{\nu+2})}{\delta_{\nu+2}(\bar t_k^{\nu+2}, t_k^{\nu+2}) f_{[\nu+2]}(t_k^{\nu+2}, \bar t_j^\nu)},$$
$$\gamma_{\nu\cdot\ 2}^{(\text{n pe})}(t_k^{\nu\cdot\ 2}) = (\cdot\ 2)^{\lambda_{\nu\cdot\ 2,m}(r_m\cdot\ 2)}\frac{\delta_{\nu\cdot\ 2}(t_k^{\nu\cdot\ 2}, \bar t_k^{\nu\cdot\ 2}) f_{[\nu]}(\bar t_j^\nu, t_k^{\nu\cdot\ 2})}{\delta_{\nu\cdot\ 2}(\bar t_k^{\nu\cdot\ 2}, t_k^{\nu\cdot\ 2}) f_{[\nu\cdot\ 2]}(t_k^{\nu\cdot\ 2}, \bar t^{\nu\cdot\ 3})}. \qquad)8/9*$$

Ui f pui fs Cfui f fr vbujpot gps $\gamma_\xi^{(\text{n pe})}$ x jui $|\xi\cdot\ \nu| > 2$ ep opudi bohf/ Ui vt- x f bssjwf buui f qspqfsuz)jw*gps ui f gvodujpo $\mathbf{P}^{(\mathbf{s})}(\bar X;\bar t)$/

Gjobmm- qspqfsuz)w* dbo cf efevdfe bt gpmpx t/ Tjodf bmmui f qprfit pgui f I D jo)4/26*bsf tjn qrfi- jujt fopvhi up efwfmpq gvodujpot $\gamma_\xi(s_j^\xi)$ vq up ui f flstupsefs pwfs ui f ejggfsfodf $s_j^\xi\cdot\ t_j^\xi$ gps ubl joh ui f rjn ju$\bar s\to\bar t$;

$$\gamma_\xi(s_j^\xi) = \gamma_\xi(t_j^\xi) + (s_j^\xi\cdot\ t_j^\xi)\frac{d\gamma_\xi(z)}{dz}\bigg|_{z=t_j^\xi} + O\Big)(s_j^\xi\cdot\ t_j^\xi)^3\Big[. \qquad)8/:*$$

Jgbmm$X_j^\xi = 1$-ui fo ui f efsjwbujwft pgγ_ξ wbojti -boe x f dbo tvctujuvuf $\gamma_\xi(s_j^\xi) = \gamma_\xi(t_j^\xi)$ joup)4/26* jo ui f rjn ju$\bar s\to\bar t$/ Ui jt rfibet vt up

$$\underset{\bar s\to\bar t}{\text{rjn}}\, S(\bar s|\bar t) = \prod_{\xi=2}^{N}\gamma_\xi(\bar t^\xi)\underset{\bar s\to\bar t}{\text{rjn}}\sum\frac{\prod_{\xi=2}^{N}\delta_\xi(\bar s_{\text{JJ}}^\xi, \bar s_{\text{J}}^\xi)\delta_\xi(\bar t_{\text{J}}^\xi, \bar t_{\text{JJ}}^\xi)}{\prod_{\xi=2}^{N\cdot\ 2} f_{[\xi+2]}(\bar s_{\text{JJ}}^{\xi+2}, \bar s_{\text{J}}^\xi) f_{[\xi+2]}(\bar t_{\text{J}}^{\xi+2}, \bar t_{\text{JJ}}^\xi)}$$
$$*\ Z^{m|n}(\bar s_{\text{J}}|\bar t_{\text{J}})\ Z^{m|n}(\bar t_{\text{JJ}}|\bar s_{\text{JJ}}). \qquad)8/21*$$

I px fwfs-evf up)6/3*ui f tvn pwfs qbsujujpot jo)8/21*wbojti ft gps bscjusbsz dpn qrfiy $\bar s$ boe $\bar t$/ Jo ui jt x bz x f bssjwf buui f qspqfsuz)w*/ □

Evf up Qspqptjujpo 5/2 xf dpodmvef uibu

$$\mathbf{P}^{(\mathbf{s})}(\bar{X};\bar{t}) = \mathrm{efu}G, \qquad)8/22*$$

mfbejoh up)5/5*/

A1 Epodmvtjpo

Xf dpotjefsfe b hfofsbmjafe rvbouvn jouhsbcmf n pefm xjui $\mathfrak{gl}(m|n)$.jowbsjbou R.n busjy/ Xf tipxfe uibu uif trvbsf pg uif opsn pg po.tifmm Cfuif wfdupst pg uijt n pefm jt qspqpsujpobm up b Kbdpcjbo pg uif tztufn pg Cfuif frvbujpot/ Uijt sftvmu dpn qmfufmz n budift uif psjhjobm Hbvejo izqpuiftjt po uif opsn pg uif Ibn jmupojbo fjhfowfdups/ Pof dbo fyqfdu uibu uijt izqpuiftjt dbo cf gvsuifs hfofsbmjafe/ Jo qbsujdvmbs- ju jt rvjuf obuvsbmmz ibwf b tjn jmbs gpsn vmb gps uif n pefmt cbtfe po $U_q(\widehat{\mathfrak{gl}}(m))$ boe $U_q(\widehat{\mathfrak{gl}}(m|n))$ bmhfcsbt/ Uijt xjmm cf uif tvckfdu pg pvs gvsuifs qvcmjdbujpot/

Uif qspcmfn pg uif opsn pg po.tifmm Cfuif wfdupst jt wfsz jn qpsubou gps uif dbmdvmbujpo pg gpsn gbdupst boe dpssfmbujpo gvodujpot jo uif n pefmt pg qiztjdbm jouf sftu/ Gvsuifs efwfmpqn foujo uijt ejsfdujpo sfrvjsft n psf efubjmfe bobmztjt pg uif Cfuif wfdupst tdbmbs qspevdut/ Gpsn bmmz- uif tvn gpsn vmb hjwft bo fyqmjdju sfqsftfoubujpo gps uif tdbmbs qspevdu pg hfofsjd Cfuif wfdupst- ipxfwfs- uijt sfqsftfoubujpo jt opu dpowfojfou gps bqqmjdbujpot jo n boz dbtft/ Bu uif tbn f ujn f- pof dbo ipqf up gjoe n psf dpn qbdu sfqsftfoubujpot gps qbsujdvmbs dbtft pg uif tdbmbs qspevdu- bt ju xbt epof jo uif n pefm xjui $\mathfrak{gl}(3|2)$.tzn n fusz]37‘/ Bu qsftfou- xpsl jo uijt ejsfdujpo jt voefs xbz/

Bdl opxmfehfn fout

Uif xpsl pg B/M/ ibt cffo gvoefe cz Svttjbo Bdbefn jd Fydfmmfodf Qspkfdu 6.211- cz Zpvoh Svttjbo Nbuifn bujdt bxbse boe cz kpjou OBTV.DOST qspkfdu G25.3128/ Uif xpsl pg T/Q/ xbt tvqqpsufe jo qbsu cz uif SGCS hsbou 27.12.11673.b/

Bqqfoejy B1 $Y(\mathfrak{gl}(m|n))$ sfqsftfoubujpot joevdfe gspn $\mathfrak{gl}(m|n)$ poft

B xjef dmbtt pg sfqsftfoubujpot gps uif Zbohjbo $Y\big(\mathfrak{gl}(m|n)\big)$ dbo cf dpotusvdufe gspn sfqsftfo. ubujpot pg $\mathfrak{gl}(m|n)$/ Uif dpotusvdujpo sfmjft po uif opujpo pg fwbmvbujpo n psqijtn boe fwbmvbujpo sfqsftfoubujpot]38-39‘/ Cfgpsf efubjmjoh ju- xf n bl f b tipsu tvn n bsz po jssfevdjcmf sfqsftfoub. ujpot pg $\mathfrak{gl}(m|n)$/

B/2/ I jhiftu xfjhiu sfqsftfoubujpot pg uif Mjf tvqfsbmhfcsb $\mathfrak{gl}(m|n)$

Gps tjn qmjdjuz- xf qsftfou ijhiftu xfjhiu sfqsftfoubujpot gps uif Mjf tvqfsbmhfcsb $\mathfrak{gl}(m|n)$ xjui $m \neq n$- cvu n ptu pg uif ejtdvttjpo bqqmjft bmtp up uif dbtf $m = n$/ I jhiftu xfjhiu sfqsf. tfoubujpot xfsf tuvejfe jo]3: -41‘- tff bmtp]42‘ gps b sfwjfx po tvqfsbmhfcsbt/ Xf jouspevdf uif $\mathfrak{gl}(m|n)$ hfofsbupst e_{ij} pcfzjoh

$$[\mathrm{e}_{ij}, \mathrm{e}_{kl}\} = \lambda_{kj}\,\mathrm{e}_{il} \cdot (\cdot\ 2)^{([i]+[j])([k]+[l])}\,\lambda_{il}\,\mathrm{e}_{kj}. \qquad)B/2*$$

I jhiftu xfjhiu sfqsftfoubujpot pg uif Mjf tvqfsbmhfcsb $\mathfrak{gl}(m|n)$ bsf di bsbdufsjafe cz b xfjhiu $= (\mu_2, ..., \mu_{m+n}) \in \mathbf{E}^{m+n}$ boe b ijhiftu xfjhiu wfdups $|1\rangle$ tvdi uibu

$$e_{ii}|1\rangle = \mu_i|1\rangle \quad \text{boe} \quad e_{ij}|1\rangle = 1, \quad i < j, \qquad)B/3*$$

x i fsf e_{ij} bsf ui f sfqsftfoubujwft pg ui f $\mathfrak{gl}(m|n)$ hfofsbupst/ Ui f i jhi ftu x fjhi u wfdups $|1\rangle$ x jmm qspevdf ui f qtfvepwbdvvn)4/2* ui spvhi ui f fwbmvbujpo n psqi jtn - tff tfdujpo B/3 cfmpx/ Jo pui fs x pset- jg ρ efopuft ui f n bqqjoh gspn ui f tvqfsbmhfcsb up b sfqsftfoubujpo tqbdf $\mathcal{V}$ - ui fo $e_{ij} = \rho$ (e_{ij}) jt b n busjy)ps bo pqfsbups gps joflojuf ejn fotjpobmsfqsftfoubujpot*bdujoh po wfdupst jo $\mathcal{V}$ / Ui f bttpdjbufe Lbd n pevmf jt pcubjofe ui spvhi ui f)n vmujqmf* bqqmjdbujpot pg ui f sfqsftfoubujwft e_{ij}- $i > j$- po $|1\rangle$/

Bn poh i jhi ftu x fjhi u sfqsftfoubujpot- ui f flojuf ejn fotjpobmpoft bsf di bsbdufsjfife[3] cz *ko. uf hsbcmf epn koboux f khi ut*- tvdi ui bu

$$\mu_i \cdot \mu_{i+2} \in \mathbf{a}_+,\ i \neq m, \quad 2 \sim i \sim m+n \cdot 2 \quad \text{boe} \quad \mu_m \in \mathbf{T}.$$

P cwjpvtmz boz x fjhi u jt b mjofbs dpn cjobujpo pg ui f gvoebn fobm)epn jobou* x fjhi ut[4]

$$^{(i)} = (\underbrace{2, ..., 2}_{i}, \underbrace{1, ..., 1}_{m+n \cdot i}), \qquad i = 2, ..., m+n.$$

Gps jouf hsbcmf epn jobou x fjhi ut- ui f mjofbs dpn cjobujpo i bt opo.ofhbujwf jouf hfs dpfgfldjfout- vq up uxp sfbmovn cfst/ Ui f flstu dpssftqpoet up ui f gfsn jpojd sppu- j/f/ up μ_m/ Ui f tfdpoe jt bttpdjbufe up ui f fjhfowbmvf pg ui f $\mathfrak{gl}(2)$ qbsu ui bu ejtujohvjti ft $\mathfrak{gl}(m|n)$ gspn jut tjn qmf qbsu $\mathfrak{sl}(m|n)$/ Judbo cf sfmbufe up ui f x fjhi u $^{(m+n)}$/

Ui f sfqsftfoubujpot bttpdjbufe up gvoebn fobmx fjhi ut bsf dbmmfe gvoebn fobmsfqsftfoub. ujpot/ Ui fsf bsf $m+n \cdot 2$ pg ui fn - boe ui f flstupof- $^{(2)}$ dpssftqpoet up x i bu jt vtvbmmz dbmmfe ui f gvoebn fobmsfqsftfoubujpo/ Jujt $(m+n)$.ejn fotjpobm boe jo ui bu dbtf $\rho_{(2)}(\mathsf{e}_{ij}) = E_{ij}$/ Jut dpoukbhsfejfou sfqsftfoubujpo)x i jdi jt bmtp $(m+n)$.ejn fotjpobm* dpssftqpoet up $^{(m+n \cdot 2)}$/

B/3/ Fwbmvbujpo n bq

Ui f *fwbmvbujpo n psqi jtn* $ev(\pi)$- gps $\pi \in \mathbf{E}$- jt bo bmhfcsb n psqi jtn gspn $Y\big)\mathfrak{gl}(m|n)\big\{$ up $U(\mathfrak{gl}(m|n))$- ui f fowfmpqjoh bmhfcsb pg $\mathfrak{gl}(m|n)$/ Jujt efflofe cz

$$ev(\pi): \quad T(u) \rightarrow \mathbf{L} + \frac{c}{u \cdot \pi}\mathbf{F} \quad \text{x jui} \quad \mathbf{F} = \sum_{i,j=2}^{m+n} (\cdot\ 2)^{[i]} E_{ij} \leq \mathsf{e}_{ji}, \qquad \text{)B/4*}$$

x jui $\mathbf{L} = \mathbf{2} \leq \mathsf{1}$- x i fsf x f jouspevdfe $\mathsf{1}$ ui f vojup pg $U(\mathfrak{gl}(m|n))$ boe x f vtfe ui f tbn f opubujpo bt jo tfdujpo 3/ Jo dpn qpofout- ui f fwbmvbujpo n bq sfbet

$$ev(\pi)\big)T_{ij}(u)\big\{ = \lambda_{ij}\,\mathsf{1} + \frac{c_{[i]}}{u \cdot \pi}\,\mathsf{e}_{ji}.$$

Joeffe- tjodf ui f Mjf tvqfsbmhfcsb sfmbujpot)B/2* bsf frvjwbmfou up

$$[\mathbf{F}_2, \mathbf{F}_3] = P(\mathbf{F}_2 \cdot \mathbf{F}_3),$$

jujt fbtz up ti px ui bu $\mathbf{L} + \frac{c}{u \cdot \pi}\mathbf{F}$ pcfzt ui f Zbohjbo *RTT*.sfmbujpot)3/5*/ Sfn bsl ui bu ui f hfofs. bupst pg $\mathfrak{gl}(m|n)$ bsf sfmbufe up ui f flfsp n peft eftdsjcfe jo]31‘; $\mathsf{e}_{ij} = (\cdot\ 2)^{[j]}\,T_{ji}[1]$/

---

[3] Gps tvqfsbmhfcsbt- ui f jssfevdjcmf qbsu pg ui f sfqsftfoubujpo dbo cf b dptfu pg ui f Lbd n pevmf- evf up ui f fyjtufodf pg buzqjdbmsfqsftfoubujpot/

[4] Ui f mbtu x fjhi u $^{(m+n)}$ qspwjeft b usjwjbmsfqsftfoubujpo gps $\mathfrak{sl}(m|n)$ boe jt sfmbufe up ui f $\mathfrak{gl}(2)$ bmhfcsb x i jdi jt dfousbm jo $\mathfrak{gl}(m|n)$/

Ui fo- vtjoh ui f fwbmvbujpo n psqi jtn pof dbo dpotusvdu gspn boz $\mathfrak{gl}(m|n)$ sfqsftfoubujpo ρ - b sfqsftfoubujpo gps ui f Zbohjbo $Y\big)\mathfrak{gl}(m|n)\big\{$/ Ui f *fwbmvbujpo sfqsftfoubujpo ev* $(\pi) = \rho$ *o ev*(π) jt efflofe bt;

$$ev\ (\pi)\Big)T_{ij}(u)\Big[= \lambda_{ij}1 + \frac{c_{[i]}}{u\cdot\ \pi} e_{ji},$$

x i fsf $e_{ij} = \rho\ (\mathsf{e}_{ij})$ jt ui f n busjy sfqsftfoubujpo pg e_{ij} jo ui f wfdups tqbdf $\mathcal{V}$ boe 1 jt ui f jefoujuz n busjy jo ui jt tqbdf/ Ui f x fjhi u pg ui f Zbohjbo sfqsftfoubujpo *ev* (π) sfbe

$$T_{ii}(u)|1\rangle = \mu_i(u)|1\rangle \quad \text{x jui} \quad \mu_i(u) = 2 + \frac{c_{[i]}}{u\cdot\ \pi}\mu_i,$$

boe x f i bwf

$$T_{ij}(u)|1\rangle = \frac{c_{[i]}}{u\cdot\ \pi} e_{ji}|1\rangle = 1, \quad j < i$$

bddpsejoh up ui f sfmbujpot)B/3*/ Ui f o jujt dmfbs ui bu ui f i jhi ftu x fjhi u wfdups pg $\mathfrak{gl}(m|n)$ cfdpn ft ui f qtfvepwbdvvn wfdups)4/2*/

Mfu vt fn qi btjiff ui f ejggfsfodf cfux ffo μ_i- ui bu bsf ui f x fjhi ut gps ui f Mjf tvqfsbmhfcsb $\mathfrak{gl}(m|n)$- boe $\mu_i(u)$- ui bubsf ui f x fjhi ut gps ui f Zbohjbo $Y\big)\mathfrak{gl}(m|n)\big\{$/

B/4/ Sfqsftfoubujpot bttpdjbufe up $f_{[i]}(u,v)$

Gps boz $j = 2, 3, ..., m+n$ boe boz dpn qmfy π- x f joupevdf ui f fwbmvbujpo sfqsftfoubujpo $Ev_j(\pi)$ bttpdjbufe up ui f x fjhi u $^{(j)}$/ Judpssftqpoet up ui f Zbohjbo x fjhi ut

$$\mu_\nu(u) = \begin{cases} f_{[\nu]}(u,\pi) & \text{jg}\ \nu \sim j, \\ 2 & \text{jg}\ \nu > j. \end{cases}$$

Xf dpotjefs ui f gpmmpx joh sfqsftfoubujpo; $\leq_{j=2}^{N} \leq_{k=2}^{L^{(j)}} Ev_j(\pi_k^{(j)})$/ Tjodf x f i bwf b ufotps qspevdu pg i jhi ftu x fjhi u sfqsftfoubujpot- ui f x fjhi ut gps ui jt ufotps qspevdubsf hjwfo cz ui f qspevdu pg ui f joejwjevbmx fjhi ut gps fbdi sfqsftfoubujpot- ui bujt

$$\mu_\nu(u) = \prod_{j=\nu}^{N}\prod_{k=2}^{L^{(j)}} f_{[\nu]}(u,\pi_k^{(j)}), \quad \nu = 2,3,...,m+n.$$

Ui jt mfbet up)6/2*/

Brrfoelz C1 Tfdvstkpo gps ui f i lhi ftudpfg dkfou

P of dbo cvjme ui f I D $Z^{m|n}$ tubsujoh gspn ui f l opx o sftvmut bu $m+n=3$ wjb sfdvstjpot efsjwfe jo]31‘/ Gps $m=3$- $n=1$ x f efbmx jui ui f I D pg $\mathfrak{gl}(3)$ cbtfe n pefm- ui bujt fr vbmup ui f qbsujujpo gvodujpo pg ui f tjy.wfsufy n pefmx jui epn bjo x bmmcpvoebsz dpoejujpo]4-35‘/ Ui f dbtf $m=1$- $n=3$ cfdpn ft fr vjwbmfou up ui f qsfwjpvt pof bgufs ui f sfqmbdfn fou ui f dpotubou $c \to \cdot\ c$ jo ui f R.n busjy)3/2*/ Gjobmmz- gps $m=n=2$ ui f I D i bt ui f gpsn]36‘

$$Z^{2|2}(\bar{s}|\bar{t}) = g(\bar{s},\bar{t}). \qquad)C/2*$$

Jo sfdvstjwf dpotusvdujpo pg ui f I D- ux p dbtft ti pvme cf ejtujohvjti fe;)2* $n>1$ boe $m>1$@)3* $n=1$ ps $m=1$/ Xf ﬂstu dpotjefs ui f dbtf $n>1$ boe $m>1$/ Ui fo- ui f sfdvstjwf qspdfevsf jt cbtfe po ui f gpmmpx joh sfevdujpot]31‘;

$$Z^{m|n}(\emptyset,\bar{s}^3,\dots,\bar{s}^N|\emptyset,\bar{t}^3,\dots,\bar{t}^N)=Z^{m\cdot\,2|n}(\bar{s}^3,\dots,\bar{s}^N|\bar{t}^3,\dots,\bar{t}^N),$$
$$Z^{m|n}(\bar{s}^2,\dots,\bar{s}^{N\cdot\,2},\emptyset|\bar{t}^2,\dots,\bar{t}^{N\cdot\,2},\emptyset)=Z^{m|n\cdot\,2}(\bar{s}^2,\dots,\bar{s}^{N\cdot\,2}|\bar{t}^2,\dots,\bar{t}^{N\cdot\,2}), \qquad)C/3*$$

boe x f sfdbmmi bu $N=m+n\cdot\,2$/ Ui vt- jo qbsujdvmbs- l opx joh $Z^{m\cdot\,2|n}$ gps tpn f m boe n x f bvupn bujdbmmz l opx $Z^{m|n}$ x jui $'\bar{s}^2='\bar{t}^2=1$/ Ui fo- up pcubjo $Z^{m|n}$ x jui $'\bar{s}^2='\bar{t}^2>1$ x f dbo vtf b sfdvstjpo]31‘

$$Z^{m|n}(\bar{s}|\bar{t})=\sum_{\sigma=3}^{N+2}\sum_{\substack{\text{qbsu}(\bar{s}^3,\dots,\bar{s}^{\sigma\cdot\,2})\\ \text{qbsu}(\bar{t}^2,\dots,\bar{t}^{\sigma\cdot\,2})}} Z^{m|n}(\{\bar{s}^{\phi}_{JJ}\}_2^{\sigma\cdot\,2},\{\bar{s}^{\phi}\}_{\sigma}^{N}|\{\bar{t}^{\phi}_{JJ}\}_2^{\sigma\cdot\,2},\{\bar{t}^{\phi}\}_{\sigma}^{N})\left(\frac{g(\bar{s}^2_{JJ},\bar{s}^2_{J})}{f(\bar{s}^2_{JJ},\bar{s}^2_{J})}\right)^{\lambda_{m,2}}$$
$$*\ \frac{g_{[3]}(\bar{t}^2_J,\bar{s}^2_J)\delta_2(\bar{t}^2_J,\bar{t}^2_{JJ})f(\bar{t}^2_{JJ},\bar{s}^2_J)}{f_{[\sigma]}(\bar{s}^{\sigma},\bar{s}^{\sigma\cdot\,2}_J)}\prod_{\xi=3}^{\sigma\cdot\,2}\frac{g_{[\xi+2]}(\bar{t}^{\xi}_J,\bar{t}^{\xi\cdot\,2}_J)g_{[\xi]}(\bar{s}^{\xi}_J,\bar{s}^{\xi\cdot\,2}_J)\delta_{\xi}(\bar{t}^{\xi}_J,\bar{t}^{\xi}_{JJ})\delta_{\xi}(\bar{s}^{\xi}_{JJ},\bar{s}^{\xi}_J)}{f_{[\xi]}(\bar{s}^{\xi},\bar{s}^{\xi\cdot\,2}_J)f_{[\xi]}(\bar{t}^{\xi}_J,\bar{t}^{\xi\cdot\,2})}. \qquad)C/4*$$

I fsf

$$Z^{m|n}(\{\bar{s}^{\phi}_{JJ}\}_2^{\sigma\cdot\,2},\{\bar{s}^{\phi}\}_{\sigma}^{N}|\{\bar{t}^{\phi}_{JJ}\}_2^{\sigma\cdot\,2},\{\bar{t}^{\phi}\}_{\sigma}^{N})$$
$$=Z^{m|n}(\bar{s}^2_{JJ},\dots,\bar{s}^{\sigma\cdot\,2}_{JJ},\bar{s}^{\sigma},\dots,\bar{s}^N|\bar{t}^2_{JJ},\dots,\bar{t}^{\sigma\cdot\,2}_{JJ},\bar{t}^{\sigma},\dots,\bar{t}^N). \qquad)C/5*$$

Gps fwfsz flyfe $\sigma\in\{3,\dots,N+2\}$ jo)C/4* ui f tvn t bsf ubl fo pwfs qbsujujpot $\bar{t}^{\phi}\Rightarrow\{\bar{t}^{\phi}_J,\bar{t}^{\phi}_{JJ}\}$ x jui $\phi=2,\dots,\sigma\cdot\,2$ boe $\bar{s}^{\phi}\Rightarrow\{\bar{s}^{\phi}_J,\bar{s}^{\phi}_{JJ}\}$ x jui $\phi=3,\dots,\sigma\cdot\,2$- tvdi ui bu $'\bar{t}^{\phi}_J='\bar{s}^{\phi}_J=2$/ Ui f tvctfu $\bar{s}^2_J$ jt b flyfe Cfui f qbsbn fufs gspn ui f tfu $\bar{s}^2$/ Ui fsf jt op tvn pwfs qbsujujpot pg ui f tfu $\bar{s}^2$ jo)C/4*/

Tjn jmbsmz- l opx joh $Z^{m|n\cdot\,2}$ gps tpn f m boe n x f bvupn bujdbmmz l opx $Z^{m|n}$ x jui $'\bar{s}^N='\bar{t}^N=1$/ Ui fo- up pcubjo $Z^{m|n}$ x jui $'\bar{s}^N='\bar{t}^N>1$ x f dbo vtf ui f tfdpoe sfdvstjpo

$$Z^{m|n}(\bar{s}|\bar{t})=\sum_{\sigma=2}^{N}\sum_{\substack{\text{qbsu}(\bar{s}^{\sigma},\dots,\bar{s}^{N})\\ \text{qbsu}(\bar{t}^{\sigma},\dots,\bar{t}^{N\cdot\,2})}} Z^{m|n}(\}\bar{s}^{\phi}|_2^{\sigma\cdot\,2},\}\bar{s}^{\phi}_{JJ}|_{\sigma}^{N}|\}\bar{t}^{\phi}|_2^{\sigma\cdot\,2};\}\bar{t}^{\phi}_{JJ}|_{\sigma}^{N})\left(\frac{g(\bar{t}^N_{JJ},\bar{t}^N_J)}{f(\bar{t}^N_{JJ},\bar{t}^N_J)}\right)^{\lambda_{m,N}}$$
$$*\ \frac{g(\bar{s}^N_J,\bar{t}^N_J)\delta_N(\bar{s}^N_{JJ},\bar{s}^N_J)f(\bar{s}^N_{JJ},\bar{t}^N_J)}{f_{[\sigma]}(\bar{t}^{\sigma}_J,\bar{t}^{\sigma\cdot\,2})}\prod_{\xi=\sigma}^{N\cdot\,2}\frac{g_{[\xi]}(\bar{s}^{\xi+2}_J,\bar{s}^{\xi}_J)g_{[\xi]}(\bar{t}^{\xi+2}_J,\bar{t}^{\xi}_J)\delta_{\xi}(\bar{s}^{\xi}_{JJ},\bar{s}^{\xi}_J)\delta_{\xi}(\bar{t}^{\xi}_J,\bar{t}^{\xi}_{JJ})}{f_{[\xi+2]}(\bar{s}^{\xi+2},\bar{s}^{\xi}_J)f_{[\xi+2]}(\bar{t}^{\xi+2}_J,\bar{t}^{\xi})}. \qquad)C/6*$$

I fsf

$$Z^{m|n}(\}\bar{s}^{\phi}|_2^{\sigma\cdot\,2},\}\bar{s}^{\phi}_{JJ}|_{\sigma}^{N}|\}\bar{t}^{\phi}|_2^{\sigma\cdot\,2};\}\bar{t}^{\phi}_{JJ}|_{\sigma}^{N})$$
$$=Z^{m|n}(\bar{s}^2,\dots,\bar{s}^{\sigma\cdot\,2},\bar{s}^{\sigma}_{JJ},\dots,\bar{s}^N_{JJ}|\bar{t}^2,\dots,\bar{t}^{\sigma\cdot\,2},\bar{t}^{\sigma}_{JJ},\dots,\bar{t}^N_{JJ}). \qquad)C/7*$$

Gps fwfsz flyfe $\sigma\in\{2,\dots,N\}$ jo)C/6* ui f tvn t bsf ubl fo pwfs qbsujujpot $\bar{t}^{\phi}\Rightarrow\{\bar{t}^{\phi}_J,\bar{t}^{\phi}_{JJ}\}$ x jui $\phi=\sigma,\dots,N\cdot\,2$ boe $\bar{s}^{\phi}\Rightarrow\{\bar{s}^{\phi}_J,\bar{s}^{\phi}_{JJ}\}$ x jui $\phi=\sigma,\dots,N$- tvdi ui bu $'\bar{t}^{\phi}_J='\bar{s}^{\phi}_J=2$/ Ui f tvctfu $\bar{t}^N_J$ jt b flyfe Cfui f qbsbn fufs gspn ui f tfu $\bar{t}^N$/ Ui fsf jt op tvn pwfs qbsujujpot pg ui f tfu $\bar{t}^N$ jo)C/6*/

Opx - mfu vt eftdsjcf ui f tjuvbujpo jo ui f dbtf $n=1$/ Ui f gpsn vmbt)C/4*-)C/6* sfn bjo wbmje jo ui jt dbtf- i px fwfs- ui fz bsf tjn qmjgjfe/ Gjstu pg bmm $\lambda_{m,2}=\lambda_{m,N}=1$ jo ui jt dbtf/ Ui jt mfbet up ui f ejtbqqfbsbodf pg ui f gbdupst jo ui f flstu mjoft pg)C/4*-)C/6*/ Tfdpoe- bmm ui f

δ.gvodujpot ti pvme cf sfqmbdfe cz ui f f.gvodujpot/ Gjobmmz- bmmui f tvctdsjqut pgui f g.gvodujpot boe f.gvodujpot ejtbqqfbs; $g_{[\xi]}(x,y)\to g(x,y)$- $f_{[\xi]}(x,y)\to f(x,y)$/

I px fwfs- ui f n bjo qfdvmjbsjuz pgui jt dbtf jt ui bu ui f sfevdujpot)C/3* ubl f ui f gpsn

$$\begin{aligned} Z^{m|1}(\emptyset,\bar s^3,\dots,\bar s^{m\cdot 2}|\emptyset,\bar t^3,\dots,\bar t^{m\cdot 2}) &= Z^{m\cdot 2|1}(\bar s^3,\dots,\bar s^{m\cdot 2}|\bar t^3,\dots,\bar t^{m\cdot 2}),\\ Z^{m|1}(\bar s^2,\dots,\bar s^{m\cdot 3},\emptyset|\bar t^2,\dots,\bar t^{m\cdot 3},\emptyset) &= Z^{m\cdot 2|1}(\bar s^2,\dots,\bar s^{m\cdot 3}|\bar t^2,\dots,\bar t^{m\cdot 3}).\end{aligned} \quad)C/8*$$

Ui vt- jgfjui fs $\bar s^2=\bar t^2=\emptyset$ ps $\bar s^{m\cdot 2}=\bar t^{m\cdot 2}=\emptyset$- ui fo jo cpui dbtft $Z^{m|1}$ sfevdft up $Z^{m\cdot 2|1}$/

Gjobmmz- ui f dbtf pg gl$(1|n)$ bmhfcsbt sfevdft up ui f dbtf dpotjefsfe bcpwf bgufs ui f sfqmbdfn fou ui f dpotubou $c\to\cdot\, c$ jo ui f R.n busjy)3/2*/ Ui fsfgpsf- x f ep opu dpotjefs ui jt dbtf cfmpx/

Brrfoekz E1 Tftkevft ko ui f rpmft pgui f i khi ftudpfg dkfou

Xf hjwf b efubjmfe qsppg pg Qspqptjujpo 4/2 gps ui f dbtf $m>1$ boe $n>1$/ Ui f dbtf $m=1$ ps $n=1$ dbo cf dpotjefsfe fybdumz jo ui f tbn f n boofs/

Ui f qsppg jt cbtfe po ui f sfevdujpot)C/3*- sfdvstjpot)C/4*-)C/6*- boe fyqmjdju sfqsftfoubujpo)C/2* gps $Z^{2|2}(\bar s|\bar t)$/ Gjstu- pof dbo fbtjmz tff ui bu evf up)C/2*

$$Z^{2|2}(\bar s|\bar t)\Big|_{s_j\to t_j} = g(s_j,t_j)g(\bar s_j,s_j)g(t_j,\bar t_j)Z^{2|2}(\bar s_j|\bar t_j)+reg. \quad)D/2*$$

Ui jt fyqsfttjpo pcwjpvtmz dpjodjeft x jui)4/28* gps $m=n=2$/ Fr vbujpo)D/2* tfswft bt ui f cbtjt pg joevdujpo/[5]

Bttvn f ui bu)4/28* jt wbmje gps bmm m' boe n'- tvdi ui bu $m'+n'$ jt flyfe/ Ui fo evf up)C/3* ui f sftjevf gpsn vmb)4/28* i pmet gps $Z^{m|n}$ x jui $m=m'+2$- $n'=n$ bu $r_2=1$)ui bu jt- $\bar s^2=\bar t^2=\emptyset$* boe gps $Z^{m|n}$ x jui $m=m'$- $n=n'+2$ bu $r_N=1$)ui bu jt- $\bar s^N=\bar t^N=\emptyset$*/ Ui fo vtjoh sfdvstjpot)C/4* boe)C/6* x f ti pvme qspwf ui bu)4/28* sfn bjot usvf gps $r_2>1$ boe $r_N>1$/ Jutp i bqqfot ui bu sfdvstjpo)C/4* bmmpx t pof up qspwf)4/28* gps $\bar s^\nu$ boe $\bar t^\nu$ x jui $\nu=3,\dots,N$/ Bu ui f tbn f ujn f sfdvstjpo)C/6* qspwjeft ui f qsppg gps $\bar s^\nu$ boe $\bar t^\nu$ x jui $\nu=2,\dots,N\cdot 2$/ Dpn cjojoh cpui sfdvstjpot x f qspwf ui f sftjevf gpsn vmb)4/28* gps bmm $\bar s^\nu$ boe $\bar t^\nu$/

Mfuvt ti px i px ui jt n fui pe x psl t/ Dpotjefs- gps fybn qmf- ui f sfdvstjpo)C/4*/ Jujt dpowf. ojfou up x sjuf jujo ui f gpmmpx joh gpsn ;

$$Z^{m|n}(\bar s|\bar t)=\sum_{\sigma=3}^{N+2}\mathcal{Z}_\sigma^{m|n}(\bar s|\bar t), \quad)D/3*$$

x i fsf

$$\mathcal{Z}_\sigma^{m|n}(\bar s|\bar t)=\sum_{\substack{\text{qbsu}(\bar s^3,\dots,\bar s^{\sigma\cdot 2})\\ \text{qbsu}(\bar t^2,\dots,\bar t^{\sigma\cdot 2})}} Z^{m|n}(\{\bar s^\phi_{\rm JJ}\}_2^{\sigma\cdot 2},\{\bar s^\phi\}_\sigma^N|\{\bar t^\phi_{\rm JJ}\}_2^{\sigma\cdot 2},\{\bar t^\phi\}_\sigma^N)\;\left(\frac{g(\bar s^2_{\rm JJ},\bar s^2_{\rm J})}{f(\bar s^2_{\rm JJ},\bar s^2_{\rm J})}\right)^{\lambda_{m,2}}$$
$$*\;\frac{g_{[3]}(\bar t^2_{\rm J},\bar s^2_{\rm J})\delta_2(\bar t^2_{\rm J},\bar t^2_{\rm JJ})f(\bar t^2_{\rm JJ},\bar s^2_{\rm J})}{f_{[\sigma]}(\bar s^\sigma,\bar s^{\sigma\cdot 2}_{\rm J})}\prod_{\xi=3}^{\sigma\cdot 2}\frac{g_{[\xi+2]}(\bar t^\xi_{\rm J},\bar t^{\xi\cdot 2}_{\rm J})g_{[\xi]}(\bar s^\xi_{\rm J},\bar s^{\xi\cdot 2}_{\rm J})\delta_\xi(\bar t^\xi_{\rm J},\bar t^\xi_{\rm JJ})\delta_\xi(\bar s^\xi_{\rm JJ},\bar s^\xi_{\rm J})}{f_{[\xi]}(\bar s^\xi,\bar s^{\xi\cdot 2}_{\rm J})f_{[\xi]}(\bar t^\xi_{\rm J},\bar t^{\xi\cdot 2})}. \quad)D/4*$$

[5] Gps dpn qmfufoftt pgui f qsppg pof ti pvme bmtp di fdl)4/28* gps $m=3$ boe $n=1$/ Ui jt x bt epof jo]4‘/

Xf flstu dpotjefs ui f dbtf $r_2 = '\bar{s}^2 = '\bar{t}^2 = 2/$ Ui fo $'\bar{s}^2_{\rm JJ} = '\bar{t}^2_{\rm JJ} = 1$- i fodf- x f bduvbmz i bwf $Z^{m\cdot 2|n}$ jo ui f si t pg)D/4*/ Bddpsejoh up ui f joevdujpo bttvn qujpo ui f sftjevf gpsn vrb)4/28* jt wbmje gps ui ftf I D/

Mfu $s^\nu_j = t^\nu_j$ gps $\nu > 2$ jo ui f rnt pg)D/3*/ Jo ui f si t pg ui jt fr vbujpo pof ti pvme dpotjefs tfqbsbufmz ui f ufsn t x jui ejggfsfou σ/ Obn fmz- pof ti pvme ejtujohvjti cfux ffo gpvs dbtft; $\sigma < \nu$ @ $\sigma > \nu + 2$@ $\sigma = \nu + 2$@ $\sigma = \nu$/

Mfu $\sigma < \nu$/ Ui f qpmf bu $s^\nu_j = t^\nu_j$ jo ui f si t pg)D/4* pddvst jo ui f I D pomz/ Ui fo evf up ui f joevdujpo bttvn qujpo ui f sftjevf pg ui f I D jo ui f si t pg)D/4* hjwft ui f gbdups

$$\mathcal{A}_\nu = \frac{g_{[\nu+2]}(t^\nu_j, s^\nu_j)\delta_\nu(\bar{t}^\nu_j, t^\nu_j)\delta_\nu(s^\nu_j, \bar{s}^\nu_j)}{f_{[\nu+2]}(\bar{t}^{\nu+2}, t^\nu_j) f_{[\nu]}(s^\nu_j, \bar{s}^{\nu\cdot 2})}. \qquad)D/5*$$

Ui jt dpfggldjfou epft opu efqfoe po ui f qbsujujpot/ Ui f sfn bjojoh tvn pwfs qbsujujpot pcwjpvtmz sfevdft up $\mathcal{Z}^{m|n}_\sigma(\bar{s}\setminus\{s^\nu_j\}|\bar{t}\setminus\{t^\nu_j\})$/ Ui vt- gps $\sigma < \nu$ x f bssjwf bu

$$\mathcal{Z}^{m|n}_\sigma(\bar{s}|\bar{t})\Big|_{s^\nu_j = t^\nu_j} = \mathcal{A}_\nu \mathcal{Z}^{m|n}_\sigma(\bar{s}\setminus\{s^\nu_j\}|\bar{t}\setminus\{t^\nu_j\}) + reg. \qquad)D/6*$$

Dpotjefs opx ui f ufsn t x jui $\sigma > \nu + 2$/ Ui f qpmf jo ui f si t pg)D/4* pddvst jo ui f I D qsp. wjefe $s^\nu_j \in \bar{s}^\nu_{\rm JJ}$ boe $t^\nu_j \in \bar{t}^\nu_{\rm JJ}$/ Mfu $\bar{s}^\nu_{\rm JJ} = \{s^\nu_j, \bar{s}^\nu_{\rm JJ'}\}$ boe $\bar{t}^\nu_{\rm JJ} = \{t^\nu_j, \bar{t}^\nu_{\rm JJ'}\}$/ Ui fo ui f sftjevf pg ui f i jhi ftu dpfggldjfou hjwft ui f gbdups

$$\frac{g_{[\nu+2]}(t^\nu_j, s^\nu_j)\delta_\nu(\bar{t}^\nu_{\rm JJ'}, t^\nu_j)\delta_\nu(s^\nu_j, \bar{s}^\nu_{\rm JJ'})}{f_{[\nu+2]}(\bar{t}^{\nu+2}_{\rm JJ}, t^\nu_j) f_{[\nu]}(s^\nu_j, \bar{s}^{\nu\cdot 2}_{\rm JJ})}. \qquad)D/7*$$

Ui f tfdpoe mjof pg)D/4* hjwft beejujpobm gbdupst efqfoejoh po s^ν_j boe t^ν_j;

$$\frac{\delta_\nu(\bar{t}^\nu_{\rm J}, t^\nu_j)\delta_\nu(s^\nu_j, \bar{s}^\nu_{\rm J})}{f_{[\nu+2]}(\bar{t}^{\nu+2}_{\rm J}, t^\nu_j) f_{[\nu]}(s^\nu_j, \bar{s}^{\nu\cdot 2}_{\rm J})}. \qquad)D/8*$$

Uphfui fs x jui)D/7* ui fz hjwf $\mathcal{A}_\nu$)D/5*/ Ui f sftu pg)D/4* epft opu efqfoe po s^ν_j boe t^ν_j- i fodf- x f bhbjo pcubjo)D/6*- cvu opx gps $\sigma > \nu + 2$/

Ui f ui jse dbtf jt $\sigma = \nu + 2$/ Bhbjo- ui f qpmf pddvst jo ui f I D- boe x f tfu $\bar{s}^\nu_{\rm JJ} = \{s^\nu_j, \bar{s}^\nu_{\rm JJ'}\}$- $\bar{t}^\nu_{\rm JJ} = \{t^\nu_j, \bar{t}^\nu_{\rm JJ'}\}$/ Opx ui f gbdups dpn joh gspn ui f I D jt

$$\frac{g_{[\nu+2]}(t^\nu_j, s^\nu_j)\delta_\nu(\bar{t}^\nu_{\rm JJ'}, t^\nu_j)\delta_\nu(s^\nu_j, \bar{s}^\nu_{\rm JJ'})}{f_{[\nu+2]}(\bar{t}^{\nu+2}, t^\nu_j) f_{[\nu]}(s^\nu_j, \bar{s}^{\nu\cdot 2}_{\rm JJ})}. \qquad)D/9*$$

Xf bmtp i bwf gspn ui f tfdpoe mjof pg)D/4*

$$\frac{\delta_\nu(\bar{t}^\nu_{\rm J}, t^\nu_j)\delta_\nu(s^\nu_j, \bar{s}^\nu_{\rm J})}{f_{[\nu]}(s^\nu_j, \bar{s}^{\nu\cdot 2}_{\rm J})}, \qquad)D/: *$$

boe bmphfui fs x f bhbjo pcubjo)D/5*/ Ui vt- fr vbujpo)D/6* i pmet gps $\sigma = \nu + 2$/

Gjobmmz- mfu $\sigma = \nu$/ Ui fo x f i bwf gspn ui f I D

$$\frac{g_{[\nu+2]}(t^\nu_j, s^\nu_j)\delta_\nu(\bar{t}^\nu_j, t^\nu_j)\delta_\nu(s^\nu_j, \bar{s}^\nu_j)}{f_{[\nu+2]}(\bar{t}^{\nu+2}, t^\nu_j) f_{[\nu]}(s^\nu_j, \bar{s}^{\nu\cdot 2}_{\rm JJ})}. \qquad)D/21*$$

Ui f beejujpobmgbdups $f_{[\nu]}(s_j^{\nu}, \bar{s}_J^{\nu\cdot 2})$ dpn ft gspn ui f tfdpoe mjof pg)D/4*- boe x f bhbjo pcubjo ui f $\mathcal{A}_\nu$ dpfgfldjfou)D/5*/ Ui f sfn bjojoh tvn pwfs qbsujujpot tujmmhjwft $\mathcal{Z}_\sigma^{m|n}(\bar{s} \setminus \{s_j^\nu\}|\bar{t} \setminus \{t_j^\nu\})$/

Ui vt- fr vbujpo)D/6* jt qspwfe gps bmm σ/ Evf up)D/3* ui jt jn n fejbufmz zjfmet ui f sftjevf gpsn vmb)4/28* gps $\mathcal{Z}^{m|n}$/

Bt tppo bt)4/28* jt qspwfe gps $r_2 = 2$ x f dbo vtf jubt b ofx cbtjt pgjoevdujpo/ X f bttvn f ui bu)4/28* jt wbmje gps tpn f $r_2 > 1$ boe ui fo qspwf ui bujusfn bjot usvf gps $r_2 + 2$/ Bmmdpotjefsbujpot bsf fybdumz ui f tbn f bt jo ui f dbtf $r_2 = 2$- ui fsfgpsf x f pn ju ui fn /

Jo ui jt x bz x f qspwf ui f sftjevf gpsn vmb gps bmm $\bar{s}^\nu$ boe $\bar{t}^\nu$ fydfqu $\bar{s}^2$ boe $\bar{t}^2$/ Up qspwf)4/28* gps ui f sftjevf bu $s_j^2 = t_j^2$ x f ti pvme vtf ui f tfdpoe sfdvstjpo)C/6* boe qfsgpsn tjn jmbs dbmdvmbujpot/

Tfgfsfodft

]2‘ N/ Hbvejo- N pe mft fybdut fo n dbojr vf tubujtujr vf; mb n ui pef ef Cfui f futf t h o sbmjtbujpot- Qsfqsjou- Dfousf e(Fuveft Ovdmbjsft ef Tbdmbz- DFB.O.266: ;2- 2: 83/

]3‘ N/ Hbvejo- Mb Gpodujpo e(P oef ef Cfui f- N bttpo- Qbsjt- 2: 94/

]4‘ WF/ Lpsfqjo- Dbmdvmbujpo pgopsn t pgCfui f x bwf gvodujpot- Dpn n / N bui / Qi zt/ 97)2: 93*4: 2 529/

]5‘ ME/ Gbeeffw- F/L/ Tl mzbojo- MB/ Ubl i ubkbo- Rvbouvn jowfstf qspcmfn / J- Ui fps/ N bui / Qi zt/ 51)2: 8: *799 817/

]6‘ ME/ Gbeeffw- MB/ Ubl i ubkbo- Ui f r vbouvn n fui pe pgui f jowfstf qspcmfn boe ui f I fjtfocfsh XYZ n pefm- Svt. tjbo N bui / Tvswfzt 45)2: 8: *22)Fohmjti usbotm/*/

]7‘ WF/ Lpsfqjo- O/N/ Cphpmjvcpw- B/H/ Jfiffshjo- Rvbouvn Jowfstf Tdbuufsjoh N fui pe boe Dpssfmbujpo Gvodujpot- Dbn csjehf Vojw/ Qsftt- Dbn csjehf- 2: : 4/

]8‘ ME/ Gbeeffw- jo; B/ Dpooft- fu bm/)Fet/*- Mft I pvdi ft Mfduvsft Rvbouvn Tzn n fusjft- Opsui I pmmboe- 2: : 9- q/ 25: /

]9‘ O/Zv/ Sfti fujl i jo- Dbmdvmbujpo pgui f opsn pgCfui f wfdupst jo n pefmt x jui $SU(4)$.tzn n fusz- [bq/ Obvdi o/ Tfn / MP N J 261)2: 97*2: 7 324 @K/ N bui / Tdj/ 57)2: 9: *27: 5 2817)Fohmjti usbotm/*/

]: ‘ O/B/ Tmbwopw- Tdbmbs qspevdut jo HM)4*.cbtfe n pefmt x jui usjhpopn fusjd R.n busjy/ Efufsn jobou sfqsftfoubujpo- K/ Tubu/ N fdi / Ui fpsz Fyq/ 2614)3126* Q1412: - bsYjw;2612/17364/

]21‘ W Ubsbtpw- B/ Wbsdi fol p- Kbdl tpo joufhsbmsfqsftfoubujpot pgtpmvujpot pgui f r vbouj{fe L ojfii ojl [bn pmpedi jl pw fr vbujpo- Bmhfcsb j Bobmj{ 7)3*)2: : 5* : 1 248 @Tu/ Qfufstcvsh N bui / K/ 7)3*)2: : 6* 386 424)Fohmjti usbotm/*- bsYjw;i fq.ui 0 422151/

]22‘ W Ubsbtpw- B/ Wbsdi fol p- Btzn qupujd tpmvujpot up ui f r vbouj{fe L ojfii ojl [bn pmpedi jl pw fr vbujpo boe Cfui f wfdupst- Bn fs/ N bui / Tpdjfuz Usbotm- Tfs/ 3)285*)2: : 7*346 384- bsYjw;i fq.ui 0 517171/

]23‘ F/ N vl i jo- B/ Wbsdi fol p- Opsn pg b Cfui f wfdups boe ui f I fttjbo pgui f n btufs gvodujpo- Dpn qptjujp N bui / 252)3116*2123 2139- bsYjw;n bui 0151345: /

]24‘ G/ H i n boo- WF/ Lpsfqjo- Ui f I vccbse di bjo; Mjfc X v fr vbujpot boe opsn pgui f fjhfogvodujpot- Qi zt/ Mfuu/ B 374)2: : : *3: 4 3: 9- bsYjw;dpoe.n bu0 : 19225/

]25‘ C/ Cbttp- G/ Dpspobep- T/ Lpn but v- I /U/ Mbn - Q/ Wjfjsb- E/ [i poh- Btzn qupujd gpvs qpjou gvodujpot- bsYjw;2812/ 15573/

]26‘ Q/Q/ Lvmjti - O/Zv/ Sfti fujl i jo- Ejbhpobmj{bujpo pg $GL(N)$ jowbsjbou usbotgfs n busjdft boe r vbouvn N.x bwf tztufn)Mff n pefm*- K/ Qi zt/ B 27)2: 94* M6: 2 M6: 7/

]27‘ Q/Q/ Lvmjti - O/Zv/ Sfti fujl i jo- Hfofsbmj{fe I fjtfocfsh gfsspn bhofu boe ui f Hsptt Ofwfv n pefm- Tpw/ Qi zt/ KFUQ 64)2*)2: 92*219 225)Fohmjti usbotm/*/

]28‘ Q/Q/ Lvmjti - O/Zv/ Sfti fujl i jo- HM)4*.jowbsjbou tpmvujpot pg ui f Zboh.Cbyufs fr vbujpo boe bttpdjbufe r vbouvn tztufn t- [bq/ Obvdi o/ Tfn / MP N J/ 231)2: 93*: 3 232 @K/ Tpw/ N bui / 45)6*)2: 97*2: 59 2: 82)Fohmjti usbotm/*/

]29‘ WF/ Lpsfqjo- Bobmztjt pg b cjmjofbs sfmbujpo gps ui f tjy.wfsufy n pefm- Tpw/ Qi zt/ Epl m/ 38)2: 93*723 724)Fohmjti usbotm/*/

]2: ‘ B/H/ Jfiffshjo- WF/ Lpsfqjo- Ui f qspcmfn pgeftdsjqujpo pg bmm L.pqfsbupst gps R.n busjdft pgui f n pefmt XXX boe XXZ)jo Svttjbo*- [bq/ Obvdi o/ Tfn / MP N J 242)2: 94*91 98/

]31‘ B/B/ I vutbmzvl - B/ Mjbti zl - T/[/ Qbl vmjbl - F/ Sbhpvdz- O/B/ Tmbwopw- Tdbmbs qspevdut pg Cfui f wfdupst jo ui f n pefmt x jui $\mathfrak{gl}(m|n)$ tzn n fusz- Ovdm/ Qi zt/ C : 34)3128*388 422- bsYjw;2815/19284/

]32‘ Q/Q/ Lvmjti - F/L/ Tl mzbojo- P o ui f tpmvujpo pg ui f Zboh Cbyufs fr vbujpo- [bq/ Obvdi o/ Tfn jo/ MP N J : 6)2: 91* 23: 271 @K/ Tpw/ N bui / 2:)2: 93*26: 7 2731)Fohmjti usbotm/*/

]33‘ B/B/ I vutbmzvl - B/ Mjbti zl - T/[/ Qbl vmjbl - F/ Sbhpvdz- O/B/ Tmbwopw- Dvssfou qsftfoubujpo gps ui f epvcmf tvqfs. Zbohjbo $DY(\mathfrak{gl}(m|n))$ boe Cfui f wfdupst- Svtt/ N bui / Tvswfzt 83)2*)3128*44 : :)Fohmjti usbotm/*/

]34‘ T/ Qbl vrjibl - F/ Sbhpvdz- O/B/ Trhwopw- Cfui f wfdupst gps n pefm cbtfe po ui f tvqfs.Zbohjbo $Y(gl(m|n))$- K/ Jouf. hsbcmf Tztufn t 3)3128*2 42- bsYjw;2715/13422/
]35‘ B/H/ Jfifshjo- Qbsujujpo gvodujpo pg ui f tjy.wfsufy n pefmjo b flojuf wpmvn f- Tpw/ Qi zt/ Epl m/ 43)2: 98* 989 98:)Fohmjtu usbotmf*/
]36‘ B/ I vutbmvl - B/ Mjbti zl - T/ Qbl vrjibl - F/ Sbhpvdz- O/B/ Trhwopw- Tdbmbs qspevdut pg Cfui f wfdupst jo n pefmt x jui gl(3|2) tzn n fusz 2/ Tvqfs.bobmph pg Sfti fujl i jo gpsn vmb- K/ Qi zt/ B; N bui / Ui fps/ 5:)56*)3127* 565116- bsYjw;2716/1: 29: /
]37‘ B/B/ I vutbmvl - B/ Mjbti zl - T/[/ Qbl vrjibl - F/ Sbhpvdz- O/B/ Trhwopw- Tdbmbs qspevdut pg Cfui f wfdupst jo n pe. fmt x jui gl(3|2) tzn n fusz 3/ Efufsn jobou sfqsftfoubujpo- K/ Qi zt/ B; N bui / Ui fps/ 61)4*)3128* 45115- bsYjw; 2717/14684/
]38‘ W Di bsj- B/O/ Qsfttmfz- Ofx vojubsz sfqsftfoubujpot pg mppq hspvqt- N bui / Boo/ 386)2: 97*98 215/
]39‘ N /M/ Obfibspw- Rvbouvn Cfsf ffijojbo boe ui f dmbttjdbm Dbqf mjnjefoujuz- Mfuu/ N bui / Qi zt/ 32)2: : 2*234 242/
]3: ‘ WH/ Lbd- Sfqsftfoubujpot pg dmbttjdbm Mjf tvqfsbmhfcsbt- Mfduvsf Opuft jo N bui / 787)2: 89*6: 8 737/
]41‘ K/Q/ I vsoj- C/ N psfm- Jssfevdjcmf sfqsftfoubujpot pg $SU(m|n)$- K/ N bui / Qi zt/ 35)2: 94*268 274/
]42‘ M/ Gsbqqbu- B/ Tdjbsjop- Q/ Tpscb- Ejdujpobsz po Mjf Bmhfcsbt boe Tvqfsbmhfcsbt- Bdbefn jd Qsftt- Tbo Ejfhp- 3111/

Chapter 5

Scalar products and norm of Bethe vectors for integrable models based on $U_q(\hat{\mathfrak{gl}}_m)$

Introduction:

This Chapter generalizes results of two previous Chapters to the case of quantum affine algebra $U_q(\hat{\mathfrak{gl}}_m)$.

Contribution:

Using antimorphism Ψ (3.14) I proved recurrent relations for dual Bethe vectors. We used these formulas to calculate scalar product of Bethe vectors. Using the scalar scalar product of Bethe vectors I proved generalization of Gaudin theorem for norm of Bethe vectors to case of quantum affine algebra $U_q(\hat{\mathfrak{gl}}_m)$. As in super-Yangian case $Y(\mathfrak{gl}_{\mathfrak{n}|\mathfrak{m}})$, it is a key object for calculation of correlation functions.

Scalar products and norm of Bethe vectors for integrable models based on $U_q(\widehat{\mathfrak{gl}}_m)$

Arthur Hutsalyuk[1,2⋆], **Andrii Liashyk**[3,4,5♣], **Stanislav Z. Pakuliak**[1,6†], **Eric Ragoucy**[7‡] and **Nikita A. Slavnov**[8⋄]

1 Moscow Institute of Physics and Technology, Dolgoprudny, Moscow reg., Russia
2 Fachbereich C Physik, Bergische Universität Wuppertal, 42097 Wuppertal, Germany
3 Bogoliubov Institute for Theoretical Physics, NAS of Ukraine, Kiev, Ukraine
4 National Research University Higher School of Economics, Faculty of Mathematics, Moscow, Russia
5 Skolkovo Institute of Science and Technology, Moscow, Russia
6 Laboratory of Theoretical Physics, JINR, Dubna, Moscow reg., Russi
7 Laboratoire de Physique Théorique LAPTh, CNRS and USMB, BP 110, 74941 Annecy-le-Vieux Cedex, France
8 Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia

⋆ hutsalyuk@gmail.com, ♣ a.liashyk@gmail.com, † stanislav.pakuliak@jinr.ru, ‡ eric.ragoucy@lapth.cnrs.fr, ⋄ nslavnov@mi.ras.ru

Abstract

We obtain recursion formulas for the Bethe vectors of models with periodic boundary conditions solvable by the nested algebraic Bethe ansatz and based on the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_m)$. We also present a sum formula for their scalar products. This formula describes the scalar product in terms of a sum over partitions of the Bethe parameters, whose factors are characterized by two highest coefficients. We provide different recursions for these highest coefficients.

In addition, we show that when the Bethe vectors are on-shell, their norm takes the form of a Gaudin determinant.

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1 Introduction

Integrable models have the striking property that their physical data are exactly computable, without the use of any perturbative expansion or asymptotic behavior. For this reason, they have always attracted the attention of researchers. In the twentieth century, quantum integrable models have been the source of many developments originating in the so-called Bethe ansatz, introduced by H. Bethe [1]. In a few words, the Bethe ansatz is an expansion of Hamiltonian eigenvectors over some clever basis (similar to planar waves) using some parameters (the Bethe parameters, which play the role of momenta). Demanding the vectors to be eigenvectors of the Hamiltonian leads to a quantization of the Bethe parameters which takes the form of a system of coupled algebraic equations called the Bethe equations. Knowing the form of the Bethe ansatz and the Bethe equations is in general enough to get a large number of information on the physical data of the system.

In continuity to the Bethe ansatz technics, the Quantum Inverse Scattering Method (QISM), mainly elaborated by the Leningrad/St-Petersburg School [2–5], has been the core of a wide range of progress. These developments were performed in continuity with (or parallel to) the works of C. N. Yang, R. Baxter, M. Gaudin, and many others, see e.g. [6–12].

The Bethe ansatz and QISM have provided a lot of interesting results for the models based on $\mathfrak{gl}_2$ symmetry and its quantum deformations. Among them, we can mention the determinant representations for the norm and the scalar products of Bethe vectors [13, 14]. Focusing on spin chains with periodic boundary conditions, it is worth mentioning the explicit solution of the quantum inverse scattering problem [15–17]. These results were used to study correlation functions of quantum integrable models in the thermodynamic limit via multiple integral representations [18–20] or form factor expansion [21–23].

For higher rank algebras, that is to say for multicomponent systems, $\mathfrak{gl}_m$ spin chains and their quantum deformation, or their $\mathbb{Z}_2$-graded versions, results are scarcer, although the general ground has been settled many years ago [24–29]. Nevertheless, some steps have been done, in particular for models with periodic boundary conditions: an explicit expression for Bethe vectors of models based on $Y(\mathfrak{gl}(m|n))$ and on $U_q(\widehat{\mathfrak{gl}}_m)$ can be found in [30–32] and [33–37]. The calculation of scalar product and form factors have been addressed for some specific algebras. The case of the $Y(\mathfrak{gl}_3)$ algebra has been studied in a series of works presenting some explicit forms of Bethe vectors [38], the calculation of their scalar product [39–43] and the expression of the form factors as determinants [44, 45]. Results for models based on the deformed version $U_q(\widehat{\mathfrak{gl}}_3)$ have been also obtained: explicit forms of Bethe vectors can be found in [46], their scalar products in [47–49] and a determinant expression for scalar products and form factors of diagonal elements was presented in [50]. The supersymmetric counterpart of $Y(\mathfrak{gl}_3)$, the superalgebra $Y(\mathfrak{gl}(2|1))$ has been dealt in [51–54]. Some partial results were also obtained for superalgebras in connection with the Super-Yang–Mills theories [55–57]. However a full understanding of the general approach to compute correlation functions is still lacking. Recently, some general results on the scalar product and the norm of Bethe vectors for $Y(\mathfrak{gl}(m|n))$ models have been obtained in [58, 59], in parallel to the original results described in [13, 39]. The present paper contains similar results for models based on the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_m)$.

It is known (see e.g. [13, 14, 60]) that most of the results concerning the scalar products of Bethe vectors in the models described by the $Y(\mathfrak{gl}_2)$ and $U_q(\widehat{\mathfrak{gl}}_2)$ algebras can be formulated in a sole universal form. This is because the R-matrices in both cases correspond to the six-vertex model. An analogous similarity takes place in the general $Y(\mathfrak{gl}_m)$ and $U_q(\widehat{\mathfrak{gl}}_m)$ cases. In spite of some differences between the R-matrices of $Y(\mathfrak{gl}_m)$ and $U_q(\widehat{\mathfrak{gl}}_m)$ based models the general structure for the recursions on Bethe vectors, their scalar products, and the properties of the scalar product highest coefficients, is almost identical. Moreover, most proofs literally mimic each other for both cases. Thus, we do not reproduce the proofs entirely, referring the reader to the works [58, 59] for the details. Instead, we mostly focus on the differences between these two cases.

The plan of the article is as follows. We describe our general framework in the two first sections: section 2 contains the algebraic framework used to handle integrable models, and section 3 gathers some properties of the Bethe vectors of $U_q(\widehat{\mathfrak{gl}}_m)$ based models. Section 4 presents our results, which are of two types. Firstly, we show results obtained for generic Bethe vectors: several recursion formulas for the Bethe vectors (section 4.1); a sum formula for their scalar products (section 4.2); and properties of the scalar product highest coefficients (section 4.3). Secondly, considering on-shell Bethe vectors, we give a determinant form *à la Gaudin* for their norm (section 4.4). The following sections are devoted to the proofs of our results. Section 5 deals with the Bethe vectors constructed within the algebraic Bethe ansatz and presents the proofs for the results given in section 4.1. Section 6 contains the proof of the sum formula, and in section 7 we consider the symmetry properties of the highest coefficients. Appendix A presents the explicit construction of Bethe vectors in a particular simple case. Some of the results obtained in the present paper were already presented in the case of $U_q(\widehat{\mathfrak{gl}}_3)$ in different articles: we make the connection with them in appendix B. A coproduct property for dual Bethe vectors is proven in appendix C.

2 Description of the model

2.1 The $U_q(\widehat{\mathfrak{gl}}_m)$ based quantum integrable model

Let $R(u,v)$ be a matrix associated with the vector representation of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_m)$:

$$\begin{aligned} R(u,v) \;=\; f(u,v) \sum_{1\le i\le m} E_{ii}\otimes E_{ii} \;+ \sum_{1\le i<j\le m}\Big(E_{ii}\otimes E_{jj}+E_{jj}\otimes E_{ii}\Big) \\ + \sum_{1\le i<j\le m} g(u,v)\Big(uE_{ij}\otimes E_{ji}+vE_{ji}\otimes E_{ij}\Big), \end{aligned} \tag{2.1}$$

where $(E_{ij})_{lk}=\delta_{il}\delta_{jk}$, $i,j,l,k=1,\dots,m$ are elementary unit matrices and the rational functions $f(u,v)$ and $g(u,v)$ are

$$f(u,v)=\frac{qu-q^{-1}v}{u-v}, \quad g(u,v)=\frac{q-q^{-1}}{u-v}, \tag{2.2}$$

with q a complex parameter not equal to zero. This matrix acts in the tensor product $\mathbf{C}^m\otimes\mathbf{C}^m$ and defines commutation relations

$$R(u,v)\big(T(u)\otimes\mathbf{1}\big)\big(\mathbf{1}\otimes T(v)\big)=\big(\mathbf{1}\otimes T(v)\big)\big(T(u)\otimes\mathbf{1}\big)R(u,v) \tag{2.3}$$

for the quantum monodromy matrix $T(u)$ of some quantum integrable model.

Equation (2.3) holds in the tensor product $\mathbf{C}^m\otimes\mathbf{C}^m\otimes\mathscr{H}$, where $\mathscr{H}$ is a Hilbert space of the model. Being projected onto specific matrix element the commutation relation (2.3) can be written as the relation for the monodromy matrix elements acting in the Hilbert space $\mathscr{H}$

$$\begin{aligned} [T_{i,j}(u),T_{k,l}(v)] \;=\; \big(f(u,v)-1\big)\Big\{\delta_{lj}\,T_{k,j}(v)T_{i,l}(u)-\delta_{ik}\,T_{k,j}(u)T_{i,l}(v)\Big\} \\ +g(u,v)\Big\{\big(u\delta_{l<j}+v\delta_{j<l}\big)T_{k,j}(v)T_{i,l}(u)-\big(u\delta_{i<k}+v\delta_{k<i}\big)T_{k,j}(u)T_{i,l}(v)\Big\}, \end{aligned} \tag{2.4}$$

where $\delta_{i<j}=1$ if $i<j$ and 0 otherwise.

The transfer matrix is defined as the trace of the monodromy matrix

$$\mathscr{T}(u)=\operatorname{tr}T(u)=\sum_{j=1}^{m}T_{j,j}(u). \tag{2.5}$$

It follows from the RTT-relation (2.3) that $[\mathscr{T}(u),\mathscr{T}(v)]=0$. Thus the transfer matrix can be used as a generating function of integrals of motion of an integrable system.

We call such a model $U_q(\widehat{\mathfrak{gl}}_m)$ based quantum integrable model because of the R-matrix used in definition of the commutation relations (2.3) and also because the centerless quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_m)$ itself can be defined using the commutation relations (2.3) by identification of the quantum monodromy matrix $T(u)$ with the generating series of the Borel subalgebra elements in $U_q(\widehat{\mathfrak{gl}}_m)$.

Assume that the operator

$$\mathscr{L}=\lim_{u\to\infty}T(u) \quad \text{with} \quad \mathscr{L}=\sum_{i,j=1}^{m}E_{ij}\otimes\mathscr{L}_{i,j}$$

is well defined. We call such operators $\mathscr{L}_{i,j}$ zero modes operators[1] and it follows from the commutation relations (2.4) that[2]

$$\mathscr{L}_{i,i}\,T_{k,l}(u)=q^{\delta_{il}-\delta_{ik}}\,T_{k,l}(u)\,\mathscr{L}_{i,i}. \tag{2.6}$$

[1]In fact the zero mode generators exist whatever is the asymptotic behavior of $T(u)$ at $u=\infty$. We have taken this particular behavior to simplify the presentation.

[2]To get this result one needs to assume that the zero mode matrix $\mathscr{L}$ is upper-triangular.

Matrix elements $T_{i,j}(u)$ of the monodromy matrix $T(u)$ form the algebra with the commutation relations (2.4) which we denote as $\mathcal{A}^q_m$. Further on we will consider certain morphisms which relate algebras $\mathcal{A}^q_m$ and $\mathcal{A}^{q^{-1}}_m$ (see section 3) as well as embeddings of the smaller rank algebra $\mathcal{A}^q_{m-1}$ into the bigger rank algebra $\mathcal{A}^q_m$.

We wish here to make some comments on the distinction between $\mathcal{A}^q_m$ and $U_q(\widehat{\mathfrak{gl}}_m)$ algebras. The R-matrix we use is definitely the one associated to the $U_q(\widehat{\mathfrak{gl}}_m)$ algebra. However, in order to define this algebra, more elements are needed, such as the Lax operator(s) and their expansion with respect to the spectral parameter. On the other hand, the definition of an integrable model 'only' needs a monodromy matrix obeying an RTT-relation. Hence, we refer to the $\mathcal{A}^q_m$ algebra when dealing with this monodromy matrix, while the denomination $U_q(\widehat{\mathfrak{gl}}_m)$ will be used when mentioning the underlying models.

Most of the time, one may identify the $\mathcal{A}^q_m$ algebra with a Borel subalgebra in the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_m)$. This allows to define the model and its Bethe vectors. However, when considering dual Bethe vectors and the morphism Ψ (see section 3.2) the situation is more delicate. This is particularly acute when the central charge is not zero, and we use the $\mathcal{A}^q_m$ algebra to bypass these subtleties. In particular, the morphism Ψ maps $\mathcal{A}^q_m$ to $\mathcal{A}^{q^{-1}}_m$, while it maps U^+_q to $U^-_{q^{-1}}$, where U^+_q and U^-_q are dual Borel subalgebras in $U_q(\widehat{\mathfrak{gl}}_m)$.

A similar discussion can be found in [32] on the Yangian case.

2.2 Notation

In this paper we use notation and conventions of the work [58]. Besides the functions $g(u,v)$ and $f(u,v)$ (2.2), we introduce the rational functions

$$g^{(r)}(u,v)=v\,g(u,v), \quad g^{(l)}(u,v)=u\,g(u,v). \tag{2.7}$$

Let us formulate now a convention on the notation. We denote sets of variables by bar, for example, $\bar{u}$. When dealing with several of them, we may equip these sets or subsets with additional superscript: $\bar{s}^i$, $\bar{t}^\nu$, etc. Individual elements of the sets or subsets are denoted by Latin subscripts, for instance, u_j is an element of $\bar{u}$, t^i_k is an element of $\bar{t}^i$ etc. Subsets complementary to the elements u_j (resp. t^i_k) are denoted by bar, i.e. $\bar{u}_j$ (resp. $\bar{t}^i_k$). Thus, $\bar{u}_j=\bar{u}\setminus\{u_j\}$ and $\bar{t}^i_k=\bar{t}^i\setminus\{t^i_k\}$. For any set $\bar{u}$, we will note $\#\bar{u}$ the cardinality of the set $\bar{u}$. As a rule, the number of elements in the sets is not shown explicitly in the equations, however we give these cardinalities in special comments to the formulas.

We use a shorthand notation for products of functions f, g or $g^{(l,r)}$: if some function depends on a set of variables (or two sets of variables), this means that one should take the product over the corresponding set (or double product over the two sets). For example,

$$g^{(l)}(\bar{u},v)=\prod_{u_j\in\bar{u}} g^{(l)}(u_j,v), \quad f(\bar{t}^\mu_j,t^\mu_j)=\prod_{\substack{t^\mu_\ell\in\bar{t}^\mu\\ \ell\neq j}} f(t^\mu_\ell,t^\mu_j), \quad f(\bar{s}^j,\bar{t}^i)=\prod_{s^j_k\in\bar{s}^j}\prod_{t^i_\ell\in\bar{t}^i} f(s^j_k,t^i_\ell). \tag{2.8}$$

The same convention is applied to the products of commuting operators. Note that (2.4) implies in particular that

$$[T_{i,j}(u)\,,\,T_{i,j}(v)]=0, \quad \forall\, i,j=1,\dots,m. \tag{2.9}$$

Thus, the notation

$$T_{i,j}(\bar{u})=\prod_{u_k\in\bar{u}} T_{i,j}(u_k) \tag{2.10}$$

is well defined.

By definition, any product over the empty set is equal to 1. A double product is equal to 1 if at least one of the sets is empty. Below we will extend this convention to the products of eigenvalues of the diagonal monodromy matrix entries and their ratios (see (3.3)).

3 Bethe vectors

Pseudovacuum vector. The entries $T_{i,j}(u)$ of the monodromy matrix $T(u)$ act in a Hilbert space $\mathcal{H}$. We do not specify $\mathcal{H}$, but we assume that it contains a *pseudovacuum vector* $|0\rangle$, such that

$$\begin{aligned} T_{i,i}(u)|0\rangle &= \lambda_i(u)|0\rangle, \qquad && i=1,\dots,m, \\ T_{i,j}(u)|0\rangle &= 0, && i>j\,, \end{aligned} \tag{3.1}$$

where $\lambda_i(u)$ are some scalar functions. In the framework of the generalized model [13] considered in this paper, the scalar functions $\lambda_i(u)$ remain free functional parameters. Let us briefly recall that the generalized model is a class of models possessing the same R-matrix (2.1) and having a pseudovacuum vector with the properties (3.1) (see [13, 58] for more details). Any representative of this class can be characterized by a set of functional parameters that are the ratios of the vacuum eigenvalues λ_i:

$$\alpha_i(u) = \frac{\lambda_i(u)}{\lambda_{i+1}(u)}, \qquad i=1,\dots,m-1. \tag{3.2}$$

We extend to these functions the convention on the shorthand notation (2.8), for instance:

$$\lambda_k(\bar u) = \prod_{u_j\in\bar u} \lambda_k(u_j)\,, \qquad \alpha_i(\bar t^i) = \prod_{t^i_\ell\in\bar t^i} \alpha_i(t^i_\ell). \tag{3.3}$$

Coloring. In physical models, the space $\mathcal{H}$ is generated by states with quasiparticles of different types (colors). In $U_q(\widehat{\mathfrak{gl}}_m)$ based models quasiparticles may have $N=m-1$ colors. For any set $\{r_1,\dots,r_N\}$ of non-negative integers, we say that a state has coloring $\{r_1,\dots,r_N\}$, if it contains r_i quasiparticles of the color i. This definition can be formalized at the level of the quantum algebra $U_q(\widehat{\mathfrak{gl}}_m)$ through the diagonal zero modes operators $\mathcal{L}_{k,k}$ (2.6). The colors correspond to the eigenvalues under the commuting generators[3]

$$\mathfrak{h}_j = \prod_{k=1}^{j} \mathcal{L}_{k,k}\,, \quad j=1,...,m-1. \tag{3.4}$$

Indeed, one can check from (2.6) that

$$\mathfrak{h}_j\, T_{k,l}(z) = q^{\varepsilon_j(k,l)}\, T_{k,l}(z)\, \mathfrak{h}_j \quad \text{with} \quad \begin{cases} \varepsilon_j(k,l) = -1, & \text{if} \quad k\le j<l, \\ \varepsilon_j(k,l) = +1, & \text{if} \quad l\le j<k, \\ \varepsilon_j(k,l) = 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

The eigenvalues $\varepsilon_j(k,l)$ just correspond to the coloring mentioned above.

To get a zero coloring of the vector $|0\rangle$, one needs to shift $\mathfrak{h}_j$ to $h_j = \mathfrak{h}_j \prod_{k=1}^{j} \lambda_k[0]^{-1}$, where $\lambda_k[0]$ is the eigenvalue of $|0\rangle$ under $\mathcal{L}_{k,k}$. Then, all states in $\mathcal{H}$ have positive (or null) colors. A state with a given coloring can be obtained by successive application of the creation operators $T_{i,j}$ with $i<j$ to the vector $|0\rangle$. Acting on a state, an operator $T_{i,j}$ with $i<j$ adds one quasiparticle of each colors $i,\dots,j-1$. In particular, the operator $T_{i,i+1}$ creates one quasiparticle of the color i, the operator $T_{1,m}$ creates N quasiparticles of N different colors. The diagonal operators $T_{i,i}$ are neutral, the matrix elements $T_{i,j}$ with $i>j$ play the role of annihilation operators. They remove from any state the quasiparticles with the colors $j,\dots,i-1$, one particle of each color. In particular, if $j-1<k<i$, and the annihilation operator $T_{i,j}$ acts on a state in which there are no particles of the color k, then its action yields zero.

[3] The last generator $\mathfrak{h}_m$ is central, see (3.5).

Bethe vectors. Bethe vectors belong to the space $\mathscr{H}$. Their distinctive feature is that when Bethe equations are fulfilled (see section 3.3) they become eigenvectors of the transfer matrix (2.5). Several explicit forms for Bethe vectors can be found in [37]. We do not use them in the present paper, however, in section 4.1 we give a recursion that formally allows the Bethe vectors to be explicitly constructed. In the present section, we only fix their normalization.

Generically, Bethe vectors are certain polynomials in the creation operators $T_{i,j}$ applied to the vector $|0\rangle$. These polynomials are eigenvectors under the Cartan generators $\mathscr{L}_{k,k}$, and hence they are also eigenvectors of the color generators h_j. Thus, Bethe vectors have a definite coloring and contain only terms with the same coloring.

A generic Bethe vector of $U_q(\widehat{\mathfrak{gl}}_m)$ based model depends on $N = m-1$ sets of variables $\bar{t}^1, \bar{t}^2, \dots, \bar{t}^N$ called Bethe parameters. We denote Bethe vectors by $\mathbb{B}(\bar{t})$, where

$$\bar{t} = \{t_1^1, \dots, t_{r_1}^1; t_1^2, \dots, t_{r_2}^2; \dots; t_1^N, \dots, t_{r_N}^N\}, \tag{3.6}$$

and the cardinalities r_i of the sets $\bar{t}^i$ coincide with the coloring. Thus, each Bethe parameter t_k^i can be associated with a quasiparticle of the color i.

Bethe vectors are symmetric over permutations of the parameters t_k^i within the set $\bar{t}^i$ (see e.g. [37]). However, they are not symmetric over permutations over parameters belonging to different sets $\bar{t}^i$ and $\bar{t}^j$.

We have already mentioned that a generic Bethe vector has the form of a polynomial in $T_{i,j}$ with $i < j$ applied to the pseudovacuum $|0\rangle$. Among all the terms of this polynomial, there is one monomial that contains the operators $T_{i,j}$ with $j - i = 1$ only. Let us call this term the *main term* and denote it by $\widetilde{\mathbb{B}}(\bar{t})$. Then

$$\mathbb{B}(\bar{t}) = \widetilde{\mathbb{B}}(\bar{t}) + \dots, \tag{3.7}$$

where the ellipsis stands for all the terms with the same coloring that contain at least one operator $T_{i,j}$ with $j - i > 1$. We fix the normalization of the Bethe vectors by requiring the following form of the main term

$$\widetilde{\mathbb{B}}(\bar{t}) = \frac{T_{1,2}(\bar{t}^1) \dots T_{N,N+1}(\bar{t}^N)|0\rangle}{\prod_{i=1}^{N} \lambda_{i+1}(\bar{t}^i) \prod_{i=1}^{N-1} f(\bar{t}^{i+1}, \bar{t}^i)}. \tag{3.8}$$

Recall that we use here the shorthand notation for the products of the functions λ_{j+1} and f, as well as for a set of commuting operators $T_{i,i+1}$. Let us stress that this normalization is different from the one used in [37] where the coefficient of the operator product in the definition of $\widetilde{\mathbb{B}}(\bar{t})$ was just 1. This additional normalization factor is convenient, in particular because the scalar products of the Bethe vectors depend on the ratios α_i (3.2) only.

Since the operators $T_{i,i+1}$ and $T_{j,j+1}$ do not commute for $i \neq j$, the main term can be written in several forms corresponding to different ordering of the monodromy matrix entries. The ordering in (3.8) naturally arises if we construct Bethe vectors via the nesting procedure corresponding to the embedding of $\mathscr{A}_{m-1}^q$ in $\mathscr{A}_m^q$ to the lower-right corner of the monodromy matrix $T(u)$.

3.1 Morphism of Bethe vectors

The quantum algebras $\mathscr{A}_m^q$ and $\mathscr{A}_m^{q^{-1}}$ are related by a morphism φ [37]:

$$\varphi\big(T(u)\big) = U\,\widetilde{T}^t(u)\,U^{-1}, \quad \text{i.e.} \quad \varphi\big(T_{a,b}(u)\big) = \widetilde{T}_{m+1-b,m+1-a}(u), \tag{3.9}$$

where $U = \sum_{i=1}^m E_{i,m+1-i}$ and we put a tilde on the generators of $\mathscr{A}_m^{q^{-1}}$ to distinguish them from those of $\mathscr{A}_m^q$. φ defines an idempotent isomorphism from $\mathscr{A}_m^q$ to $\mathscr{A}_m^{q^{-1}}$. This mapping also acts on the vacuum eigenvalues $\lambda_i(u)$ (3.1) and their ratios $\alpha_i(u)$ (3.2)

$$\varphi: \begin{cases} \lambda_i(u) & \to \widetilde{\lambda}_{m+1-i}(u), \qquad i=1,\dots,m\,, \\ \alpha_i(u) & \to \frac{1}{\widetilde{\alpha}_{m-i}(u)}\,, \qquad i=1,\dots,m-1\,. \end{cases} \tag{3.10}$$

We can extend this morphism to representations, defining $\varphi(|0\rangle)=\widetilde{|0\rangle}$, where $|0\rangle$ and $\widetilde{|0\rangle}$ are the pseudovacua in $\mathscr{H}$ and $\widetilde{\mathscr{H}}$ respectively. It has been shown in [37] that this morphism induces the following correspondence between Bethe vectors

Lemma 3.1. *The morphism φ induces a mapping of Bethe vectors $\mathbb{B}_q(\bar t)\in\mathscr{H}$ to Bethe vectors $\mathbb{B}_{q^{-1}}(\bar t)\in\widetilde{\mathscr{H}}$:*

$$\varphi\Big(\mathbb{B}_q(\overrightarrow{t})\Big)=\frac{\mathbb{B}_{q^{-1}}(\overleftarrow{t})}{\prod_{k=1}^{N}\widetilde{\alpha}_{N+1-k}(\bar t^k)}, \tag{3.11}$$

where we have introduced the special orderings of the sets of Bethe parameters[4]

$$\overrightarrow{t}=\{\bar t^1,\bar t^2,\dots,\bar t^N\} \qquad and \qquad \overleftarrow{t}=\{\bar t^N,\dots,\bar t^2,\bar t^1\}. \tag{3.12}$$

3.2 Dual Bethe vectors

Dual Bethe vectors belong to the dual Hilbert space $\mathscr{H}^*$, and they are polynomials in $T_{i,j}$ with $i>j$ applied from the right to the dual pseudovacuum vector $\langle 0|$. This vector possesses the properties similar to (3.1)

$$\begin{aligned} &\langle 0|T_{i,i}(u)=\lambda_i(u)\langle 0|, && i=1,\dots,m,\\ &\langle 0|T_{i,j}(u)=0\,, && i<j\,, \end{aligned} \tag{3.13}$$

where the functions $\lambda_i(u)$ are the same as in (3.1).

We denote dual Bethe vectors by $\mathbb{C}(\bar t)$, where the set of Bethe parameters $\bar t$ consists of several sets $\bar t^i$ as in (3.6). As it was done for Bethe vectors, we can introduce the coloring of the dual Bethe vectors, with now the role of creation and annihilation operators reversed.

One can obtain dual Bethe vectors via the special antimorphism Ψ given by

$$\Psi\big(T(u)\big) \;=\; \widetilde{T}^t(u^{-1}), \quad \text{i.e.} \quad \Psi\big(T_{a,b}(u)\big) \;=\; \widetilde{T}_{b,a}(u^{-1}). \tag{3.14}$$

Ψ defines an idempotent antimorphism from $\mathscr{A}_m^q$ to $\mathscr{A}_m^{q^{-1}}$. Let us extend the action of this antimorphism to the pseudovacuum vectors by

$$\begin{aligned} &\Psi\big(|0\rangle\big)=\widetilde{\langle 0|}, && \Psi\big(A|0\rangle\big)=\widetilde{\langle 0|}\Psi\big(A\big),\\ &\Psi\big(\langle 0|\big)=\widetilde{|0\rangle}, && \Psi\big(\langle 0|A\big)=\Psi\big(A\big)\widetilde{|0\rangle}, \end{aligned} \tag{3.15}$$

where A is any product of $T_{i,j}$. Then it turns out that [37]

$$\Psi\big(\mathbb{B}_q(\bar t)\big)=\mathbb{C}_{q^{-1}}(\bar t^{-1}), \qquad \Psi\big(\mathbb{C}_q(\bar t)\big)=\mathbb{B}_{q^{-1}}(\bar t^{-1}), \tag{3.16}$$

where, again, we put a subscript on (dual) Bethe vectors to distinguish the ones of $\mathscr{A}_m^q$ from those of $\mathscr{A}_m^{q^{-1}}$. We used the notation

$$\bar t^{-1}\equiv\frac{1}{\bar t}\equiv\Big\{\frac{1}{t_1^1},\frac{1}{t_2^1},\dots,\frac{1}{t_{r_1}^1},\frac{1}{t_1^2},\dots,\frac{1}{t_{r_N}^N}\Big\}.$$

[4]Let us stress that the order of the Bethe parameters within every subset $\bar t^k$ is not essential.

The main term of the dual Bethe vector can be obtained from (3.8) via the mapping[5] Ψ:

$$\widetilde{\mathbb{C}}(\bar{t}) = \frac{\langle 0|T_{N+1,N}(\bar{t}^N)\dots T_{2,1}(\bar{t}^1)}{\prod_{i=1}^{N}\lambda_{i+1}(\bar{t}^i)\prod_{i=1}^{N-1}f(\bar{t}^{i+1},\bar{t}^i)}. \tag{3.17}$$

Finally, using the morphism φ we obtain a relation between dual Bethe vectors corresponding to the quantum algebras $\mathscr{A}_m^q$ and $\mathscr{A}_m^{q^{-1}}$

$$\varphi\Big(\mathbb{C}_q(\vec{t})\Big) = \frac{\mathbb{C}_{q^{-1}}(\overleftarrow{t})}{\prod_{k=1}^{N}\widetilde{\alpha}_{N+1-k}(\bar{t}^k)}. \tag{3.18}$$

3.3 On-shell Bethe vectors

For generic Bethe vectors, the Bethe parameters t_k^i are generic complex numbers. If these parameters satisfy a special system of equations (the Bethe equations, see (3.19)), then the corresponding vector becomes an eigenvector of the transfer matrix (2.5). In this case it is called *on-shell Bethe vector*. In most of the paper we consider generic Bethe vectors. However, for the calculation of the norm of Bethe vectors we will consider on-shell Bethe vectors. In that case, the parameters $\bar{t}$ and α_μ will be related by the following system of Bethe equations

$$\alpha_\nu(t_j^\nu) = \frac{f(t_j^\nu,\bar{t}_j^\nu)f(\bar{t}^{\nu+1},t_j^\nu)}{f(\bar{t}_j^\nu,t_j^\nu)f(t_j^\nu,\bar{t}^{\nu-1})}, \qquad \nu=1,\dots,N, \quad j=1,\dots,r_\nu, \tag{3.19}$$

and we recall that $\bar{t}_j^\nu = \bar{t}^\nu \setminus \{t_j^\nu\}$. Usually, when the functions α_μ are given (and define a physical model), one considers these equations as a way to determine the allowed values for the Bethe parameters $\bar{t}$. For the generalized models, where the functions α_μ are not fixed, the Bethe equations form a set of relations between the functional parameters $\alpha_\mu(t_j^\mu)$ and the Bethe parameters t_k^ν.

3.4 Coproduct property and composite models

The proofs for the results shown in the present paper rely on a coproduct property for Bethe vectors, which connects the Bethe vectors belonging to the spaces $\mathscr{H}^{(1)}$ and $\mathscr{H}^{(2)}$ to the Bethe vectors in the space $\mathscr{H}^{(1)}\otimes\mathscr{H}^{(2)}$. This property is intimately related to the notion of composite model, that we introduce now. It is important to point out that in this section we consider Bethe vectors corresponding to different monodromy matrices. We stress it by adding the monodromy matrix to the list of the Bethe vectors arguments. Namely, the notation $\mathbb{B}(\bar{t}|T)$ means that the Bethe vector $\mathbb{B}(\bar{t})$ corresponds to the monodromy matrix T.

In a composite model, the monodromy matrix $T(u)$ is presented as a product of two partial monodromy matrices [32, 62–64]:

$$T(u) = T^{(2)}(u)T^{(1)}(u). \tag{3.20}$$

Here every $T^{(l)}(u)$ satisfies the RTT-relation (2.3) and has its own pseudovacuum vector $|0\rangle^{(l)}$ and dual vector $\langle 0|^{(l)}$, such that $|0\rangle = |0\rangle^{(1)}\otimes|0\rangle^{(2)}$ and $\langle 0| = \langle 0|^{(1)}\otimes\langle 0|^{(2)}$. The operators $T_{i,j}^{(2)}(u)$ and $T_{k,l}^{(1)}(v)$ act in different spaces, and hence, they commute with each other. We assume that

$$\begin{aligned} &T_{i,i}^{(l)}(u)|0\rangle^{(l)} = \lambda_i^{(l)}(u)|0\rangle^{(l)}, \\ &\langle 0|^{(l)}T_{i,i}^{(l)}(u) = \lambda_i^{(l)}(u)\langle 0|^{(l)}, \end{aligned} \qquad i=1,\dots,m, \qquad l=1,2, \tag{3.21}$$

[5] To get a dual Bethe vector in $U_q(\widehat{\mathfrak{gl}}_m)$ one should start from $U_{q^{-1}}(\widehat{\mathfrak{gl}}_m)$, see [37] where these considerations are detailed.

where $\lambda_i^{(l)}(u)$ are new free functional parameters. We also introduce

$$\alpha_k^{(l)}(u) = \frac{\lambda_k^{(l)}(u)}{\lambda_{k+1}^{(l)}(u)}, \qquad l = 1,2, \qquad k = 1,\dots,N. \tag{3.22}$$

Obviously

$$\lambda_i(u) = \lambda_i^{(1)}(u)\lambda_i^{(2)}(u), \qquad \alpha_k(u) = \alpha_k^{(1)}(u)\alpha_k^{(2)}(u). \tag{3.23}$$

The partial monodromy matrices $T^{(l)}(u)$ have the corresponding Bethe vectors $\mathbb{B}(\bar t|T^{(l)})$ and dual Bethe vectors $\mathbb{C}(\bar s|T^{(l)})$. A Bethe vector $\mathbb{B}(\bar t|T)$ of the total monodromy matrix $T(u)$ can be expressed in terms partial Bethe vectors $\mathbb{B}(\bar t|T^{(l)})$ via coproduct formula [34, 35]

$$\mathbb{B}(\bar t|T) = \sum \frac{\prod_{\nu=1}^{N} \alpha_\nu^{(2)}(\bar t_{\rm i}^\nu) f(\bar t_{\rm ii}^\nu, \bar t_{\rm i}^\nu)}{\prod_{\nu=1}^{N-1} f(\bar t_{\rm ii}^{\nu+1}, \bar t_{\rm i}^\nu)} \mathbb{B}(\bar t_{\rm i}|T^{(1)}) \otimes \mathbb{B}(\bar t_{\rm ii}|T^{(2)}). \tag{3.24}$$

Here all the sets of the Bethe parameters $\bar t^\nu$ are divided into two subsets $\bar t^\nu \Rightarrow \{\bar t_{\rm i}^\nu, \bar t_{\rm ii}^\nu\}$, and the sum is taken over all possible partitions.

A similar formula exists for the dual Bethe vectors $\mathbb{C}(\bar s|T)$ (see appendix C)

$$\mathbb{C}(\bar s|T) = \sum \frac{\prod_{\nu=1}^{N} \alpha_\nu^{(1)}(\bar s_{\rm ii}^\nu) f(\bar s_{\rm i}^\nu, \bar s_{\rm ii}^\nu)}{\prod_{\nu=1}^{N-1} f(\bar s_{\rm i}^{\nu+1}, \bar s_{\rm ii}^\nu)} \mathbb{C}(\bar s_{\rm ii}|T^{(2)}) \otimes \mathbb{C}(\bar s_{\rm i}|T^{(1)}), \tag{3.25}$$

where the sum is organised in the same way as in (3.24).

4 Main results

In this section we present the main results of the paper. For generic Bethe vectors, we provide recursion formulas (section 4.1), sum formulas for their scalar products (section 4.2), and recursions for the highest coefficients (section 4.3). For on-shell Bethe vectors, we exhibit a Gaudin determinant form for their norm (section 4.4).

We would like to stress that all the results are given in terms of rational functions $f(u,v)$ (2.2), $g^{(l,r)}(u,v)$ (2.7), and ratios of the eigenvalues $\alpha_i(u)$ (3.2). Therefore, they can easily be compared with the results obtained in [58, 59] for the models with the Yangian R-matrix. This comparison shows that in both cases the results have completely the same structure. The only slight difference consists in the fact that in the case of the Yangian the functions $g^{(l)}(u,v)$ and $g^{(r)}(u,v)$ degenerate into one function $g(u,v)$. As we have already mentioned in Introduction, this similarity of the results is not accidental. It is explained by the similarity of the corresponding R-matrices. Due to this reason the proofs of most of the results listed above for the $U_q(\widehat{\mathfrak{gl}}_m)$ based models are identical to the corresponding proofs in the Yangian case. To show this we give a detailed proof of the sum formula (4.11). However, for the proofs of other statements we refer the reader to the works [58, 59].

The essential difference between models that are described by $Y(\mathfrak{gl}_m)$ and $U_q(\widehat{\mathfrak{gl}}_m)$ algebras is the action of morphisms φ (3.9) and Ψ (3.14). In particular, in the case of the Yangian, the antimorphism (3.14) turns into an endomorphism, while in the $U_q(\widehat{\mathfrak{gl}}_m)$ case this mapping connects two different algebras. Therefore, all the proofs based on the application of the mappings φ and Ψ, are given in details.

4.1 Recursion for Bethe vectors

Here we give recursions for (dual) Bethe vectors. The corresponding proofs are given in section 5.

Proposition 4.1. *Bethe vectors of $U_q(\widehat{\mathfrak{gl}}_m)$ based models satisfy a recursion*

$$\mathbb{B}(\{z,\bar t^1\};\{\bar t^k\}_2^N)=\sum_{j=2}^{N+1}\frac{T_{1,j}(z)}{\lambda_2(z)}\sum_{\text{part}(\bar t^2,\dots,\bar t^{j-1})}\mathbb{B}(\{\bar t^1\};\{\bar t^k_{\rm II}\}_2^{j-1};\{\bar t^k\}_j^N)\\ \times\frac{\prod_{\nu=2}^{j-1}\alpha_\nu(\bar t^\nu_{\rm I})\,g^{(l)}(\bar t^\nu_{\rm I},\bar t^{\nu-1}_{\rm I})f(\bar t^\nu_{\rm II},\bar t^\nu_{\rm I})}{\prod_{\nu=1}^{j-1}f(\bar t^{\nu+1},\bar t^\nu_{\rm I})}. \qquad (4.1)$$

Here for $j>2$ the sets of Bethe parameters $\bar t^2,\dots,\bar t^{j-1}$ are divided into disjoint subsets $\bar t^\nu_{\rm I}$ and $\bar t^\nu_{\rm II}$ ($\nu=2,\dots,j-1$) such that the subset $\bar t^\nu_{\rm I}$ consists of one element only: $\#\bar t^\nu_{\rm I}=1$. The sum is taken over all partitions of this type. We set $\bar t^1_{\rm I}\equiv z$ and $\bar t^{N+1}=\emptyset$. Recall also that $N=m-1$.

We used the following notation in proposition 4.1

$$\mathbb{B}(\{z,\bar t^1\};\{\bar t^k\}_2^N)=\mathbb{B}(\{z,\bar t^1\};\bar t^2;\dots;\bar t^N),\\ \mathbb{B}(\{\bar t^1\};\{\bar t^k_{\rm II}\}_2^{j-1};\{\bar t^k\}_j^N)=\mathbb{B}(\bar t^1;\bar t^2_{\rm II};\dots;\bar t^{j-1}_{\rm II};\bar t^j;\dots;\bar t^N). \qquad (4.2)$$

Similar notation will be used throughout the paper.

Remark. We stress that each of the subsets $\bar t^2_{\rm I},\dots,\bar t^N_{\rm I}$ in (4.1) must consist of exactly one element. However, this condition cannot be achieved if the original Bethe vector $\mathbb{B}(t)$ contains an empty set $\bar t^k=\emptyset$ for some $k\in[2,\dots,N]$. In this case, the sum over j in (4.1) ends at $j=k$. If $\mathbb{B}(t)$ contains several empty sets $\bar t^{k_1},\dots,\bar t^{k_\ell}$, then the sum finishes at $j=\min(k_1,\dots,k_\ell)$.

Using the mapping (3.9) one can obtain a second recursion for the Bethe vectors:

Proposition 4.2. *Bethe vectors of $U_q(\widehat{\mathfrak{gl}}_m)$ based models satisfy a recursion*

$$\mathbb{B}(\{\bar t^k\}_1^{N-1};\{z,\bar t^N\})=\sum_{j=1}^{N}\frac{T_{j,N+1}(z)}{\lambda_{N+1}(z)}\sum_{\text{part}(\bar t^j,\dots,\bar t^{N-1})}\mathbb{B}(\{\bar t^k\}_1^{j-1};\{\bar t^k_{\rm II}\}_j^{N-1};\bar t^N)\\ \times\frac{\prod_{\nu=j}^{N-1}g^{(r)}(\bar t^{\nu+1}_{\rm I},\bar t^\nu_{\rm I})f(\bar t^\nu_{\rm I},\bar t^\nu_{\rm II})}{\prod_{\nu=j}^{N}f(\bar t^\nu_{\rm I},\bar t^{\nu-1})}. \qquad (4.3)$$

Here for $j<N$ the sets of Bethe parameters $\bar t^j,\dots,\bar t^{N-1}$ are divided into disjoint subsets $\bar t^\nu_{\rm I}$ and $\bar t^\nu_{\rm II}$ ($\nu=j,\dots,N-1$) such that the subset $\bar t^\nu_{\rm I}$ consists of one element: $\#\bar t^\nu_{\rm I}=1$. The sum is taken over all partitions of this type. We set by definition $\bar t^N_{\rm I}\equiv z$ and $\bar t^0=\emptyset$.

Remark. If the Bethe vector $\mathbb{B}(t)$ contains several empty sets $\bar t^{k_1},\dots,\bar t^{k_\ell}$, then the sum over j in (4.3) begins with $j=\max(k_1,\dots,k_\ell)+1$.

Acting with the antimorphism (3.14) onto equations (4.1) and (4.3) we arrive at

Corollary 4.3. *Dual Bethe vectors of $U_q(\widehat{\mathfrak{gl}}_m)$ based models satisfy recursions*

$$\mathbb{C}(\{z,\bar s^1\};\{\bar s^k\}_2^N)=\sum_{j=2}^{N+1}\sum_{\text{part}(\bar s^2,\dots,\bar s^{j-1})}\mathbb{C}(\{\bar s^1\};\{\bar s^k_{\rm II}\}_2^{j-1};\{\bar s^k\}_j^N)\frac{T_{j,1}(z)}{\lambda_2(z)}\\ \times\frac{\prod_{\nu=2}^{j-1}\alpha_\nu(\bar s^\nu_{\rm I})\,g^{(r)}(\bar s^\nu_{\rm I},\bar s^{\nu-1}_{\rm I})f(\bar s^\nu_{\rm II},\bar s^\nu_{\rm I})}{\prod_{\nu=1}^{j-1}f(\bar s^{\nu+1},\bar s^\nu_{\rm I})}, \qquad (4.4)$$

and

$$\mathbb{C}(\{\bar s^k\}_1^{N-1};\{z,\bar s^N\})=\sum_{j=1}^{N}\sum_{\text{part}(\bar s^j,\dots,\bar s^{N-1})}\mathbb{C}(\{\bar s^k\}_1^{j-1};\{\bar s^k_{\rm II}\}_j^{N-1};\bar s^N)\frac{T_{N+1,j}(z)}{\lambda_{N+1}(z)}\\ \times\frac{\prod_{\nu=j}^{N-1}g^{(l)}(\bar s^{\nu+1}_{\rm I},\bar s^\nu_{\rm I})f(\bar s^\nu_{\rm I},\bar s^\nu_{\rm II})}{\prod_{\nu=j}^{N}f(\bar s^\nu_{\rm I},\bar s^{\nu-1})}. \qquad (4.5)$$

Here the summation over the partitions occurs as in the formulas (4.1) *and* (4.3). *The subsets* $\bar{s}^{\nu}_{\rm I}$ *consist of one element:* $\#\bar{s}^{\nu}_{\rm I}=1$. *If* $\mathbb{C}(\bar{s})$ *contains empty sets of Bethe parameters, then the sum cuts similarly to the case of the Bethe vectors* $\mathbb{B}(\bar{t})$. *By definition* $\bar{s}^{1}_{\rm I}\equiv z$ *in* (4.4), $\bar{s}^{N}_{\rm I}\equiv z$ *in* (4.5), *and* $\bar{s}^{0}=\bar{s}^{N+1}=\emptyset$.

Applying successively the recursion (4.1), we eventually express a Bethe vector with $\#\bar{t}^1=r_1$ as a linear combination of Bethe vectors with $\#\bar{t}^1=0$. The latter effectively correspond to the quantum algebra $\mathcal{A}^q_{m-1}$:

$$\mathbb{B}^{(m)}(\emptyset;\{\bar{t}^k\}_2^N)=\mathbb{B}^{(m-1)}(\bar{t})\Big|_{\bar{t}^k\to\bar{t}^{k+1}}, \tag{4.6}$$

where we put a superscript to distinguish the Bethe vectors in $\mathcal{A}^q_m$ from those of $\mathcal{A}^q_{m-1}$. Thus, continuing this process we formally can reduce Bethe vectors of $\mathcal{A}^q_m$ to the known ones of $\mathcal{A}^q_2$. Similarly, one can build dual Bethe vectors via (4.4), (4.5). Unfortunately, these procedures are too cumbersome for explicit calculations. However, they can be used to prove various assertions by induction.

4.2 Sum formula for the scalar product

In this section we collect some results concerning scalar products of generic Bethe vectors. The proofs of propositions 4.4 and 4.5 literally coincide with the ones given in [58] for the Yangian case. Nevertheless, to illustrate this similarity we present one of these proofs (proposition 4.5) in section 6.

Let $\mathbb{B}(\bar{t})$ be a generic Bethe vector and $\mathbb{C}(\bar{s})$ be a generic dual Bethe vector. Then their scalar product is defined by

$$S(\bar{s}|\bar{t})=\mathbb{C}(\bar{s})\mathbb{B}(\bar{t}). \tag{4.7}$$

Note that if $\#\bar{t}^k\neq\#\bar{s}^k$ for some $k\in\{1,\dots,N\}$, then the scalar product vanishes. Indeed, in this case the numbers of creation and annihilation operators of the color k in $\mathbb{B}(\bar{t})$ and $\mathbb{C}(\bar{s})$ respectively do not coincide. Thus, in the following we will assume that $\#\bar{t}^k=\#\bar{s}^k=r_k$, $k=1,\dots,N$.

Due to the normalizations (3.8) and (3.17), the scalar product of Bethe vectors depends on the functions λ_i only through the ratios α_i. The following proposition specifies this dependence.

Proposition 4.4. *Let* $\mathbb{B}(\bar{t})$ *be a generic Bethe vector and* $\mathbb{C}(\bar{s})$ *be a generic dual Bethe vector such that* $\#\bar{t}^k=\#\bar{s}^k=r_k$, $k=1,\dots,N$. *Then their scalar product is given by*

$$S(\bar{s}|\bar{t})=\sum W_{\rm part}(\bar{s}_{\rm I},\bar{s}_{\rm II}|\bar{t}_{\rm I},\bar{t}_{\rm II})\prod_{k=1}^{N}\alpha_k(\bar{s}^k_{\rm I})\alpha_k(\bar{t}^k_{\rm II}). \tag{4.8}$$

Here all the sets of the Bethe parameters $\bar{t}^k$ *and* $\bar{s}^k$ *are divided into two subsets* $\bar{t}^k\Rightarrow\{\bar{t}^k_{\rm I},\bar{t}^k_{\rm II}\}$ *and* $\bar{s}^k\Rightarrow\{\bar{s}^k_{\rm I},\bar{s}^k_{\rm II}\}$, *such that* $\#\bar{t}^k_{\rm I}=\#\bar{s}^k_{\rm I}$. *The sum is taken over all possible partitions of this type. The rational coefficients* $W_{\rm part}$ *depend on the partition of* $\bar{t}$ *and* $\bar{s}$, *but not on the vacuum eigenvalues* λ_k. *They are completely determined by the R-matrix of the model.*

Proposition 4.4 states that in the scalar product (4.7), the Bethe parameters of the type k (t^k_j or s^k_j) are arguments of the functions α_k only. This property has been proven for the case of Bethe vectors associated to the Yangian $Y(\mathfrak{gl}(m|n))$ in [58], and the proof for $\mathcal{A}^q_m$ follows exactly the same lines. The only difference lies in the relation (7.7) which now relates scalar products in different quantum algebras. However, this does not affect the functional dependence stated in proposition 4.4. Simply, one has to work the proof simultaneously in $\mathscr{A}_m^q$ and in $\mathscr{A}_m^{q^{-1}}$. We refer the interested reader to [58] for more details.

We would like to stress that the rational functions $W_{\rm part}$ are model independent. Thus, if two different models share the same R-matrix (2.1), then the scalar products of Bethe vectors in these models are given by (4.8) with the same coefficients $W_{\rm part}$. In other words, the model dependent part of the scalar product entirely lies in the α_k functions.

The Highest Coefficient (HC) of the scalar product is defined as the rational coefficient corresponding to the partition $\bar{s}_{\rm I} = \bar{s}$, $\bar{t}_{\rm I} = \bar{t}$, and $\bar{s}_{\rm II} = \bar{t}_{\rm II} = \emptyset$. We denote the HC by $Z(\bar{s}|\bar{t})$:

$$W_{\rm part}(\bar{s},\emptyset|\bar{t},\emptyset) = Z(\bar{s}|\bar{t}). \tag{4.9}$$

It corresponds to the coefficient of $\prod_{k=1}^N \alpha_k(\bar{s}^k)$ in the formula (4.8).

Similarly one can define a conjugated HC $\overline{Z}(\bar{s}|\bar{t})$ as the coefficient corresponding to the partition $\bar{s}_{\rm II} = \bar{s}$, $\bar{t}_{\rm II} = \bar{t}$, and $\bar{s}_{\rm I} = \bar{t}_{\rm I} = \emptyset$.

$$W_{\rm part}(\emptyset,\bar{s}|\emptyset,\bar{t}) = \overline{Z}(\bar{s}|\bar{t}). \tag{4.10}$$

In the following, when speaking of both HC and conjugated HC, we will loosely call them *the HCs*.

The following proposition determines the general coefficient $W_{\rm part}$ in terms of the HCs.

Proposition 4.5. *For a fixed partition $\bar{t}^k \Rightarrow \{\bar{t}^k_{\rm I}, \bar{t}^k_{\rm II}\}$ and $\bar{s}^k \Rightarrow \{\bar{s}^k_{\rm I}, \bar{s}^k_{\rm II}\}$ in* (4.8) *the rational coefficient $W_{\rm part}$ has the following presentation in terms of the HCs:*

$$W_{\rm part}(\bar{s}_{\rm I},\bar{s}_{\rm II}|\bar{t}_{\rm I},\bar{t}_{\rm II}) = Z(\bar{s}_{\rm I}|\bar{t}_{\rm I})\,\overline{Z}(\bar{s}_{\rm II}|\bar{t}_{\rm II})\,\frac{\prod_{k=1}^N f(\bar{s}^k_{\rm II},\bar{s}^k_{\rm I})f(\bar{t}^k_{\rm I},\bar{t}^k_{\rm II})}{\prod_{j=1}^{N-1} f(\bar{s}^{j+1}_{\rm II},\bar{s}^j_{\rm I})f(\bar{t}^{j+1}_{\rm I},\bar{t}^j_{\rm II})}. \tag{4.11}$$

Note that this proposition was already proven in the case of $\mathscr{A}_2^q$ in [13] and $\mathscr{A}_3^q$ in [48]. A comparison with the previous results obtained for $m = 3$ is given in appendix B. The proof for $\mathscr{A}_m^q$ is given in section 6.

4.3 Properties of the highest coefficient

In this section we list several useful properties of the HCs. Most of them are quite analogous to the properties of the HC in the Yangian case (see [58, 59]). The exception is the symmetry properties given in the following proposition.

Proposition 4.6. *The HC and conjugated HC in the quantum algebras $U_q(\widehat{\mathfrak{gl}}_m)$ and $U_{q^{-1}}(\widehat{\mathfrak{gl}}_m)$ are connected through the relations:*

$$Z_q(\vec{s}|\vec{t}) = \overline{Z}_{q^{-1}}(\overleftarrow{s}|\overleftarrow{t}), \tag{4.12}$$

$$\overline{Z}_q(\bar{s}|\bar{t}) = Z_{q^{-1}}(\bar{t}^{\,-1}|\bar{s}^{\,-1}), \tag{4.13}$$

where again we put a subscript to indicate to which algebra the HC corresponds to.

The HC possesses also the symmetry

$$Z_q(\vec{s}|\vec{t}) = Z_q\big(\overleftarrow{t}^{\,-1}|\overleftarrow{s}^{\,-1}\big). \tag{4.14}$$

The proof of this proposition is given in section 7.

Explicit expressions for the HC are known for $m = 2, 3$ [49, 60], but they become very ponderous when m is generic. Fortunately, one can use relatively simple recursions described in the subsequent propositions.

Proposition 4.7. *The HC $Z(\bar{s}|\bar{t})$ possesses the following recursion over the set $\bar{s}^1$:*

$$
\begin{aligned}
Z(\bar{s}|\bar{t}) = \sum_{p=2}^{N+1} \sum_{\substack{\text{part}(\bar{s}^2,\dots,\bar{s}^{p-1})\\ \text{part}(\bar{t}^1,\dots,\bar{t}^{p-1})}} & \frac{g^{(l)}(\bar{t}^1_{\text{I}},\bar{s}^1_{\text{I}}) f(\bar{t}^1_{\text{I}},\bar{t}^1_{\text{II}}) f(\bar{t}^1_{\text{II}},\bar{s}^1_{\text{I}})}{f(\bar{s}^p,\bar{s}^{p-1}_{\text{I}})} \\
& \times \prod_{\nu=2}^{p-1} \frac{g^{(r)}(\bar{s}^\nu_{\text{I}},\bar{s}^{\nu-1}_{\text{I}}) g^{(l)}(\bar{t}^\nu_{\text{I}},\bar{t}^{\nu-1}_{\text{I}}) f(\bar{s}^\nu_{\text{II}},\bar{s}^\nu_{\text{I}}) f(\bar{t}^\nu_{\text{I}},\bar{t}^\nu_{\text{II}})}{f(\bar{s}^\nu,\bar{s}^{\nu-1}_{\text{I}}) f(\bar{t}^\nu_{\text{I}},\bar{t}^{\nu-1})} \\
& \times Z\big(\{\bar{s}^k_{\text{II}}\}_1^{p-1},\{\bar{s}^k\}_p^N \big| \{\bar{t}^k_{\text{II}}\}_1^{p-1};\{\bar{t}^k\}_p^N\big).
\end{aligned} \tag{4.15}
$$

In (4.15)*, for every fixed $p \in \{2,\dots,N+1\}$ the sums are taken over partitions $\bar{t}^k \Rightarrow \{\bar{t}^k_{\text{I}},\bar{t}^k_{\text{II}}\}$ with $k = 1,\dots,p-1$ and $\bar{s}^k \Rightarrow \{\bar{s}^k_{\text{I}},\bar{s}^k_{\text{II}}\}$ with $k = 2,\dots,p-1$, such that $\#\bar{t}^k_{\text{I}} = \#\bar{s}^k_{\text{I}} = 1$ for $k = 2,\dots,p-1$. The subset $\bar{s}^1_{\text{I}}$ is a fixed Bethe parameter from the set $\bar{s}^1$. There is no sum over partitions of the set $\bar{s}^1$ in* (4.15)*.*

The proof of this proposition coincides with the corresponding proof in [58].

Corollary 4.8. *The HC $Z(\bar{s}|\bar{t})$ satisfies the following recursion over the set $\bar{t}^N$:*

$$
\begin{aligned}
Z(\bar{s}|\bar{t}) = \sum_{p=1}^{N} \sum_{\substack{\text{part}(\bar{s}^p,\dots,\bar{s}^N)\\ \text{part}(\bar{t}^p,\dots,\bar{t}^{N-1})}} & \frac{g^{(l)}(\bar{t}^N_{\text{I}},\bar{s}^N_{\text{I}}) f(\bar{s}^N_{\text{II}},\bar{s}^N_{\text{I}}) f(\bar{t}^N_{\text{I}},\bar{s}^N_{\text{II}})}{f(\bar{t}^p_{\text{I}},\bar{t}^{p-1})} \\
& \times \prod_{\nu=p}^{N-1} \frac{g^{(l)}(\bar{s}^{\nu+1}_{\text{I}},\bar{s}^\nu_{\text{I}}) g^{(r)}(\bar{t}^{\nu+1}_{\text{I}},\bar{t}^\nu_{\text{I}}) f(\bar{s}^\nu_{\text{II}},\bar{s}^\nu_{\text{I}}) f(\bar{t}^\nu_{\text{I}},\bar{t}^\nu_{\text{II}})}{f(\bar{s}^{\nu+1},\bar{s}^\nu_{\text{I}}) f(\bar{t}^{\nu+1}_{\text{I}},\bar{t}^\nu)} \\
& \times Z\big(\{\bar{s}^k\}_1^{p-1},\{\bar{s}^k_{\text{II}}\}_p^N \big| \{\bar{t}^k\}_1^{p-1};\{\bar{t}^k_{\text{II}}\}_p^N\big).
\end{aligned} \tag{4.16}
$$

In (4.16)*, for every fixed $p \in \{1,\dots,N\}$ the sums are taken over partitions $\bar{t}^k \Rightarrow \{\bar{t}^k_{\text{I}},\bar{t}^k_{\text{II}}\}$ with $k = p,\dots,N$ and $\bar{s}^k \Rightarrow \{\bar{s}^k_{\text{I}},\bar{s}^k_{\text{II}}\}$ with $k = p,\dots,N-1$, such that $\#\bar{t}^k_{\text{I}} = \#\bar{s}^k_{\text{I}} = 1$ for $k = p,\dots,N-1$. The subset $\bar{t}^N_{\text{I}}$ is a fixed Bethe parameter from the set $\bar{t}^N$. There is no sum over partitions for the set $\bar{t}^N$ in* (4.16)*.*

This recursion follows from (4.15) and equation (4.14).

Remark. Similarly to the recursions for the Bethe vectors the sums over p in (4.15), (4.16) break off, if HC $Z(\bar{s}|\bar{t})$ contains empty sets of the Bethe parameters with the colors $\{k_1,\dots,k_\ell\}$, such that $k_1 < \cdots < k_\ell$. Namely, the sum over p in (4.15) ends at $p = k_1$, while in (4.16) it begins at $p = k_\ell + 1$. These restrictions follow from the corresponding restrictions in the recursions for the Bethe vectors.

Using proposition 4.7 one can built the HC with $\#\bar{s}^1 = \#\bar{t}^1 = r_1$ in terms of the HC with $\#\bar{s}^1 = \#\bar{t}^1 = r_1 - 1$. Iterating the process, $Z(\bar{s}|\bar{t})$ with $\#\bar{s}^1 = \#\bar{t}^1 = r_1$ can be expressed in terms of $Z(\bar{s}|\bar{t})$ with $\#\bar{s}^1 = \#\bar{t}^1 = 0$. Moreover it is obvious, due to (4.6), that

$$
Z^{(m)}(\emptyset,\{\bar{s}^k\}_2^N|\emptyset,\{\bar{t}^k\}_2^N) = Z^{(m-1)}(\{\bar{s}^k\}_2^N|\{\bar{t}^k\}_2^N), \tag{4.17}
$$

where the superscript indicates for which algebra, $\mathscr{A}^q_m$ or $\mathscr{A}^q_{m-1}$, the HC is computed. Thus, equation (4.15) allows one to perform recursion over m as well.

Similarly, corollary 4.8 allows one to find the HC with $\#\bar{s}^N = \#\bar{t}^N = r_N$ in terms of the HC with $\#\bar{s}^N = \#\bar{t}^N = r_N - 1$ and to perform another recursion over m. In both cases, the initial condition corresponds to the $\mathscr{A}^q_2$ case, where the HC is nothing but the Izergin–Korepin determinant [13, 60].

To conclude this section we describe the properties of HC in the poles.

Proposition 4.9. *The HC has poles at $s^\mu_j = t^\mu_j$, $\mu = 1,\dots,N$, $j = 1,\dots,r_\mu$. The residues in these poles are proportional to $Z(\bar s \setminus \{s^\mu_j\}|\bar t \setminus \{t^\mu_j\})$:*

$$Z(\bar s|\bar t)\Big|_{s^\mu_j \to t^\mu_j} = g^{(l)}(t^\mu_j, s^\mu_j)\,\frac{f(\bar t^\mu_j, t^\mu_j) f(s^\mu_j, \bar s^\mu_j)\, Z(\bar s \setminus \{s^\mu_j\}|\bar t \setminus \{t^\mu_j\})}{f(\bar t^{\mu+1}, t^\mu_j) f(s^\mu_j, \bar s^{\mu-1})} + reg, \tag{4.18}$$

where reg means regular terms.

This property is in complete analogy with the Yangian case [59] and can be proved via induction and recursions (4.15), (4.16). In its turn, the residues of the HC play a crucial role in the proof of the Gaudin formula for the norm of on-shell Bethe vectors.

4.4 Norm of on-shell Bethe vectors and Gaudin matrix

The Gaudin matrix G for $U_q(\widehat{\mathfrak{gl}}_m)$ based models is an $N \times N$ block-matrix. The sizes of the blocks $G^{(\mu,\nu)}$ are $r_\mu \times r_\nu$, where $r_\mu = \#\bar t^\mu$. To describe the entries $G^{(\mu,\nu)}_{jk}$ we introduce a function

$$\Phi^{(\mu)}_j = \alpha_\mu(t^\mu_j)\frac{f(\bar t^\mu_j, t^\mu_j)}{f(t^\mu_j, \bar t^\mu_j)}\frac{f(t^\mu_j, \bar t^{\mu-1})}{f(\bar t^{\mu+1}, t^\mu_j)}. \tag{4.19}$$

It is easy to see that Bethe equations (3.19) can be written in terms of $\Phi^{(\mu)}_j$ as

$$\Phi^{(\nu)}_j = 1, \qquad j = 1,\dots,r_\nu, \quad \nu = 1,\dots,N. \tag{4.20}$$

The entries of the Gaudin matrix are defined as

$$G^{(\mu,\nu)}_{jk} = -(q-q^{-1})\, t^\nu_k \frac{\partial \log \Phi^{(\mu)}_j}{\partial t^\nu_k}. \tag{4.21}$$

Explicitly, the diagonal blocks $G^{(\mu,\mu)}$ read

$$G^{(\mu,\mu)}_{jk} = \delta_{jk}\Big[X^\mu_j - \sum_{p=1}^{r_\mu} \mathscr{K}(t^\mu_j, t^\mu_p) + \sum_{q=1}^{r_{\mu-1}} \mathscr{J}(t^\mu_j, t^{\mu-1}_q) + \sum_{r=1}^{r_{\mu+1}} \mathscr{J}(t^{\mu+1}_r, t^\mu_j)\Big] + \mathscr{K}(t^\mu_j, t^\mu_k), \tag{4.22}$$

while the off-diagonal blocks are given by

$$\begin{aligned} &G^{(\mu,\mu-1)}_{jk} = -\mathscr{J}(t^\mu_j, t^{\mu-1}_k), \qquad G^{(\mu,\mu+1)}_{jk} = -\mathscr{J}(t^{\mu+1}_k, t^\mu_j),\\ &G^{(\mu,\nu)}_{jk} = 0 \quad \text{if} \quad |\mu-\nu| > 1. \end{aligned} \tag{4.23}$$

In (4.22) and (4.23), we have introduced the functions

$$X^\mu_j = -(q-q^{-1})\, z \frac{d}{dz} \log \alpha_\mu(z)\Big|_{z=t^\mu_j}, \tag{4.24}$$

$$\mathscr{K}(x,y) = \frac{(q+q^{-1})(q-q^{-1})^2\, xy}{(qx-q^{-1}y)(q^{-1}x-qy)}, \quad \text{and} \quad \mathscr{J}(x,y) = \frac{(q-q^{-1})^2\, xy}{(qx-q^{-1}y)(x-y)}. \tag{4.25}$$

Theorem 4.10. *The square of the norm of the on-shell Bethe vector reads*

$$\mathbb{C}(\bar t)\mathbb{B}(\bar t) = \prod_{k=1}^{N}\Big(f(\bar t^{k+1}, \bar t^k)^{-1} \prod_{\substack{p,q=1\\ p\neq q}}^{r_k} f(t^k_p, t^k_q)\Big) \det G, \tag{4.26}$$

where the matrix G is given by (4.21), or explicitly in (4.22) and (4.23).

The proof of the similar theorem for the models described by the $Y(\mathfrak{gl}_m)$ and $Y(\mathfrak{gl}(m|n))$ R-matrices can be found in [59]. Despite the fact that in the case of $U_q(\widehat{\mathfrak{gl}}_m)$ algebra the proof is completely identical, we will briefly outline the main steps.

The main idea is to prove that the norm of on-shell Bethe vector satisfies several properties called *Korepin criteria*. Namely, let $\mathbf{F}^{(\mathbf{r})}(\bar{X};\bar{t})$ be a function depending on $\mathbf{r}$ variables X_j^μ and $\mathbf{r}$ variables t_j^μ. It is assumed that this function satisfies Korepin criteria, if it possesses the following properties.

(i) The function $\mathbf{F}^{(\mathbf{r})}(\bar{X};\bar{t})$ is symmetric over the replacement of the pairs $(X_j^\mu, t_j^\mu) \longleftrightarrow (X_k^\mu, t_k^\mu)$.

(ii) It is a linear function of each X_j^μ.

(iii) $\mathbf{F}^{(1)}(X_1^1; t_1^1) = X_1^1$ for $\mathbf{r} = 1$.

(iv) The coefficient of X_j^μ is given by a function $\mathbf{F}^{(\mathbf{r}-1)}$ with modified parameters X_k^ν

$$\frac{\partial \mathbf{F}^{(\mathbf{r})}(\bar{X};\bar{t})}{\partial X_j^\mu} = \mathbf{F}^{(\mathbf{r}-1)}(\{\bar{X}^{\rm mod} \setminus X_j^{{\rm mod};\mu}\}; \{\bar{t} \setminus t_j^\mu\}), \tag{4.27}$$

where the original variables X_k^ν should be replaced by $X_k^{{\rm mod};\nu}$:

$$\begin{aligned} X_k^{{\rm mod};\mu} &= X_k^\mu - \mathscr{K}(t_j^\mu, t_k^\mu), \\ X_k^{{\rm mod};\mu+1} &= X_k^{\mu+1} + \mathscr{J}(t_k^{\mu+1}, t_j^\mu), \\ X_k^{{\rm mod};\mu-1} &= X_k^{\mu-1} + \mathscr{J}(t_j^\mu, t_k^{\mu-1}), \\ X_k^{{\rm mod};\nu} &= X_k^\nu, \qquad |\nu-\mu|>1. \end{aligned} \tag{4.28}$$

Here $\mathscr{K}(x,y)$ and $\mathscr{J}(x,y)$ are some two-variables functions. Their explicit forms are not essential.

(v) $\mathbf{F}^{(\mathbf{r})}(\bar{X};\bar{t}) = 0$, if all $X_j^\nu = 0$.

The properties (i)–(v) fix function $\mathbf{F}^{(\mathbf{r})}(\bar{X};\bar{t})$ uniquely (see [13, 59]). On the other hand, one can easily show that these properties are enjoyed by the determinant of the matrix G given by equations (4.22), (4.23). Thus, $\mathbf{F}^{(\mathbf{r})}(\bar{X};\bar{t}) = \det G$.

The proof that the norm of the on-shell vector satisfies Korepin criteria is realized within the framework of the generalized model. In this model, Bethe parameters and logarithmic derivatives X_j^μ (4.24) are independent variables. Then properties (i)–(iii) are fairly obvious. Property (v) follows from the analysis of a special scalar product in which all $X_j^\mu = 0$. Finally, property (iv) is a consequence of the recursions of the highest coefficients with coinciding arguments (4.18). These recursions allow us to establish a recursion for the scalar product, which in turn implies property (iv) for the norm.

5 Proof of recursion for Bethe vectors

5.1 Proofs of proposition 4.1

One can prove proposition 4.1 via direct application of the nested algebraic Bethe ansatz. Let us briefly recall the basic notions of this method and introduce the necessary notation.

The nested algebraic Bethe ansatz relates Bethe vectors of $\mathscr{A}_m^q$ and $\mathscr{A}_{m-1}^q$ invariant systems. To distinguish objects associated to the $\mathscr{A}_{m-1}^q$ algebra from those from the $\mathscr{A}_m^q$ one, we use a special font for the former, keeping the usual style for the later. For example, we denote the basis vectors in $\mathbf{C}^m$ by e_k, where $\big(e_k\big)_j = \delta_{jk}$, and $j,k = 1,\dots,m$, while the basis vectors in $\mathbf{C}^{m-1}$ are denoted by e_k, where $\big(\mathsf{e}_k\big)_j = \delta_{jk}$, and $j,k = 2,\dots,m$. Note that the enumeration of the basis vectors e_k starts at 2, not 1. We will use the same prescription for the other objects related to the $\mathscr{A}_{m-1}^q$ algebra and the $\mathbf{C}^{m-1}$ space.

We present the original monodromy matrix in the block form

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{5.1}$$

where $D(u)$ is a $(m-1)\times(m-1)$ matrix with elements $D_{i,j}(u)$, $i,j = 2,\dots,m$.

Obviously, the elements $D_{i,j}(u)$ enjoy the commutation relations (2.4). Hence, the matrix $D(u)$ satisfies the RTT-relation

$$\mathsf{r}(u,v)\cdot(D(u)\otimes\mathbf{1})\cdot(\mathbf{1}\otimes D(v)) = (\mathbf{1}\otimes D(v))\cdot(D(u)\otimes\mathbf{1})\cdot \mathsf{r}(u,v), \tag{5.2}$$

where $\mathsf{r}(u,v)$ is the R-matrix corresponding to the vector representation of the algebra $U_q(\widehat{\mathfrak{gl}}_{m-1})$

$$\begin{aligned} \mathsf{r}(u,v) \;=\; f(u,v)\sum_{2\le i\le m}\mathsf{E}_{ii}\otimes\mathsf{E}_{ii} \;+\; &\sum_{2\le i<j\le m}\Big(\mathsf{E}_{ii}\otimes\mathsf{E}_{jj}+\mathsf{E}_{jj}\otimes\mathsf{E}_{ii}\Big) \\ +\sum_{2\le i<j\le m} g(u,v)&\Big(u\,\mathsf{E}_{ij}\otimes\mathsf{E}_{ji}+v\,\mathsf{E}_{ji}\otimes\mathsf{E}_{ij}\Big). \end{aligned} \tag{5.3}$$

In (5.3), E_{ij}, $i,j = 2,\dots,m$, are elementary units acting in $\mathbf{C}^{m-1}$, in accordance with the style convention described above.

Now we are in position to describe the main procedure of the nested algebraic Bethe ansatz. Let $\mathbb{B}(\bar t|T) = \mathbb{B}(\bar t^1,\dots,\bar t^{m-1}|T)$ be a Bethe vector of the $U_q(\widehat{\mathfrak{gl}}_m)$ based monodromy matrix $T(u)$ such that $\#\bar t^\nu = r_\nu$. Let us introduce a Hilbert space

$$\mathscr{H}^{(r_1)} = \underbrace{\mathbf{C}^{m-1}\otimes\cdots\otimes\mathbf{C}^{m-1}}_{r_1}, \tag{5.4}$$

and an inhomogeneous monodromy matrix

$$\mathsf{T}_{[r_1]}(u,\bar t^1) = \mathsf{r}_{0,r_1}(u,t^1_{r_1})\dots\mathsf{r}_{0,1}(u,t^1_1). \tag{5.5}$$

Remark that $\mathsf{T}_{[r_1]}(u,\bar t^1)$ corresponds to a $U_q(\widehat{\mathfrak{gl}}_{m-1})$ model. Indeed, in (5.5), $\mathsf{r}_{0,k}(u,t^1_k)$ are the R-matrices (5.3) and they act in $\mathbf{C}^{m-1}\otimes\mathscr{H}^{(r_1)}$. The first subscript refers to an auxiliary space $\mathbf{C}^{m-1}$, while the second subscript refers to the k-th copy of $\mathbf{C}^{m-1}$ in the definition (5.4) of $\mathscr{H}^{(r_1)}$. It is clear that $\mathsf{T}_{[r_1]}(u,\bar t^1)$ satisfies the RTT-relation (5.2).

Consider a monodromy matrix

$$\widetilde{\mathsf{T}}_{[r_1]}(u,\bar t^1) = D(u)\mathsf{T}_{[r_1]}(u,\bar t^1). \tag{5.6}$$

The entries of this matrix act in the space $\mathscr{H}\otimes\mathscr{H}^{(r_1)}$, where $\mathscr{H}$ is the space where the elements of the original monodromy matrix (5.1) act. It is clear that $\widetilde{\mathsf{T}}_{[r_1]}(u,\bar t^1)$ satisfies the RTT relation, because both $D(u)$ and $\mathsf{T}_{[r_1]}(u,\bar t^1)$ satisfy this relation and their matrix elements act in the different quantum spaces (respectively in $\mathscr{H}$ and $\mathscr{H}^{(r_1)}$). The space of states of $\widetilde{\mathsf{T}}_{[r_1]}$ has a pseudovacuum vector $|0\rangle\otimes\Omega_{r_1}$, where

$$\Omega_{r_1} = \underbrace{\mathsf{e}_2\otimes\cdots\otimes\mathsf{e}_2}_{r_1} \in \big(\mathbf{C}^{m-1}\big)^{\otimes r_1}. \tag{5.7}$$

The subscript r_1 on Ω_{r_1} shows the number of copies of $\mathbf{C}^{m-1}$ in the space $\mathscr{H}^{(r_1)}$.

Let $\mathbb{B}(\bar t|\widetilde T_{[r_1]}) = \mathbb{B}(\bar t^2,\dots,\bar t^{m-1}|\widetilde T_{[r_1]})$ be Bethe vectors of the monodromy matrix (5.6), and let $\widetilde\alpha_\nu^{(r_1-1)}(u)$ be the ratios of the vacuum eigenvalues of $\widetilde{\mathsf{T}}_{[r_1-1]}(u)$. Then the Bethe vector $\mathbb{B}(\bar t|T)$ has the following presentation [29, 65]

$$\mathbb{B}(\bar t|T) = \sum_{k_1,\dots,k_{r_1}=2}^{m} \frac{T_{1,k_1}(t_1^1)\dots T_{1,k_{r_1}}(t_{r_1}^1)}{\lambda_2(\bar t^1)f(\bar t^2,\bar t^1)} \big[\mathbb{B}(\bar t|\widetilde T_{[r_1]})\big]_{k_1,\dots,k_{r_1}}, \tag{5.8}$$

where $\big[\mathbb{B}(\bar t|\widetilde T_{[r_1]})\big]_{k_1,\dots,k_{r_1}}$ are components of the vector $\mathbb{B}(\bar t|\widetilde T_{[r_1]})$ in the space $\mathscr{H}^{(r_1)}$.

Representation (5.8) allows us to obtain a recursion for the Bethe vector. This can be done in the framework of a composite model. Indeed, we have

$$\widetilde T_{[r_1]}(u) = \widetilde T_{[r_1-1]}(u)\,\mathsf{r}_{0,1}(u,t_1^1), \tag{5.9}$$

where

$$\widetilde T_{[r_1-1]}(u) = D(u)T_{[r_1-1]}(u) = D(u)\,\mathsf{r}_{0,r_1}(u,t_{r_1}^1)\dots\mathsf{r}_{0,2}(u,t_2^1). \tag{5.10}$$

We can associate the monodromy matrices $\widetilde T_{[r_1-1]}(u)$ and $\mathsf{r}_{0,1}(u,t_1^1)$ respectively with $T^{(2)}(u)$ and $T^{(1)}(u)$ in (3.20). Then the partial Bethe vectors respectively are $\mathbb{B}(\bar t|\widetilde T_{[r_1-1]})$ and $\mathbb{B}(\bar t|\mathsf{r}_{0,1})$. Using the coproduct formula (3.24) we obtain

$$\mathbb{B}(\bar t|T) = \sum_{k_1,\dots,k_{r_1}=2}^{m} \frac{T_{1,k_1}(t_1^1)\dots T_{1,k_{r_1}}(t_{r_1}^1)}{\lambda_2(\bar t^1)f(\bar t^2,\bar t^1)} \times \sum_{\mathrm{part}(\bar t^2,\dots,\bar t^{m-1})} \frac{\prod_{\nu=2}^{m-1}\widetilde\alpha_\nu^{(r_1-1)}(\bar t_{\rm I}^\nu)f(\bar t_{\rm II}^\nu,\bar t_{\rm I}^\nu)}{\prod_{\nu=2}^{m-2}f(\bar t_{\rm II}^{\nu+1},\bar t_{\rm I}^\nu)} \big[\mathbb{B}(\bar t_{\rm II}|\widetilde T_{[r_1-1]})\big]_{k_2,\dots,k_{r_1}} \big[\mathbb{B}(\bar t_{\rm I}|\mathsf{r}_{0,1})\big]_{k_1}. \tag{5.11}$$

The sum is taken over partitions of the sets $\{\bar t^2,\dots,\bar t^{m-1}\}$ as it is described in (3.24). The functions $\widetilde\alpha_\nu^{(r_1-1)}(u)$ are the ratios of the vacuum eigenvalues of $\widetilde{\mathsf{T}}_{[r_1-1]}(u)$

$$\widetilde\alpha_\nu^{(r_1-1)}(u) = \frac{\widetilde\lambda_\nu^{(r_1-1)}(u)}{\widetilde\lambda_{\nu+1}^{(r_1-1)}(u)}, \tag{5.12}$$

where

$$\Big(\widetilde{\mathsf{T}}_{[r_1-1]}(u)\Big)_{\nu,\nu}|0\rangle\otimes\Omega_{r_1-1} = \widetilde\lambda_\nu^{(r_1-1)}(u)|0\rangle\otimes\Omega_{r_1-1}, \tag{5.13}$$

and Ω_{r_1-1} is defined similarly to (5.7). It is convenient to divide the set $\bar t^1$ into two subsets $\bar t^1 = \bar t_{\rm I}^1\cup\bar t_{\rm II}^1$, where $\bar t_{\rm I}^1$ consists of one element t_1^1, and $\bar t_{\rm II}^1 = \{t_2^1,\dots,t_{r_1}^1\}$ is the complementary subset. Then it is easy to see from the definition (5.6) that

$$\begin{aligned}\widetilde\lambda_2^{(r_1-1)}(u) &= \lambda_2(u)f(u,\bar t_{\rm II}^1),\\ \widetilde\lambda_\nu^{(r_1-1)}(u) &= \lambda_\nu(u), \qquad \nu>2,\end{aligned} \tag{5.14}$$

and hence,

$$\begin{aligned}\widetilde\alpha_2^{(r_1-1)}(u) &= \alpha_2(u)f(u,\bar t_{\rm II}^1),\\ \widetilde\alpha_\nu^{(r_1-1)}(u) &= \alpha_\nu(u), \qquad \nu>2.\end{aligned} \tag{5.15}$$

Due to (5.8) we see that

$$\sum_{k_2,\dots,k_{r_1}=2}^{m} \frac{T_{1,k_2}(t_2^1)\dots T_{1,k_{r_1}}(t_{r_1}^1)}{\lambda_2(\bar t_{\rm II}^1)f(\bar t_{\rm II}^2,\bar t_{\rm II}^1)} \big[\mathbb{B}(\bar t_{\rm II}|\widetilde T_{[r_1-1]})\big]_{k_2,\dots,k_{r_1}} = \mathbb{B}(\bar t_{\rm II}|T). \tag{5.16}$$

Substituting this into (5.11) we find

$$\mathbb{B}(\bar t|T)=\sum_{\mathrm{part}(\bar t^2,\dots,\bar t^{m-1})}\sum_{k=2}^{m}\frac{T_{1,k}(t^1_{\rm I})}{\lambda_2(t^1_{\rm I})}\mathbb{B}(\bar t_{\rm II}|T)\frac{\prod_{\nu=2}^{m-1}\alpha_\nu(\bar t^\nu_{\rm I})f(\bar t^\nu_{\rm II},\bar t^\nu_{\rm I})}{\prod_{\nu=2}^{m-2}f(\bar t^{\nu+1}_{\rm II},\bar t^\nu_{\rm I})}\frac{\big[\mathbb{B}(\bar t_{\rm I}|\mathrm{r}_{0,1})\big]_k}{f(\bar t^2,\bar t^1_{\rm I})}. \tag{5.17}$$

The components of the vector $\mathbb{B}(\bar t_{\rm I}|\mathrm{r}_{0,1})$ are computed in appendix A (see (A.4)). It follows from these formulas that the k-th component of this vector corresponds to the partitions for which the subsets $\bar t^k_{\rm I},\dots,\bar t^{m-1}_{\rm I}$ are empty, while the subsets $\bar t^\nu_{\rm I}$ with $2\le\nu<k$ consist of one element. This gives us

$$\mathbb{B}(\bar t|T)=\sum_{\mathrm{part}(\bar t^2,\dots,\bar t^{m-1})}\sum_{k=2}^{m}\frac{T_{1,k}(t^1_{\rm I})}{\lambda_2(t^1_{\rm I})}\mathbb{B}(\{\bar t^\nu_{\rm II}\}_1^{k-1};\{\bar t^\nu\}_k^{m-1}|T)\frac{\prod_{\nu=2}^{k-1}\alpha_\nu(\bar t^\nu_{\rm I})g^{(l)}(\bar t^\nu_{\rm I},\bar t^{\nu-1}_{\rm I})f(\bar t^\nu_{\rm II},\bar t^\nu_{\rm I})}{\prod_{\nu=1}^{k-1}f(\bar t^{\nu+1},\bar t^\nu_{\rm I})}. \tag{5.18}$$

Recall that here by definition the subsets $\bar t^1_{\rm I}$ and $\bar t^1_{\rm II}$ are fixed: $\bar t^1_{\rm I}\equiv t^1_1$ and $\bar t^1_{\rm II}\equiv\bar t^1_1=\bar t^1\setminus t^1_1$. Then, replacing $\bar t^1\to\{z,\bar t^1\}$ and setting $\bar t^1_{\rm I}=z$ we arrive at (4.1).

5.2 Proofs of proposition 4.2

Let us derive now the recursion (4.3) starting with (4.1) and using the morphism (3.9). The proof mimics the one done in [58], and we just point out the differences. Since the mapping (3.9) relates two different quantum algebras $\mathcal{A}^q_m$ and $\mathcal{A}^{q^{-1}}_m$, we use here an additional subscript for the different rational functions, to denote the value of the deformation parameter. For instance

$$f_q(u,v)=\frac{qu-q^{-1}v}{u-v},\quad\text{and}\quad g_q(u,v)=\frac{q-q^{-1}}{u-v}, \tag{5.19}$$

while

$$f_{q^{-1}}(u,v)=\frac{q^{-1}u-qv}{u-v},\quad\text{and}\quad g_{q^{-1}}(u,v)=\frac{q^{-1}-q}{u-v}. \tag{5.20}$$

It is easy to see that

$$g^{(r)}_{q^{-1}}(u,v)=g^{(l)}_q(v,u)\quad\text{and}\quad f_{q^{-1}}(u,v)=f_q(v,u). \tag{5.21}$$

We act with φ onto (4.1) using (3.9)–(3.11). It implies in particular

$$\varphi\left(\mathbb{B}_q(\{\bar t^1\};\{\bar t^k_{\rm II}\}_2^{j-1};\{\bar t^k\}_j^N)\prod_{\nu=2}^{j-1}\alpha_\nu(\bar t^\nu_{\rm I})\right)=\frac{\mathbb{B}_{q^{-1}}(\{\bar t^k\}_N^j;\{\bar t^k_{\rm II}\}_{j-1}^2;\bar t^1)}{\prod_{k=1}^N\tilde\alpha_{N+1-k}(\bar t^k)}. \tag{5.22}$$

Remark that the functions α_ν play a non-trivial role in the game. Then, the action of the morphism φ onto (4.1) gives

$$\mathbb{B}_{q^{-1}}(\{\bar t^k\}_N^2;\{z,\bar t^1\})=\sum_{j=2}^{N+1}\frac{\widetilde T_{N+2-j,N+1}(z)}{\widetilde\lambda_{N+1}(z)}\sum_{\mathrm{part}(\bar t^2,\dots,\bar t^{j-1})}\mathbb{B}_{q^{-1}}(\{\bar t^k\}_N^j;\{\bar t^k_{\rm II}\}_{j-1}^2;\bar t^1)$$
$$\times\frac{\prod_{\nu=2}^{j-1}g^{(l)}_q(\bar t^\nu_{\rm I},\bar t^{\nu-1}_{\rm I})f_q(\bar t^\nu_{\rm II},\bar t^\nu_{\rm I})}{\prod_{\nu=1}^{j-1}f_q(\bar t^{\nu+1},\bar t^\nu_{\rm I})}. \tag{5.23}$$

Using the relations (5.21), relabeling the sets of the Bethe parameters $\bar t^k\to\bar t^{N+1-k}$, changing indices $j\to N+2-j$, $\nu\to N+1-\nu$ and replacing $q^{-1}\to q$ (which means going from $\mathcal{A}^{q^{-1}}_m$ to $\mathcal{A}^q_m$) we get (4.3). □

5.3 Proofs of corollary 4.3

The proof for corollary 4.3 follows the same steps as in section 5.2, but using the antimorphism Ψ instead of the morphism φ. Thus, we just sketch the proof.

One starts with relation (4.1) and applies Ψ, to get in $\mathscr{A}_m^{q^{-1}}$:

$$\mathbb{C}_{q^{-1}}\Big(\big\{\frac{1}{z},\frac{1}{\bar t^1}\big\};\big\{\frac{1}{\bar t^k}\big\}_2^N\Big)=\sum_{j=2}^{N+1}\sum_{\text{part}(\bar t^2,\dots,\bar t^{j-1})}\mathbb{C}_{q^{-1}}\Big(\big\{\frac{1}{\bar t^1}\big\};\big\{\frac{1}{\bar t^k_{\rm II}}\big\}_2^{j-1};\big\{\frac{1}{\bar t^k}\big\}_j^N\Big)\frac{\widetilde T_{j,1}(\frac{1}{z})}{\widetilde\lambda_2(\frac{1}{z})}$$
$$\times\frac{\prod_{\nu=2}^{j-1}\widetilde\alpha_\nu(\frac{1}{\bar t^\nu_{\rm I}})\,g_q^{(l)}(\bar t^\nu_{\rm I},\bar t^{\nu-1}_{\rm I})f_q(\bar t^\nu_{\rm II},\bar t^\nu_{\rm I})}{\prod_{\nu=1}^{j-1}f_q(\bar t^{\nu+1},\bar t^\nu_{\rm I})}. \tag{5.24}$$

Now, renaming the parameters $t^\nu_k\to\frac{1}{t^\nu_k}$, $z\to\frac{1}{z}$ and using the relations

$$g_q^{(r)}\Big(\frac{1}{x},\frac{1}{y}\Big)=g_{q^{-1}}^{(l)}(x,y)\quad\text{and}\quad f_q\Big(\frac{1}{x},\frac{1}{y}\Big)=f_{q^{-1}}(x,y) \tag{5.25}$$

we obtain

$$\mathbb{C}_{q^{-1}}(\{z,\bar t^1\};\{\bar t^k\}_2^N)=\sum_{j=2}^{N+1}\sum_{\text{part}(\bar t^2,\dots,\bar t^{j-1})}\mathbb{C}_{q^{-1}}(\{\bar t^1\};\{\bar t^k_{\rm II}\}_2^{j-1};\{\bar t^k\}_j^N)\frac{\widetilde T_{j,1}(z)}{\widetilde\lambda_2(z)}$$
$$\times\frac{\prod_{\nu=2}^{j-1}\widetilde\alpha_\nu(\bar t^\nu_{\rm I})\,g_{q^{-1}}^{(r)}(\bar t^\nu_{\rm I},\bar t^{\nu-1}_{\rm I})f_{q^{-1}}(\bar t^\nu_{\rm II},\bar t^\nu_{\rm I})}{\prod_{\nu=1}^{j-1}f_{q^{-1}}(\bar t^{\nu+1},\bar t^\nu_{\rm I})}. \tag{5.26}$$

It remains to change $q^{-1}\to q$ to get relation (4.4). Similar considerations lead to (4.5). □

6 Proof of proposition 4.5

In this section we provide an explicit representation of the rational coefficients $W_{\rm part}$ (4.8) in terms of the HC. For this we consider the original monodromy matrix $T(u)$ as a monodromy matrix of a composite model (3.20). Then we should use the representation (3.24) for the Bethe vector $\mathbb{B}(\bar t)$ and the representation (3.25) for the dual vector $\mathbb{C}(\bar s)$. As a consequence, the scalar product $S(\bar s|\bar t)=\mathbb{C}(\bar s)\mathbb{B}(\bar t)$ takes the form

$$S(\bar s|\bar t)=\sum\frac{\prod_{\nu=1}^{N}\alpha_\nu^{(1)}(\bar s^\nu_{\rm ii})\alpha_\nu^{(2)}(\bar t^\nu_{\rm i})f(\bar s^\nu_{\rm i},\bar s^\nu_{\rm ii})f(\bar t^\nu_{\rm ii},\bar t^\nu_{\rm i})}{\prod_{\nu=1}^{N-1}f(\bar s^{\nu+1}_{\rm i},\bar s^\nu_{\rm ii})f(\bar t^{\nu+1}_{\rm ii},\bar t^\nu_{\rm i})}S^{(1)}(\bar s_{\rm i}|\bar t_{\rm i})S^{(2)}(\bar s_{\rm ii}|\bar t_{\rm ii}), \tag{6.1}$$

where

$$S^{(1)}(\bar s_{\rm i}|\bar t_{\rm i})=\mathbb{C}(\bar s_{\rm i}|T^{(1)})\mathbb{B}(\bar t_{\rm i}|T^{(1)}),\qquad S^{(2)}(\bar s_{\rm ii}|\bar t_{\rm ii})=\mathbb{C}(\bar s_{\rm ii}|T^{(2)})\mathbb{B}(\bar t_{\rm ii}|T^{(2)}). \tag{6.2}$$

Note that in this formula $\#\bar s^\nu_{\rm i}=\#\bar t^\nu_{\rm i}$, (and hence, $\#\bar s^\nu_{\rm ii}=\#\bar t^\nu_{\rm ii}$), otherwise the scalar products $S^{(1)}$ and $S^{(2)}$ vanish. Let $\#\bar s^\nu_{\rm i}=\#\bar t^\nu_{\rm i}=k'_\nu$, where $k'_\nu=0,1,\dots,r_\nu$. Then $\#\bar s^\nu_{\rm ii}=\#\bar t^\nu_{\rm ii}=r_\nu-k'_\nu$.

Now let us turn to equation (4.8). Our goal is to express the rational coefficients $W_{\rm part}$ in terms of the HC. For this we use the fact that $W_{\rm part}$ are model independent. Therefore, we can find them in some special model whose monodromy matrix satisfies the RTT-relation.

Let us fix some partitions of the Bethe parameters in (4.8): $\bar s^\nu\Rightarrow\{\bar s^\nu_{\rm I},\bar s^\nu_{\rm II}\}$ and $\bar t^\nu\Rightarrow\{\bar t^\nu_{\rm I},\bar t^\nu_{\rm II}\}$ such that $\#\bar s^\nu_{\rm I}=\#\bar t^\nu_{\rm I}=k_\nu$, for some $k_\nu=0,1,\dots,r_\nu$. Hence, $\#\bar s^\nu_{\rm II}=\#\bar t^\nu_{\rm II}=r_\nu-k_\nu$. Consider a concrete model, in which

$$\begin{aligned}\alpha_\nu^{(1)}(z)&=0,\quad\text{if}\quad z\in\bar s^\nu_{\rm II},\\ \alpha_\nu^{(2)}(z)&=0,\quad\text{if}\quad z\in\bar t^\nu_{\rm I}.\end{aligned} \tag{6.3}$$

Due to (3.23) these conditions imply

$$\alpha_\nu(z) = 0, \quad \text{if} \quad z \in \bar{s}^\nu_{\rm II} \cup \bar{t}^\nu_{\rm I}. \tag{6.4}$$

Then the scalar product is proportional to the coefficient $W_{\rm part}(\bar{s}_{\rm I}, \bar{s}_{\rm II}|\bar{t}_{\rm I}, \bar{t}_{\rm II})$, because all other terms in the sum over partitions (4.8) vanish due to the condition (6.4). Thus,

$$S(\bar{s}|\bar{t}) = W_{\rm part}(\bar{s}_{\rm I}, \bar{s}_{\rm II}|\bar{t}_{\rm I}, \bar{t}_{\rm II}) \prod_{k=1}^{N} \alpha_k(\bar{s}^k_{\rm I})\alpha_k(\bar{t}^k_{\rm II}). \tag{6.5}$$

On the other hand, (6.3) implies that a non-zero contribution in (6.1) occurs if and only if $\bar{s}^\nu_{\rm ii} \subset \bar{s}^\nu_{\rm I}$ and $\bar{t}^\nu_{\rm i} \subset \bar{t}^\nu_{\rm II}$. Hence, $r_\nu - k'_\nu \le k_\nu$ and $k'_\nu \le r_\nu - k_\nu$. But this is possible if and only if $k'_\nu + k_\nu = r_\nu$. Thus, $\bar{s}^\nu_{\rm ii} = \bar{s}^\nu_{\rm I}$ and $\bar{t}^\nu_{\rm i} = \bar{t}^\nu_{\rm II}$. Then, for the complementary subsets we obtain $\bar{s}^\nu_{\rm i} = \bar{s}^\nu_{\rm II}$ and $\bar{t}^\nu_{\rm ii} = \bar{t}^\nu_{\rm I}$. Thus, we arrive at

$$S(\bar{s}|\bar{t}) = \frac{\prod_{\nu=1}^{N} \alpha^{(1)}_\nu(\bar{s}^\nu_{\rm I})\alpha^{(2)}_\nu(\bar{t}^\nu_{\rm II}) f(\bar{s}^\nu_{\rm II}, \bar{s}^\nu_{\rm I}) f(\bar{t}^\nu_{\rm I}, \bar{t}^\nu_{\rm II})}{\prod_{\nu=1}^{N-1} f(\bar{s}^{\nu+1}_{\rm II}, \bar{s}^\nu_{\rm I}) f(\bar{t}^{\nu+1}_{\rm I}, \bar{t}^\nu_{\rm II})} S^{(1)}(\bar{s}_{\rm II}|\bar{t}_{\rm II}) S^{(2)}(\bar{s}_{\rm I}|\bar{t}_{\rm I}). \tag{6.6}$$

It is easy to see that calculating the scalar product $S^{(1)}(\bar{s}_{\rm II}|\bar{t}_{\rm II})$ we should take only the term corresponding to the conjugated HC. Indeed, all other terms are proportional to $\alpha^{(1)}_\nu(z)$ with $z \in \bar{s}^\nu_{\rm II}$, therefore, they vanish. Hence

$$S^{(1)}(\bar{s}_{\rm II}|\bar{t}_{\rm II}) = \prod_{\nu=1}^{N} \alpha^{(1)}_\nu(\bar{t}^\nu_{\rm II}) \cdot \overline{Z}(\bar{s}_{\rm II}|\bar{t}_{\rm II}). \tag{6.7}$$

Similarly, calculating the scalar product $S^{(2)}(\bar{s}_{\rm I}|\bar{t}_{\rm I})$ we should take only the term corresponding to the HC:

$$S^{(2)}(\bar{s}_{\rm I}|\bar{t}_{\rm I}) = \prod_{\nu=1}^{N} \alpha^{(2)}_\nu(\bar{s}^\nu_{\rm I}) \cdot Z(\bar{s}_{\rm I}|\bar{t}_{\rm I}). \tag{6.8}$$

Substituting this into (6.6) and using (3.23), (6.5) we arrive at (4.11).

The reader can easily convince himself that the above proof coincides with the one given in [58] for the $Y(\mathfrak{gl}(m|n))$ based models.

As already mentioned, the proofs for the results presented in section 4.2 and 4.4 are also similar to those of the $Y(\mathfrak{gl}(m|n))$ based models and given in [58, 59], thus we don't repeat them here. In the following section we deal with the proof for section 4.3, focusing on the parts that truly differ from the Yangian case.

7 Symmetry of the highest coefficient

To prove (4.12), we consider the sum formula (4.8)

$$S_q(\vec{s}|\vec{t}) = \sum W^q_{\rm part}(\vec{s}_{\rm I}, \vec{s}_{\rm II}|\vec{t}_{\rm I}, \vec{t}_{\rm II}) \prod_{k=1}^{N} \alpha_k(\bar{s}^k_{\rm I})\alpha_k(\bar{t}^k_{\rm II}), \tag{7.1}$$

where we have stressed the ordering (3.12) of the Bethe parameters and put a label q to distinguish scalar product for the algebra $\mathscr{A}^q_m$ from $\mathscr{A}^{q^{-1}}_m$. Let us act with the morphism φ (3.9) on this scalar product. This can be done in two ways. First, using (3.11) and (3.18) we obtain

$$\varphi\Big(S_q(\vec{s}|\vec{t})\Big) = \varphi\Big(\mathbb{C}_q(\vec{s})\mathbb{B}_q(\vec{t})\Big) = \frac{\mathbb{C}_{q^{-1}}(\overleftarrow{s})\mathbb{B}_{q^{-1}}(\overleftarrow{t})}{\prod_{k=1}^{N}\widetilde{\alpha}_{N+1-k}(\bar{s}^k)\widetilde{\alpha}_{N+1-k}(\bar{t}^k)}$$
$$= \frac{S_{q^{-1}}(\overleftarrow{s}|\overleftarrow{t})}{\prod_{k=1}^{N}\widetilde{\alpha}_{N+1-k}(\bar{s}^k)\widetilde{\alpha}_{N+1-k}(\bar{t}^k)}. \tag{7.2}$$

The scalar product $S_{q^{-1}}(\overleftarrow{s}|\overleftarrow{t})$ has the standard representation (4.8). Thus, we find

$$\varphi\Big(S_q(\vec{s}|\vec{t})\Big) = \sum_{\rm part}\frac{W_{\rm part}^{q^{-1}}(\overleftarrow{s}_{\rm I},\overleftarrow{s}_{\rm II}|\overleftarrow{t}_{\rm I},\overleftarrow{t}_{\rm II})}{\prod_{k=1}^{N}\widetilde{\alpha}_{N+1-k}(\bar{s}^k)\widetilde{\alpha}_{N+1-k}(\bar{t}^k)}\prod_{k=1}^{N}\widetilde{\alpha}_k(\bar{s}_{\rm I}^{N-k+1})\widetilde{\alpha}_k(\bar{t}_{\rm II}^{N-k+1}). \tag{7.3}$$

On the other hand, acting with φ directly on the sum formula (7.1) we have

$$\varphi\Big(S_q(\vec{s}|\vec{t})\Big) = \sum_{\rm part}W_{\rm part}^{q}(\vec{s}_{\rm I},\vec{s}_{\rm II}|\vec{t}_{\rm I},\vec{t}_{\rm II})\prod_{k=1}^{N}\Big(\widetilde{\alpha}_{N+1-k}(\bar{s}_{\rm I}^k)\widetilde{\alpha}_{N+1-k}(\bar{t}_{\rm II}^k)\Big)^{-1}. \tag{7.4}$$

Comparing (7.3) and (7.4) we arrive at

$$\sum_{\rm part}W_{\rm part}^{q^{-1}}(\overleftarrow{s}_{\rm I},\overleftarrow{s}_{\rm II}|\overleftarrow{t}_{\rm I},\overleftarrow{t}_{\rm II})\prod_{k=1}^{N}\widetilde{\alpha}_{N+1-k}(\bar{s}_{\rm I}^k)\widetilde{\alpha}_{N+1-k}(\bar{t}_{\rm II}^k)$$
$$= \sum_{\rm part}W_{\rm part}^{q}(\vec{s}_{\rm I},\vec{s}_{\rm II}|\vec{t}_{\rm I},\vec{t}_{\rm II})\prod_{k=1}^{N}\widetilde{\alpha}_{N+1-k}(\bar{s}_{\rm II}^k)\widetilde{\alpha}_{N+1-k}(\bar{t}_{\rm I}^k). \tag{7.5}$$

Since α_i are free functional parameters, the coefficients of the same products of $\widetilde{\alpha}_i$ must be equal. Hence,

$$W_{\rm part}^{q}(\vec{s}_{\rm I},\vec{s}_{\rm II}|\vec{t}_{\rm I},\vec{t}_{\rm II}) = W_{\rm part}^{q^{-1}}(\overleftarrow{s}_{\rm II},\overleftarrow{s}_{\rm I}|\overleftarrow{t}_{\rm II},\overleftarrow{t}_{\rm I}), \tag{7.6}$$

for arbitrary partitions of the sets $\bar{s}$ and $\bar{t}$. In particular, setting $\bar{s}_{\rm II} = \bar{t}_{\rm II} = \emptyset$ we obtain (4.12).

To prove (4.13), we start again with the sum formula (4.8) and use the antimorphism Ψ:

$$\Psi(S_q(\bar{s}|\bar{t})) = \mathbb{C}_{q^{-1}}(\bar{t}^{-1})\mathbb{B}_{q^{-1}}(\bar{s}^{-1}) = S_{q^{-1}}(\bar{t}^{-1}|\bar{s}^{-1}). \tag{7.7}$$

The lhs of (7.7) can be computed from the relation (4.8):

$$\Psi(S_q(\bar{s}|\bar{t})) = \sum W_{\rm part}^{q}(\bar{s}_{\rm I},\bar{s}_{\rm II}|\bar{t}_{\rm I},\bar{t}_{\rm II})\prod_{k=1}^{N}\widetilde{\alpha}_k\Big(\frac{1}{\bar{s}_{\rm I}^k}\Big)\widetilde{\alpha}_k\Big(\frac{1}{\bar{t}_{\rm II}^k}\Big). \tag{7.8}$$

The rhs of (7.7) is computed directly from (4.8) written for $\mathscr{A}_m^{q^{-1}}$:

$$S_{q^{-1}}(\bar{t}^{-1}|\bar{s}^{-1}) = \sum W_{\rm part}^{q^{-1}}(\bar{t}_{\rm I}^{-1},\bar{t}_{\rm II}^{-1}|\bar{s}_{\rm I}^{-1},\bar{s}_{\rm II}^{-1})\prod_{k=1}^{N}\widetilde{\alpha}_k\Big(\frac{1}{\bar{t}_{\rm I}^k}\Big)\widetilde{\alpha}_k\Big(\frac{1}{\bar{s}_{\rm II}^k}\Big). \tag{7.9}$$

Since α_i are free functional parameters, the comparison of these two equalities leads to

$$W_{\rm part}^{q}(\bar{s}_{\rm I},\bar{s}_{\rm II}|\bar{t}_{\rm I},\bar{t}_{\rm II}) = W_{\rm part}^{q^{-1}}(\bar{t}_{\rm II}^{-1},\bar{t}_{\rm I}^{-1}|\bar{s}_{\rm II}^{-1},\bar{s}_{\rm I}^{-1}). \tag{7.10}$$

Setting $\bar{s}_{\rm I} = \bar{t}_{\rm I} = \emptyset$, we get (4.13).

Combining (4.12) and (4.13), we get (4.14).

Applying the property (4.14) to (4.15), one obtains a new recursion written for the parameters $\bar{t}^{-1}$ and $\bar{s}^{-1}$. Using the relations

$$g^{(l)}\Big(\frac{1}{x},\frac{1}{y}\Big) = g^{(l)}(y,x) \quad \text{and} \quad f\Big(\frac{1}{x},\frac{1}{y}\Big) = f(y,x)$$

together with the replacement $\bar{t}^{-1} \to \bar{t}$ and $\bar{s}^{-1} \to \bar{s}$, we get the recursion (4.16) for the highest coefficient.

Conclusion

In this paper, we have shown how the results obtained for the scalar products and the norm of Bethe vectors for $Y(\mathfrak{gl}(m))$ based models can be generalized to the case of $U_q(\widehat{\mathfrak{gl}}_m)$ based models. In this way, we have obtained recursion formulas for the Bethe vectors of these models, as well as a sum formula for their scalar products. We have obtained different recursions for the highest coefficients, which characterize the sum formula. When the Bethe vectors are on-shell, we have also shown that their norm takes the form of a Gaudin determinant.

Comparing these results with the ones obtained for the case of $Y(\mathfrak{gl}(m))$, one can see that for the most of them the generalization is quite straightforward. The only minor difference is that in the Yangian case the highest coefficient of the scalar product coincides with its conjugated, while for the $\mathscr{A}_m^q$ algebra they are related by the transformations (4.12), (4.13). This difference was already pointed out in [49] for the particular case of the $U_q(\widehat{\mathfrak{gl}}_3)$ based models.

The sum formula itself is rather bulky, however, we recall that it is obtained for the most general case of the Bethe vectors scalar product. This formula can be used as a starting point for calculating form factors of the monodromy matrix entries. In this case we deal with scalar products involving on-shell Bethe vectors. Then, the free functional parameters $\alpha_k(u)$ disappear from the sum formula due to Bethe equations, and we obtain a possibility for additional re-summation. This re-summation might lead to compact determinant representations for form factors (see e.g. [50] for the $\mathscr{A}_3^q$ case), like in the case of the norm of on-shell Bethe vector.

One more possible simplification of the sum formula is related to consideration of specific models, in which the free functional parameters $\alpha_k(u)$ are fixed. For instance, for the spin chain based on $U_q(\widehat{\mathfrak{gl}}_m)$ fundamental representations, $\alpha_1(u)$ is a rational function, while $\alpha_k(u) = 1$ for $k > 1$. Thus, in this case most of these functional parameters also disappear from the sum formula, which gives a chance for its simplification.

These two possibilities of further development certainly are worthy of attention. Finally, we wish to note that it seems to us rather obvious that the results presented here can also be readily generalized to the case of models based on $U_q(\widehat{\mathfrak{gl}}(m|n))$. We plan to come back on this generalization in a further publication.

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A The simplest $U_q(\widehat{\mathfrak{gl}}_m)$ Bethe vectors

In this section we construct Bethe vectors for a very specific case of the $\mathscr{A}_m^q$ monodromy matrix $T(u) = R(u,\xi)$, where $R(u,\xi)$ is given by (2.1) and ξ is a complex number. In other words, we consider spin chain with only one site which carries a fundamental representation of $\mathscr{A}_m^q$. The Bethe vector construction procedure is still based on the embedding (5.1) of $\mathscr{A}_{m-1}^q$ into $\mathscr{A}_m^q$. In this appendix, to distinguish Bethe vectors corresponding to the R-matrices (2.1) and (5.3) we respectively equip them with superscripts (m) or $(m-1)$.

This case has many peculiarities which allow a simple and explicit calculation of Bethe vectors. First of all, the space of states is $\mathscr{H} = \mathbf{C}^m$ with the pseudovacuum $|0\rangle = e_1$. As usual, the Bethe vectors depend on $N = m-1$ sets of variables $\bar{t}^{\nu}$. However, due to the nilpotency of the creation operators[6] each set consists at most of one element. Furthermore, $D_{i,i}|0\rangle = |0\rangle$ for all $i = 2, \ldots, m$. Therefore, in the framework of the algebraic Bethe ansatz, the matrix D is equivalent to the identity matrix. Hence, we can omit this matrix in the definition (5.6).

Proposition A.1. *The monodromy matrix $T(u) = R(u,\xi)$ has $m-1$ Bethe vectors of the form*

$$\mathbb{B}^{(m)}(\{t^\nu\}_1^{k-1}, \{\emptyset\}_k^{m-1}) = \left(\prod_{\nu=2}^{k-1} \frac{g^{(l)}(t^\nu, t^{\nu-1})}{f(t^\nu, t^{\nu-1})}\right) g^{(l)}(t^1,\xi)\, e_k, \qquad k = 2, \ldots, m. \tag{A.1}$$

One additional Bethe vector coincides with the pseudovacuum e_1.

Proof. One can easily prove (A.1) via induction over m. Indeed, for $m = 2$ we have only two Bethe vectors: the pseudovacuum $\mathsf{e}_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathbf{C}^2$ and

$$\mathbb{B}^{(2)}(t^1) = T_{12}(t^1)\,\mathsf{e}_1 = g^{(l)}(t^1,\xi)\,\mathsf{E}_{21}\,\mathsf{e}_1 = g^{(l)}(t^1,\xi)\,\mathsf{e}_2 = g^{(l)}(t^1,\xi)\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right). \tag{A.2}$$

Assume that (A.1) holds for $m-1$. One of the $U_q(\widehat{\mathfrak{gl}}_m)$ Bethe vectors still coincides with the pseudovacuum vector $\mathbb{B}^{(m)}(\emptyset) = e_1$. The other Bethe vectors can be constructed via (5.8), where one should set $\lambda_2(u) = 1$:

$$\mathbb{B}^{(m)}(t^1, \ldots, t^{m-1}) = \sum_{k=2}^{m} T_{1,k}(t^1)\, e_1 \frac{\left[\mathbb{B}^{(m-1)}(t^2, \ldots, t^{m-1})\right]_k}{f(t^2, t^1)}. \tag{A.3}$$

Here $\left[\mathbb{B}^{(m-1)}(t^2, \ldots, t^{m-1})\right]_k$ is the k-th component of the Bethe vector $\mathbb{B}^{(m-1)}(\bar{t})$ of the monodromy matrix $\mathsf{r}(u,t^1)$ (5.3). Due to the induction assumption we have

$$\left[\mathbb{B}^{(m-1)}(\{t^\nu\}_2^{j-1}, \{\emptyset\}_j^{m-1})\right]_k = \delta_{jk}\left(\prod_{\nu=3}^{k-1} \frac{g^{(l)}(t^\nu, t^{\nu-1})}{f(t^\nu, t^{\nu-1})}\right) g^{(l)}(t^2, t^1). \tag{A.4}$$

Thus, taking into account that for $k > 1$, $T_{1,k}(u) = g^{(l)}(u,\xi) E_{k1}$ and

$$T_{1,k}(t^1) e_1 = g^{(l)}(t^1,\xi) e_k, \tag{A.5}$$

we immediately arrive at (A.1).

B Comparison with known results of $U_q(\widehat{\mathfrak{gl}}_3)$ based models

Propositions 4.4 and 4.5 were already obtained for $m = 3$ in [46, 49], but using different normalization of Bethe vectors, and a different notation and normalization for the HC. We present here the connection between the two conventions. To clarify the presentation we will put a subscript *old* for the quantities dealt in [46, 49], and a subscript *new* for the ones used in the present article.

Normalisation of (dual) Bethe vectors. By comparison of their main terms, we get the following correspondence for Bethe vectors:

$$\mathbb{B}_{new}(\bar{t}) = \frac{\lambda_2(\bar{t}^2)}{\lambda_3(\bar{t}^2)} \mathbb{B}_{old}(\bar{t}^1, \bar{t}^2) \quad \text{and} \quad \mathbb{C}_{new}(\bar{s}) = \frac{\lambda_2(\bar{s}^2)}{\lambda_3(\bar{s}^2)} \mathbb{C}_{old}(\bar{s}^1, \bar{s}^2), \tag{B.1}$$

where $\bar{s} = \{\bar{s}^1, \bar{s}^2\}$ and $\bar{t} = \{\bar{t}^1, \bar{t}^2\}$. Note that in [46, 49], the sets $\bar{s}^1, \bar{s}^2$ and $\bar{t}^1, \bar{t}^2$ were noted $\bar{u}^{\mathrm{C}}, \bar{v}^{\mathrm{C}}$ and $\bar{u}^{\mathrm{B}}, \bar{v}^{\mathrm{B}}$ respectively.

[6]Obviously, $T_{i,j}(u) = g^{(l)}(u,\xi) E_{ji}$ for $i < j$.

Sum formula. Once the normalisation is fixed, one can compare the scalar product of Bethe vectors and the expressions given in proposition 4.4. In [49], the scalar product is expressed in term of functionals $r_1(z) = \alpha_1(z)$ and $r_3(z) = \alpha_2(z)^{-1}$. Using the normalisation (B.1), we get a sum formula identical to (4.8) with

$$W_{old}\begin{pmatrix} \bar{s}^1_{\rm I} & \bar{t}^1_{\rm I} & \bar{s}^1_{\rm II} & \bar{t}^1_{\rm II} \\ \bar{s}^2_{\rm I} & \bar{t}^2_{\rm I} & \bar{s}^2_{\rm II} & \bar{t}^2_{\rm II} \end{pmatrix} = f(\bar{s}^2,\bar{s}^1)\, f(\bar{t}^2,\bar{t}^1)\, W_{new}(\bar{s}_{\rm I},\bar{s}_{\rm II}|\bar{t}_{\rm I},\bar{t}_{\rm II}) . \tag{B.2}$$

Note that in order to make the comparison, one has to exchange the subsets $\bar{s}^1_{\rm I} \longleftrightarrow \bar{s}^1_{\rm II}$ in one of the sum formulas. This change is harmless since one performs a summation over all partitions $\bar{s}^1 \Rightarrow \{\bar{s}^1_{\rm I},\bar{s}^1_{\rm II}\}$.

Expression in term of HCs. Applying the correspondence (B.2), the relation (4.11) is identical to the one obtained in [49] with

$$\begin{aligned} Z^{(l)}_{old}(\bar{s}^1,\bar{t}^1|\bar{s}^2,\bar{t}^2) &= f(\bar{s}^2,\bar{s}^1)\, f(\bar{t}^2,\bar{t}^1)\; Z_{new}(\bar{s}^1,\bar{s}^2|\bar{t}^1,\bar{t}^2), \\ Z^{(r)}_{old}(\bar{s}^1,\bar{t}^1|\bar{s}^2,\bar{t}^2) &= f(\bar{s}^2,\bar{s}^1)\, f(\bar{t}^2,\bar{t}^1)\; \overline{Z}_{new}(\bar{s}^1,\bar{s}^2|\bar{t}^1,\bar{t}^2). \end{aligned} \tag{B.3}$$

C Coproduct formula for the dual Bethe vectors

The presentation (3.24) for the Bethe vector of the composite model can be treated as a coproduct formula for the Bethe vector. Indeed, equation (3.20) formally determines a coproduct Δ of the monodromy matrix entries

$$\Delta(T_{i,j}(u)) = \sum_{k=1}^{m} T_{k,j}(u) \otimes T_{i,k}(u). \tag{C.1}$$

Then (3.24) is nothing but the action of Δ onto the Bethe vector.

The action of the coproduct onto the dual Bethe vectors can be obtained via antimorphism (3.16) thanks to the relation

$$\Delta_{q^{-1}} \circ \Psi = (\Psi \otimes \Psi) \circ \Delta'_q, \tag{C.2}$$

where

$$\Delta'_q(T_{i,j}(u)) = \sum T_{i,k}(u) \otimes T_{k,j}(u). \tag{C.3}$$

Then applying (C.2) to $\mathbb{B}_q(\bar{t})$, we get

$$\begin{aligned} \Delta_{q^{-1}}(\Psi(\mathbb{B}_q(\bar{t}))) &= \Delta_{q^{-1}}(\mathbb{C}_{q^{-1}}(\bar{t}^{\,-1})) = (\Psi \otimes \Psi) \circ \Delta'_q(\mathbb{B}_q(\bar{t})) \\ &= (\Psi \otimes \Psi)\left(\sum \frac{\prod_{\nu=1}^{N} \alpha^{(1)}_\nu(\bar{t}^\nu_{\rm I}) f_q(\bar{t}^\nu_{\rm II},\bar{t}^\nu_{\rm I})}{\prod_{\nu=1}^{N-1} f_q(\bar{t}^{\nu+1}_{\rm II},\bar{t}^\nu_{\rm I})}\, \mathbb{B}_q(\bar{t}_{\rm I}) \otimes \mathbb{B}_q(\bar{t}_{\rm II}) \right) \\ &= \sum \frac{\prod_{\nu=1}^{N} \widetilde{\alpha}^{(1)}_\nu(\frac{1}{\bar{t}^\nu_{\rm I}}) f_q(\bar{t}^\nu_{\rm II},\bar{t}^\nu_{\rm I})}{\prod_{\nu=1}^{N-1} f_q(\bar{t}^{\nu+1}_{\rm II},\bar{t}^\nu_{\rm I})}\, \mathbb{C}_{q^{-1}}(\bar{t}^{\,-1}_{\rm I}) \otimes \mathbb{C}_{q^{-1}}(\bar{t}^{\,-1}_{\rm II}). \end{aligned} \tag{C.4}$$

Relabeling the subsets $\bar{t}^\nu_{\rm I} \longleftrightarrow \frac{1}{\bar{t}^\nu_{\rm II}}$ and using (5.25), we arrive at

$$\Delta_{q^{-1}}(\mathbb{C}_{q^{-1}}(\bar{t})) = \sum \frac{\prod_{\nu=1}^{N} \widetilde{\alpha}^{(1)}_\nu(\bar{t}^\nu_{\rm II}) f_{q^{-1}}(\bar{t}^\nu_{\rm I},\bar{t}^\nu_{\rm II})}{\prod_{\nu=1}^{N-1} f_{q^{-1}}(\bar{t}^{\nu+1}_{\rm I},\bar{t}^\nu_{\rm II})}\, \mathbb{C}_{q^{-1}}(\bar{t}_{\rm II}) \otimes \mathbb{C}_{q^{-1}}(\bar{t}_{\rm I}). \tag{C.5}$$

It remains to make the change $q^{-1} \to q$ to obtain (3.25). □

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Chapter 6

New symmetries of $\mathfrak{gl}(N)$-invariant Bethe vectors

Introduction:

In this Chapter we proposed a new representation of Bethe vectors in terms of inverse monodromy matrix entries. It was proven that such representation is related to the usual one, but with the converted parameters. This relation gives important formula describing symmetry of the highest coefficient in the scalar product.

Contribution:

I proved the central result of this Chapter Theorem 4.1. The statement of the theorem is related to the symmetry of Dynkin diagram for $\mathfrak{gl}_N$. The combinatorial formula (5.12) for the highest coefficient was obtained by me.

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New symmetries of $\mathfrak{gl}(N)$-invariant Bethe vectors

A Liashyk[1,2,3,4], S Z Pakuliak[5,6], E Ragoucy[7] and N A Slavnov[8]

[1] Bogolyubov Institute for Theoretical Physics, NAS of Ukraine, Kiev, Ukraine
[2] Faculty of Mathematics, National Research University Higher School of Economics, Moscow, Russia
[3] Skolkovo Institute of Science and Technology, Moscow, Russia
[4] Institut Denis-Poisson, Universit de Tours, Parc de Grandmont, 37200 Tours, France
[5] Moscow Institute of Physics and Technology, Dolgoprudny, Moscow reg., Russia
[6] Laboratory of Theoretical Physics, JINR, Dubna, Moscow reg., Russia
[7] Laboratoire de Physique Thöorique LAPTh, CNRS and USMB, BP 110, 74941 Annecy-le-Vieux Cedex, France
[8] Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia

E-mail: a.liashyk@gmail.com, stanislav.pakuliak@jinr.ru, eric.ragoucy@lapth.cnrs.fr and nslavnov@mi-ras.ru

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Abstract. We consider quantum integrable models solvable by the nested algebraic Bethe ansatz and possessing $\mathfrak{gl}(N)$-invariant R-matrix. We study two types of Bethe vectors. The first type corresponds to the original monodromy matrix. The second type is associated to a monodromy matrix closely related to the inverse of the monodromy matrix. We show that these two types of Bethe vectors are identical up to normalization and reshuffling of the Bethe parameters. To prove this correspondence we use the current approach. This identity gives new combinatorial relations for the scalar products of the Bethe vectors. The q-deformed case, as well as the superalgebra case, are also evoked in the conclusion.

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Keywords: algebraic structures of integrable models, integrable spin chains and vertex models, quantum integrability (Bethe ansatz)

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1. Introduction

The algebraic Bethe ansatz developed by the Leningrad school [1–3] is a powerful method to investigate quantum integrable systems. One can use this approach to find the spectra of quantum Hamiltonians. Besides, this method can be used for calculating correlation functions of quantum integrable models [4–7]. In the framework of the algebraic Bethe ansatz this problem reduces to calculating scalar products of Bethe vectors.

The notion of the Bethe vector is one of the most important notions of the algebraic Bethe ansatz. These vectors belong to the physical space of states of the quantum model under consideration. They depend on a set of complex numbers called Bethe parameters. Under certain constraints imposed on the Bethe parameters, the Bethe vector becomes an eigenvector of the quantum Hamiltonian. In this case it is commonly called an *on-shell Bethe vector*. Otherwise, if the Bethe parameters are generic complex numbers, the corresponding vector sometimes is called an *off-shell Bethe vector*.

In the $\mathfrak{gl}\!+\!1$, based model, the form of the Bethe vectors is quite simple [1–4]. However, in the quantum integrable models with higher rank symmetry algebra, the construction of Bethe vectors becomes very intricate. There are several ways to specify these vectors. A recursive procedure for constructing the off-shell Bethe vectors was given in the papers [8–10]. An explicit formula for these vectors (trace formula) containing tensor products of the monodromy matrices and R-matrices was proposed in [11–13]. Another approach to this problem, based on projections in the current algebra was formulated in [14–17]. Explicit formulas for the Bethe vectors in terms of the monodromy matrix entries acting on a reference state were obtained in [18, 19].

In this paper we find a new symmetry of the Bethe vectors in the models with $\mathfrak{gl}$+>,-invariant R-matrix. It is quite natural to expect that the symmetries of the monodromy matrix should generate corresponding symmetries of the Bethe vectors [10, 11, 18, 19]. In the present paper we consider a mapping of the monodromy matrix T to a new matrix ı closely related to the inverse monodromy matrix. We study the properties of the Bethe vectors associated to both matrices. We show how these two types of Bethe vectors are related to each other. As a direct application of this correspondence, we find new symmetries of the Bethe vector scalar products.

The paper is organized as follows. We recall basic notions of the algebraic Bethe ansatz in section 2. There we also give a notation used in the paper. Section 3 is devoted to the description of the properties of the Bethe vectors. The main results of our paper are given in section 4, where we use an identification of the Bethe vectors with certain combination of the generators of the Yangian double [19] to prove the claimed symmetry of the Bethe vectors. In section 5 we study symmetry properties of the scalar products of the Bethe vectors. Several appendices gather technical details of the proofs.

2. *RTT*-algebra and notation

We consider quantum integrable models solvable by the algebraic Bethe ansatz and possessing $\mathfrak{gl}$+>,-invariant R-matrix

$$+=,\ e \quad | \quad . \quad +=, \ = \ +=,\ e \ \frac{}{\_}[\tag{2.1}$$

Here e $\ell_{t\#2}$ tt is the identity operator acting in the space ℓ, tx are > > matrices with the only nonzero entry equal to 1 at the intersection of the ith row and jth column, e $\ell_{t(x\#2}$ $\in_{tx}$ | $\in_{xt}$ is the permutation operator acting in ℓ | ℓ, c is a constant, and = are arbitrary complex parameters called spectral parameters.

The key object of the algebraic Bethe ansatz is a monodromy matrix +, with operator-valued entries $T_{ij}(u)$ acting in a Hilbert space (physical space of a quantum model). It satisfies an RTT-algebra:

$$+=,++,| \quad ,+| \quad +,,\ e \ +| \quad +,,++,| \quad , \quad +=,[\tag{2.2}$$

Equation (2.2) yields the commutation relations of the monodromy matrix entries

$$\xi \ _{tx}+,= {}_{<>}+,\mathrm{ne} \quad +=,+_{t>}+, \ {}_{<x}+,- \ {}_{t>}+, \ {}_{<x}+,,[\tag{2.3}$$

Using (2.2) it is easy to prove that

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$$ ꒰ \ +,= \ +,\mathrm{ne} \ ]= $$

where $+,$ e $_t$ $_{tt}+,$ is the transfer matrix. Thus, the transfer matrix is a generating function for the integrals of motion of the model under consideration.

We assume the following dependence of the monodromy matrix elements $T_{ij}(u)$ on the parameter u

$$ _{tx}+,\ \mathrm{e}\ u_{tx}\ .\ \ _{tx}\mathrm{g}\,\mathrm{n}^{-p-2}= \quad \underset{p\geqslant)}{} \tag{2.4} $$

where and $_{tx}\mathrm{g}\,\mathrm{n}$ are respectively the unity and nontrivial operators acting in the Hilbert space .

Remark. In fact, for our purpose, the condition (2.4) is optional. We impose this requirement on the asymptotics of $+,$ only in order to facilitate the presentation. In quantum models of physical interest, the monodromy matrix may have a different asymptotic expansion, however, it can easily be reduced to the expansion (2.4).

We also assume that the space has a pseudovacuum vector] (reference state) such that

$$ \begin{aligned} &_{tt}+,\ ]\ \ \mathrm{e}\ \ g_t+,\ ]\ \ = \\ &_{tx}+,\ ]\ \ \mathrm{e}\ \ ]=\ \ \ j\ \ = \end{aligned} \tag{2.5} $$

where $g_t+,$ are some functions depending on the concrete quantum integrable model. The action of $T_{ij}(u)$ with $i<j$ onto the pseudovacuum is nontrivial. In the models of physical interest, multiple action of these operators onto] generates a basis in the space .

Since the monodromy matrix is defined up to a common normalization scalar factor, it is convenient to deal with the ratios:

$$ N_t\!+\!U,\ \mathrm{e}\ \frac{i_{\ t}\!+\!U,}{i_{\ t+2}\!+\!U,}=\ \ \psi\mathrm{e}\ \ 0\!=\!ppp\!\Rightarrow\ -0[\tag{2.6} $$

We treat the functions $N_t\!+\!U,$ as free functional parameters (generalized model) up to the restriction which follows from (2.4).

Besides the original monodromy matrix $k+U,$ we also can consider its inverse matrix. For this, we first introduce the quantum determinant of the monodromy matrix $\mathrm{a}^{-\hat{}}\Delta(k+U,)$ [20–23] by

$$ \mathrm{a}^{-\hat{}}\Delta k+U,(\ \mathrm{e}\ \ \sum_q \mathrm{i}\#F\!+\!q,\ k_{2(q,20}\!+\!U,\ k_{\hat{}(q,\hat{}0}\!+\!U-\delta,\ ppp\,k_{\ell\ (q,\ell\ 0}\!+\!U-\Rightarrow -0,\delta,[$$

Here the sum is taken over all permutations p of the set $0\!=\!1\!=\!ppp>$, $q\!+\!i,$ being the ith element of the permutation p of the set $0\!=\!1\!=\!ppp>$. The quantum determinant belongs to the center of the RTT-algebra

$$)\ \mathrm{a}^{-\hat{}}\Delta k+U,(=k_{tx}+,\ |\ \mathrm{e}\ \ ][$$

It is also easy to see that due to (2.5)

$$ \mathrm{a}^{-\hat{}}\Delta k+U,(\ ]\ \ \mathrm{e}\ \ i_{\ 2}\!+\!U,i_{\ \hat{}}\!+\!U-\delta,\ ppp\,i_{\ \ell}\ \ +U-\Rightarrow -0,\delta,\ ]\ \ [$$

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Similarly to the quantum determinant, we can introduce quantum minors of the size $(0 \leqslant W v >)$

$$L^{1(.\,2(jjj(.\,k}_{a_1(a_2(jjj(a_k} +U, \text{ e } \underset{q}{} \text{ i}\#F+q,\ k_{.\,1(a_{s(1)}} +U,\ k_{.\,2(a_{s(2)}} +U-\delta,\ ppp\, k_{.\,k\,(a_{s(k)})} +U- +W-0,\delta, [\tag{2.7}$$

Here the sum is taken over permutations of the set $0{=}1{=}ppp\,W$, $q{+}i{,}$ being the ith element of the permutation p of the set $0{=}1{=}ppp\,W$.

Now we can introduce the inverse monodromy matrix $\hat{k}^3 +U,$

$$\hat{k}^3 +U, k +U, \text{ e } = \tag{2.8}$$

where the entries $\hat{k}^3_{tx} +U,$ are given by quantum minors divided by the quantum determinant

$$\hat{k}^3_{tx} +U, \text{ e } +-0,^{t+x} L^{2jjj\,ajjj\ell}_{2jjj\,a\,jjj\ell} +U-\delta, \text{ a}^{-\hat{}}\Delta k +U,,^{-2} [\tag{2.9}$$

Here and mean that the corresponding indices are omitted.

It is known [23] that the inverse monodromy matrix satisfies the RTT-relation with opposite sign of the constant c, that is

$$[\hat{k}^3_{tx} +U, =\hat{k}^3_{<>} + , \text{ne } s+ = U,\ \ \hat{k}^3_{t>} +U, \hat{k}^3_{<x} + , - \hat{k}^3_{t>} + , \hat{k}^3_{<x} +U,\ \ [$$

Then, defining $k_{\dagger x} +U,$ by

$$k_{\dagger x} +U, \text{ e } \hat{k}^3_{\ell\ +2-x(\ell\ +2-t} +U, = \tag{2.10}$$

we find that the elements $k_{\dagger x} +U,$ satisfy commutation relations

$$[k_{\dagger x} +U, =k_{\dagger <>} + , \text{ne } s+U = , \sum k_{\dagger >} +U, k_{\dagger <x} + , - k_{\dagger >} + , k_{\dagger <x} +U, ([$$

Since these commutation relations coincide with (2.3), we conclude that $k_{\dagger} +U,$ satisfies the RTT-algebra (2.2) with the same R-matrix (2.1).

Thus, a mapping

$$k_{tx} +U, \qquad k_{\dagger x} +U, \tag{2.11}$$

is an automorphism of the RTT-algebra. The aim of this paper is to investigate the symmetries of the off-shell Bethe vectors (see section 3) related to this automorphism.

Note that this symmetry is specific to higher rank algebras (and the existence of several simple roots). Indeed, in the $\mathfrak{gl}+1,$ case, one gets $k +U,$ e $k_{\dagger} +U,,$ and the symmetry becomes trivial, while it becomes informative as soon as the rank is higher than 1 (see e.g. section 5).

2.1. Notation

In this section we describe the notation that we use below. First, we introduce a special notation for the combination $0\ .\ s+U = ,$

$$P+U = , \text{ e } 0\ .\ s+U = , \text{ e } \frac{U - \ .\ \delta}{U -} [\tag{2.12}$$

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Second, we formulate a convention on the notation of sets of variables. We denote them by bar: $\bar t^i$, $\bar s$, and so on. Here the superscripts refer to different sets. Individual elements of the sets are denoted by subscripts: t^i_j, s_k, and so on. Thus, for example, $\bar t = \{\bar t^1, \bar t^2\}$ means that the set $\bar t$ is the union of two sets $\bar t^1$ and $\bar t^2$. At the same time, each of these two sets consists of the elements $\bar t^i = \{t^i_1, t^i_2, \dots, t^i_{r_i}\}$, where $i = 1, 2$.

Notation $\bar t^i \pm c$ means that a constant c is added to all the elements of the set $\bar t^i$. Subsets of variables are denoted by roman indices: $\bar t^i_{\rm I}$, $\bar s_{\rm II}$, and so on. In particular, we consider partitions of sets into subsets. Then the notation $\bar t^i \Rightarrow \{\bar t^i_{\rm I}, \bar t^i_{\rm II}\}$ means that the set $\bar t^i$ is divided into two disjoint subsets $\bar t^i_{\rm I}$ and $\bar t^i_{\rm II}$. The order of the elements in each subset is not essential.

To make the formulas more compact we use a shorthand notation for the products of functions depending on one or two variables. Namely, if the f-function (2.12) depends on a set of variables (or two sets of variables), this means that one should take the product over the corresponding set (or the double product over both sets). For example,

$$f(u,\bar t) = \prod_{t_j\in\bar t} f(u,t_j), \qquad f(\bar t^i,\bar s) = \prod_{t^i_j\in\bar t^i}\prod_{s_k\in\bar s} f(t^i_j,s_k). \tag{2.13}$$

We use the same prescription for the products of commuting operators, their vacuum eigenvalues λ_i (2.5), and the ratios of these eigenvalues α_i (2.6)

$$\lambda_i(\bar t) = \prod_{t_j\in\bar t}\lambda_i(t_j), \qquad \alpha_i(\bar t) = \prod_{t_j\in\bar t}\alpha_i(t_j), \qquad T_{ij}(\bar t) = \prod_{t_k\in\bar t} T_{ij}(t_k). \tag{2.14}$$

We will extend this convention for new functions that will appear later. Finally, by definition, any product over the empty set is equal to 1. A double product is equal to 1 if at least one of the sets is empty.

3. Bethe vectors

One of the main tasks of the algebraic Bethe ansatz is to find the eigenvectors of the transfer matrix, that usually are called on-shell Bethe vectors. To do this, one should first construct off-shell Bethe vectors (or equivalently, Bethe vectors), that belong to the Hilbert space $\mathcal{H}$. The latter are special polynomials in $T_{ij}(u)$ with $i<j$ acting on $|0\rangle$. In the simplest $\mathfrak{gl}(2)$ case the Bethe vectors have the form $T_{12}(\bar u)|0\rangle$, where $\bar u = \{u_1, \dots, u_n\}$, $n = 0, 1, \dots$. However, in the general $\mathfrak{gl}(N)$ case, the form of the Bethe vectors is much more involved (see e.g. [19]).

In the $\mathfrak{gl}(N)$ based models, an off-shell Bethe vector $\mathbb{B}(\bar t)$ depends on $N-1$ sets of complex numbers $\bar t = \{\bar t^1, \dots, \bar t^{N-1}\}$ called Bethe parameters. The Bethe vector $\mathbb{B}(\bar t)$ is symmetric over permutations of the Bethe parameters within each subset $\bar t^i$. However, it is not symmetric with respect to rearrangements of subsets, and also for replacements $t^i_j \leftrightarrow t^k_l$. If the Bethe parameters satisfy a special system of equations (Bethe equations), then the Bethe vector becomes an eigenstate of the transfer matrix. However, generically no constraint on the Bethe parameters t^i_k are imposed.

Given a monodromy matrix $k\,+\!U,$, the different procedures[9] to construct off-shell Bethe vectors provide, up to a global normalization factor, the same vectors, although several different explicit forms may exist due to the commutation relations (2.3). Then, it remains to fix unambiguously this normalization factor. In this paper we use the same normalization as in [24]. Namely, we have already mentioned that a generic Bethe vector has the form of a polynomial in T_{ij} with $i<j$ applied to the pseudovacuum $]$. Among all the terms of this polynomial, there is one monomial that contains the operators T_{ij} with $j-i=1$ only. We call this term the *main term* and denote it by $\mathbb{B}\!+\!L$. We fix the normalization of the Bethe vectors by fixing the numeric coefficient of the main term

$$\mathbb{B}\!+\!L\;\; \text{e}\;\; \frac{k_{\ell\;-2(\ell}\;+\!L^{\ell\;-2}, k_{\ell\;-\hat{}(\ell\;-2}+\!L^{\ell\;-\hat{}}, \otimes\!\otimes\!\otimes k_{\hat{}3}+\!L, k_{2\hat{}}+\!L^{2}, \;]}{\ell\;-2_{t\#2}\; i_{\;t+2}+\!L^{t}, \quad \ell\;-\hat{}_{t\#2}\; P\!+\!L^{t+2}=\!L^{t},}\,[\tag{3.1}$$

Recall that we use here the shorthand notation (2.13) and (2.14) for the products of the operators $T_{i,i+1}$, the vacuum eigenvalues $i_{\;t+2}$, and the f-functions.

3.1. Bethe vectors of the matrix $T_{|}(u)$

We have seen in the previous section that the matrix $k_{\Gamma}\!+\!U,$ satisfies the RTT-relation (2.2). Using the definition of $k_{\mathfrak{f}x}$ (see (2.9), (2.10) and (2.7)) one can find the action of the operators $k_{\mathfrak{f}x}$ onto the pseudovacuum. A straightforward calculation shows that

$$\begin{aligned} &k_{\mathfrak{f}x}\!+\!U,\;] \quad \text{e} \;\;]= \qquad \psi< A= \\ &k_{\mathfrak{f}t}\!+\!U,\;] \quad \text{e} \;\; \mathring{\tilde{\imath}}_{\;t}\!+\!U,\;] \;\; = \end{aligned} \tag{3.2}$$

where

$$\mathring{\tilde{\imath}}_{\;t}\!+\!U,\;\; \text{e}\;\; \frac{0}{i_{\;\ell\;-t+2}+\!U - +\!\!> -\,\psi\!\!,\delta,}\;\; {}^{\ell\;-t}_{\;r\#2}\;\; \frac{i_{\;p}+\!U - F\!\delta,}{i_{\;p}+\!U - +\!F - 0,\delta,}\,[\tag{3.3}$$

It follows from (3.3) that the ratios of the vacuum eigenvalues have the following form

$$\mathcal{N}_{t}\!+\!U,\;\; \text{e}\;\; \frac{\mathring{\tilde{\imath}}_{\;t}+\!U,}{\mathring{\tilde{\imath}}_{\;t+2}+\!U,}\;\; \text{e}\;\; N_{\ell\;-t}+\!U - +\!\!> -\,\psi\!\!,\delta,\,[\tag{3.4}$$

Finally, the operators $k_{\mathfrak{f}x}$ with $i<j$ act on $]$ as creation operators.

Thus, we can construct off-shell Bethe vectors $\mathbb{B}\!+\!L$ associated to the monodromy matrix $k_{\Gamma}\!+\!U,$. These vectors are uniquely defined provided their normalization is fixed. We do this as in (3.1). Namely, the main term $\mathring{\mathbb{B}}\!+\!L$ of the off-shell Bethe vector $\mathbb{B}\!+\!L$ reads

$$\mathring{\mathbb{B}}\!+\!L\;\; \text{e}\;\; \frac{k_{\mathfrak{f}\;-2(\ell}\;+\!L^{\ell\;-2}, k_{\mathfrak{f}\;-\hat{}(\ell\;-2}+\!L^{\ell\;-\hat{}}, \otimes\!\otimes\!\otimes k_{\Gamma 3}+\!L, k_{\mathfrak{p}\hat{}}+\!L^{2}, \;]}{\big\lfloor\, {}^{\ell\;-2}_{t\#2}\; \mathring{\tilde{\imath}}_{\;t+2}+\!L^{t}, \;\big\lfloor\, {}^{\ell\;-\hat{}}_{t\#2}\; P\!+\!L^{t+2}=\!L^{t},}\,[\tag{3.5}$$

[9] The known procedures are the nested algebraic Bethe ansatz [8–10], the trace formula [11–13], or the projection of currents [14–17].

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Here we have extended the shorthand notation (2.13) and (2.14) to the products of the operators $k_{\!|(t+2}$ and the vacuum eigenvalues $\hat{\hat{r}}_{t+2}$.

The main result of this paper is a correspondence between $\mathbb{B}+\!\!L,$ and $\mathbb{B}+\!\!\grave{L},$.

4. Correspondence between two types of Bethe vectors

In order to formulate the main result of this paper we introduce a mapping of the sets of Bethe parameters:

$$u+\!\!\grave{L}, \cdot \quad u+\!\!e\grave{L}^2=\!\!\hat{L}=\!\!ppp=\!\!\grave{L}^{\ell\ -2}\ ,\ \mathrm{e}\ \ \mathrm{c}\grave{L}^{\ell\ -2}-\delta=\!\!\grave{L}^{\ell\ -\hat{}}-1\delta=\!\!ppp=\!\!\grave{L}^2-\Rightarrow\ -0,\delta\ [\tag{4.1}$$

Thus, this mapping reorders the sets $\grave{I}^t$ and shifts every set $\grave{I}^t$ by $-\psi- > ,\delta$.

Theorem 4.1. *The off-shell Bethe vectors* $\mathbb{B}$ *and* $\mathbb{B}$ *of integrable models with* $\mathfrak{gl}+\!\!\Rightarrow,$*-invariant R-matrix are related by*

$$\mathbb{B}+\!\!\grave{L},\ \mathrm{e}\ \ +\!\!-0,^{=1}\prod_{c\#2}^{\ell\ -\hat{}}P+\!\!\grave{L}^{c+2}=\!\!\grave{L}^{c},^{-2}\quad \mathbb{B}\grave{\ }u+\!\!\grave{L},(\,[\tag{4.2}$$

Here $2\ \grave{L}$ *is the total cardinality of all the sets* $\grave{I}^t$*, and according to (4.1)*

$$\mathbb{B}\grave{\ }u+\!\!\grave{L},(\ \mathrm{e}\ \ \mathbb{B}\grave{\ }\grave{L}^{\ell\ -2}-\delta=\!\!\grave{L}^{\ell\ -\hat{}}-1\delta=\!\!ppp=\!\!\grave{L}^2-\Rightarrow\ -0,\delta(\,[\tag{4.3}$$

We prove this theorem using identification of the off-shell Bethe vectors with certain combinations of the generating series of the Yangian double generators (see [19]). The main tool of this approach relies on the Gauss coordinates of the monodromy matrix rather than considering its matrix elements $T_{ij}(u)$.

4.1. Gauss decomposition of the monodromy matrix

The idea of using the Gauss decomposition of the monodromy matrix satisfying the RTT-relation (2.2) goes back to the paper [25] where this decomposition was used to prove the isomorphism between R-matrix and current realization of the quantum affine algebras. Then the Gauss decomposition of the monodromy was used in the series of papers [14–17] to find closed and explicit formulas for the off-shell Bethe vectors. The Bethe vectors were expressed in terms of the Gauss coordinates using a projection method developed in those papers. In this section we find the relation between the Gauss coordinates of the original $k+\!\!U,$ and the 'transpose-inverse' monodromy $k_{|}+\!\!U,$. It will imply the statement of theorem 4.1.

As it was shown in the paper [19], in order to obtain the off-shell Bethe vectors in the form where the main term $\mathbb{B}+\!\!\grave{L},$ is given by (3.1), one has to use the following Gauss decomposition of the monodromy matrix $k+\!\!U,$ (for $i<j$):

$$k_{tx}+\!\!U,\ \mathrm{e}\ \ \mathrm{t}_{\,xt}+\!\!U,B_x+\!\!U,\ .\sum_{xk\ p\leqslant\ell}\mathrm{t}_{\,pt}+\!\!U,B_p+\!\!U,\mathrm{q}_{\,xp}+\!\!U,= \tag{4.4}$$

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$$k_{tt}+U, \text{ e } B_t+U, \; . \sum_{tk\, p\leqslant \ell} \text{t}_{pt}+U, B_p+U, \text{q}_{tp}+U, = \tag{4.5}$$

$$k_{xt}+U, \text{ e } B_x+U, \text{q}_{tx}+U, \; . \sum_{xk\, p\leqslant \ell} \text{t}_{px}+U, B_p+U, \text{q}_{tp}+U, [\tag{4.6}$$

These formulas are the result of product of three matrices

$$k+U, \text{ e } \; +U, \; (\; +U, \; (\; +U, [\tag{4.7}$$

In the above formula, $+U,$ is an upper-triangular matrix with unities 0 on the diagonal, $+U,$ e $^{-}$E#B_2+U,=B+U,=ppp=B_ℓ +U,, is a diagonal matrix, and $+U,$ is a lower-triangular matrix again with unities on the diagonal (see appendix B for an example of these matrices in the case $N=3$).

It is clear from the reference state definition (2.5) that the Gauss coordinates $\text{q}_{tx}+U,$ annihilate this state: $\text{q}_{tx}+U,$] e]. The definition also implies that it is a common eigenstate of the matrix $+U,$ diagonal elements: $B_t+U,$] e $i_t+U,$] and that the Gauss coordinates $\text{t}_{xt}+U,$ create non-trivial vectors in the space of states of the quantum integrable models.

In order to describe the 'transpose-inverse' monodromy matrix $k_{\Gamma}+U,$ in terms of the Gauss coordinates $\text{t}_{xt}+U,$, $\text{q}_{tx}+U,$, $k_i(u)$ we have to invert the matrices $+U,$, $+U,$ and $+U,$. The Gauss coordinates of the inverse matrices

$$\begin{aligned} &+U,^{-2} \text{ e } \; . \sum_{tk\, x\in tx} \tilde{\text{t}}_{xt}+U, = \\ \mathbf{F}\, &+U,^{-2} \text{ e } {}^{-}\text{E}\#B_2+U,^{-2}=B+U,^{-2}=ppp=B_\ell+U,^{-2}, = \\ &+U,^{-2} \text{ e } \; . \sum_{tk\, x\in xt} \tilde{\text{q}}_{tx}+U, = \end{aligned} \tag{4.8}$$

are given by the following.

Lemma 4.1. *The Gauss coordinates* $\tilde{\text{t}}_{xt}+U,$ *and* $\tilde{\text{q}}_{tx}+U,$, $0 \leqslant \psi v \; \measuredangle \leqslant >$ *are*

$$\tilde{\text{t}}_{xt}+U, \text{ e } \sum_{p\#)}^{x-t-2} +-,^{p+2} \sum_{xl\; t_\ell l\; \cdots l\; t_1 l\; t} \text{t}_{t_1(t}+U, \text{t}_{t_2(t_1}+U, \otimes\otimes\otimes \text{t}_{t_\ell(t_{\ell-1}}+U, \text{t}_{x(t_\ell}+U, = \tag{4.9}$$

$$\tilde{\text{q}}_{tx}+U, \text{ e } \sum_{p\#)}^{x-t-2} +-,^{p+2} \sum_{xl\; t_\ell l\; \cdots l\; t_1 l\; t} \text{q}_{t_\ell(x}+U, \text{q}_{t_{\ell-1}(t_\ell}+U, \otimes\otimes\otimes \text{q}_{t_1(t_2}+U, \text{q}_{t(t_1}+U, [\tag{4.10}$$

Proof of this Lemma follows from a direct verification. I

According to the assumed dependence (2.4) of the monodromy matrix $k+U,$ on the spectral parameter u we may conclude from the formulas (4.4)–(4.6) that the Gauss coordinates $\text{t}_{xt}+U,$, $\text{q}_{tx}+U,$, $k_i(u)$ have the following dependence on the parameter u

$$\text{t}_{xt}+U, \text{ e } \sum_{f\geqslant)} \text{t}_{xt}gZ\imath U^{-f-2}= \; \text{q}_{tx}+U, \text{ e } \sum_{f\geqslant)} \text{q}_{tx}gZ\imath U^{-f-2}= \; B_t+U, \text{ e } 0\,. \sum_{f\geqslant)} B_t gZ\imath U^{-f-2}[\tag{4.11}$$

The zero mode operators t_{xt}g] ŋ q_{tx}g] nand $k_i[0]$ play an important role. In particular, according to the RTT commutation relations (2.2) the Gauss coordinates with bigger difference of the indices $j-i$ may be expressed as commutators of zero-mode operators and Gauss coordinates with smaller difference $j-i$. In what follows we will need following

Lemma 4.2. *The Gauss coordinates* $\mathrm{t}_{xt}(U),\ \mathrm{q}_{tx}(U)$, *and* $\tilde{\mathrm{t}}_{xt}(U),\ \tilde{\mathrm{q}}_{tx}(U)$ *can be written as multiple commutators* ($j>i$)

$$\mathrm{t}_{xt}(U) \text{ e } \delta^{t+2-x} \otimes\otimes \ \mathrm{t}_{x(x-2}(U) = \mathrm{t}_{x-2(x-}\,\mathrm{g}\ \mathrm{n} = \mathrm{t}_{x-(x-3}\,\mathrm{g}\ \mathrm{n} = \otimes\otimes = \mathrm{t}_{t+(t+2}\,\mathrm{g}\ \mathrm{n} = \mathrm{t}_{t+2(t}\,\mathrm{g}\ \mathrm{n} =$$
$$\tilde{\mathrm{t}}_{xt}(U) \text{ e } -\delta^{t+2-x}\ \mathrm{t}_{x(x-2}\,\mathrm{g}\ \mathrm{n} = \mathrm{t}_{x-2(x-}\,\mathrm{g}\ \mathrm{n} = \otimes\otimes = \mathrm{t}_{t+3(t+}\,\mathrm{g}\ \mathrm{n} = \mathrm{t}_{t+(t+2}\,\mathrm{g}\ \mathrm{n} = \mathrm{t}_{t+2(t}(U), \quad \otimes\otimes \quad = \tag{4.12}$$

and

$$\mathrm{q}_{tx}(U) \text{ e } \delta^{t+2-x}\ \mathrm{q}_{t(t+2}\,\mathrm{g}\ \mathrm{n} = \mathrm{q}_{t+2(t+}\,\mathrm{g}\ \mathrm{n} = \otimes\otimes = \mathrm{q}_{x-3(x-}\,\mathrm{g}\ \mathrm{n} = \mathrm{q}_{x-(x-2}\,\mathrm{g}\ \mathrm{n} = \mathrm{q}_{x-2(x}(U), \quad \otimes\otimes \quad =$$
$$\tilde{\mathrm{q}}_{tx}(U) \text{ e } -\delta^{t+2-x} \quad \otimes\otimes \quad \mathrm{q}_{t(t+2}(U) = \mathrm{q}_{t+2(t+}\,\mathrm{g}\ \mathrm{n} = \mathrm{q}_{t+(t+3}\,\mathrm{g}\ \mathrm{n} = \otimes\otimes = \mathrm{q}_{x-(x-2}\,\mathrm{g}\ \mathrm{n} = \mathrm{q}_{x-2(x}\,\mathrm{g}\ \mathrm{n}\,[\tag{4.13}$$

Proof is based on the RTT-relation for the monodromy matrix $k(U)$ and its inverse $\tilde{k}(U)$. Details are given in appendix A. □

After applying the transposition with respect to the anti-diagonal to the inverse monodromy matrix $\tilde{k}(U)$, we obtain for the matrix $k_{\mathrm{l}}(U)$ a Gauss decomposition (for $i<j$)

$$k_{\mathrm{l}x}(U) \text{ e } B_{\ell+2-x}(U)^{-2}\tilde{\mathrm{t}}_{\ell+2-t(\ell+2-x}(U) \cdot \sum_{2\leqslant p\leqslant \ell+2-x} \tilde{\mathrm{q}}_{p(\ell+2-x}(U)B_p(U)^{-2}\tilde{\mathrm{t}}_{\ell+2-t(p}(U) = \tag{4.14}$$

$$k_{\mathrm{l}t}(U) \text{ e } B_{\ell+2-t}(U)^{-2} \cdot \sum_{2\leqslant p\leqslant \ell+2-t} \tilde{\mathrm{q}}_{p(\ell+2-t}(U)B_p(U)^{-2}\tilde{\mathrm{t}}_{\ell+2-t(p}(U) = \tag{4.15}$$

$$k_{\mathrm{l}xt}(U) \text{ e } \tilde{\mathrm{q}}_{\ell+2-x(\ell+2-t}(U)B_{\ell+2-x}(U)^{-2} \cdot \sum_{2\leqslant p\leqslant \ell+2-x} \tilde{\mathrm{q}}_{p(\ell+2-t}(U)B_p(U)^{-2}\tilde{\mathrm{t}}_{\ell+2-x(p}(U) = \tag{4.16}$$

similar to the Gauss decomposition (4.4)–(4.6) of the original monodromy matrix $k(U)$. The only crucial difference is the ordering of the 'new' Gauss coordinates in the formulas (4.14)–(4.16).

We call a product of the Gauss coordinates *normal ordered* if all the coordinates $\mathrm{t}_{xt}(U)$ are on the left of the product of all other Gauss coordinates and all $\mathrm{q}_{<}(U)$ are on the right. This ordering is adapted to the action of the Gauss coordinates onto reference state described above.

By construction, the expressions (4.4)–(4.6) of the monodromy matrix elements $T_{ij}(u)$ in terms of the Gauss coordinates $\mathrm{t}_{xt}(U),\ \mathrm{q}_{tx}(U),\ i<j$ and $k_i(u)$, $\imath = \prime$ e $0 = pp\jmath = >$ are written in the normal ordered form. However, the formulas (4.14)–(4.16) for the inverse monodromy matrix are not normal ordered. The normal ordering is given by the following.

Theorem 4.2. *The normal ordered Gauss decomposition of the monodromy* $k_{\mathrm{l}}(U)$ *has literally the same form as in (4.4)–(4.6) with the Gauss coordinates* $\mathrm{t}_{xt}(U),\ \mathrm{q}_{tx}(U),\ k_j(u)$ *replaced by* $\hat{\mathrm{t}}_{xt}(U),\ \hat{\mathrm{q}}_{tx}(U),\ \hat{B}_x(U)$ *where (for $i<j$)*

$$\hat{\mathrm{t}}_{xt}(U) \text{ e } \tilde{\mathrm{t}}_{\ell+2-t(\ell+2-x}(U - \Rightarrow - A.\ \ 0, \delta,) = \tag{4.17}$$

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$$\tilde{T}_x(U) = \frac{0}{B_{\ell+2-x}(U - \triangleright - A)\delta} \prod_{r\#2}^{\ell - x} \frac{B_p(U - F\delta)}{B_p(U - F - 0,\delta)} = \tag{4.18}$$

$$\delta_{tx}(U) = \delta_{\ell+2-x(\ell+2-t}(U - \triangleright - A.\ 0,\delta) [\tag{4.19}$$

Proof is based on the presentation of the Gauss coordinates as multiple commutators. The shifts of the indices in (4.17) and (4.19) can be seen from the formulas (4.14) and (4.16), while the shifts of the spectral parameters and transformation of the diagonal generating series $B_x(U)$, $\tilde{B}_x(U)$ follow from the commutation relations between Gauss coordinates. They are gathered in appendix B. Note that formulas (4.18) are in accordance with the action of the diagonal matrix elements (3.2) onto the reference state] .

4.2. Bethe vectors and currents

This section is devoted to the proof of theorem 4.1. We heavily use the results of the paper [19] where the off-shell Bethe vectors were explicitly constructed from the current generators of the super-Yangian double $g\ R(\mathfrak{gl}(W\ Z))$. In what follows we will use some results of this paper in the case $m = N$, $n = 0$.

The Yangian double associated with the algebra $\mathfrak{gl}(\triangleright)$ is a Hopf algebra of a pair of generating $>$ $>$ matrices $k\ (U)$ satisfying the commutation relations

$$\ell\ (U=,\ (k^{\kappa}(U)\ |\ \ ,+\ |\ k^{\nu}+,,\ e\ +\ |\ k^{\nu}+,,\ (k^{\kappa}(U)\ |\ \ ,\ell\ (U=,= \tag{4.20}$$

where : e . Being rewritten in terms of the Gauss coordinates $q_{tx}(U)$, $t_{xt}(U)$ and $B_t(U)$ (4.4)–(4.6) and generating series (*currents*) [25]

$$\epsilon_t(U) = t^+_{t+2(t}(U) - t^-_{t+2(t}(U) = \quad f_t(U) = q^+_{t(t+2}(U) - q^-_{t(t+2}(U) = \tag{4.21}$$

the commutation relations (4.20) can be presented in the form (so called 'new' realization of the Yangian double)

$$\begin{aligned} &B^{\to}_t(U)\epsilon_t+,B^{\to}_t(U)^{-2} = P+ =U,\ \epsilon_t+,= \\ &L^{\to}_{t+2}(U)\epsilon_t+,B^{\to}_{t+2}(U)^{-2} = P(U=,\ \epsilon_t+,= \end{aligned} \tag{4.22}$$

$$\begin{aligned} &B^{\to}_t(U)^{-2} f_t+,B^{\to}_t(U) = P+ =U,\ f_t+,= \\ &B^{\to}_{t+2}(U)^{-2} f_t+,B^{\to}_{t+2}(U) = P(U=,\ f_t+,= \end{aligned} \tag{4.23}$$

$$F(U=,\ \epsilon_t(U)\epsilon_t+, = P+ =U,\ \epsilon_t+,\epsilon_t(U)= \tag{4.24}$$

$$P+ =U,\ f_t(U)f_t+, = P(U=,\ f_t+,f_t(U)= \tag{4.25}$$

$$-U - \ - \delta,\ \epsilon_t(U)\epsilon_{t+2}+, = (U - \ ,\ \epsilon_{t+2}+,\epsilon_t(U)= \tag{4.26}$$

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$$(u-v)\,F_i(u)F_{i+1}(v) = (u-v-c)\,F_{i+1}(v)F_i(u), \tag{4.27}$$

$$[E_i(u),F_j(v)] = c\,\delta_{i,j}\,\delta(u,v)\left(k^+_i(u)k^+_{i+1}(u)^{-1} - k^-_i(v)k^-_{i+1}(v)^{-1}\right), \tag{4.28}$$

and the Serre relations for the currents $E_i(u)$ and $F_i(u)$. In (4.28) the symbol $\delta(u,v)$ means the additive δ-function given by the formal series

$$\delta(u,v) = \frac{1}{u}\sum_{p\in\mathbb{Z}} \frac{v^p}{u^p}. \tag{4.29}$$

The Borel subalgebra in the Yangian double generated by matrix $T^+(u)$ is isomorphic to the standard $\mathfrak{gl}(N)$ Yangian [23]. Then, we can identify the monodromy matrix $T(u)$ discussed in the previous sections with the generating matrix $T^+(u)$. We also identify the Gauss coordinates of these monodromy matrices

$$\begin{aligned} F^+_{j,i}(u) &= F_{j,i}(u) = \sum_{\ell\geq 0} F_{j,i}[\ell]u^{-\ell-1},\\ E^+_{i,j}(u) &= E_{i,j}(u) = \sum_{\ell\geq 0} E_{i,j}[\ell]u^{-\ell-1},\\ k^+_i(u) &= k_i(u) = 1+\sum_{\ell\geq 0} k_i[\ell]u^{-\ell-1}. \end{aligned} \tag{4.30}$$

The currents $F_i(u)$, $k^+_j(u)$ and $E_i(u)$, $k^-_j(u)$ form the so-called dual Drinfeld Borel subalgebras with their own Drinfeld coproduct properties. According to the general theory of projections developed in [26] one can define the projections P^+_f and P^-_f onto intersections of these current Borel subalgebras with the standard Borel subalgebras formed by the Gauss coordinates $F^+_{j,i}(u)$, $E^+_{i,j}(u)$, $k^+_j(u)$ and $F^-_{j,i}(u)$, $E^-_{i,j}(u)$, $k^-_j(u)$.

Due to the results of the papers [14, 19] the off-shell Bethe vectors can be identified with the normalized projection of the product of the currents. In order to formulate this result we need to introduce some notation. For any scalar function $\gamma(u,v)$ of two variables and any set $\bar u = \{u_1,\ldots,u_n\}$ we define the product

$$\Delta_\gamma(\bar u) = \prod_{i<j}\gamma(u_j,u_i). \tag{4.31}$$

Let $\mathcal{F}_i(\bar u)$, $i=1,\ldots,N-1$ be the ordered product of the currents

$$\mathcal{F}_i(\bar u) = F_i(u_n)F_i(u_{n-1})\cdots F_i(u_2)F_i(u_1). \tag{4.32}$$

Note that this product is not symmetric with respect to permutation of the parameters u_i, as it follows from the commutation relation (4.24).

One of the main result of the papers [14, 19] is the identification of the off-shell Bethe vectors with the projections of the product of the currents:

$$\mathbb{B}(\bar t) = \frac{\prod_{\nu=1}^{N-1}\Delta_f(\bar t^\nu)}{\prod_{\nu=1}^{N-2} f(\bar t^{\nu+1},\bar t^\nu)}\,P^+_f\left(\mathcal{F}_{N-1}(\bar t^{N-1})\mathcal{F}_{N-2}(\bar t^{N-2})\cdots\mathcal{F}_2(\bar t^2)\mathcal{F}_1(\bar t^1)\right)|0\rangle. \tag{4.33}$$

Observe that the product $(\,_{\jmath} +\!\bar{L}^p, \mathcal{H}_p +\!\bar{L}^p,$ is symmetric with respect to permutations within the set $\bar{l}^p$, due to the commutation relations (4.24). As a result, the Bethe vector given by equation (4.33) is symmetric with respect to the permutations of the Bethe parameters of the same type.

Mathematically rigorous definitions of the projections onto different type Borel subalgebras intersections can be found in the paper [26]. They use the different Hopf structures associated with different type of Borel subalgebras in the Yangian double. However, one may understand the projection entering the equation (4.33) in a more simple way. In order to calculate this projection one has to replace each current by the difference of the Gauss coordinates (4.21) and then use the commutation relations in the Yangian double (4.20) between 'positive' and 'negative' Gauss coordinates sending all 'negative' coordinates to the left and all 'positive' coordinates to the right. After such ordering the action of the projection amounts to remove all the terms containing at least one 'negative' Gauss coordinate on the left. Of course, practical implementation of this program is rather heavy. Fortunately, there exist effective methods to perform this procedure [14, 19].

In this paper we are not going to describe the methods which allow to calculate the projection in (4.33) and re-express the result of this calculation in terms of the original monodromy matrix element. We refer the interested reader to the paper [19]. In order to prove the statement of theorem 4.1 we will need only the closed expression (4.33).

The main trick in the calculation of the projection in (4.33) is the appearance of the so called composed currents $F_{ji}(u)$, $i<j$ in the commutation relations of the currents $F_{j,s+1}(u)$ and $F_{si}(u)$ for (e ɿ . 0: ppɿ : ⁄ − 0. Then the rewriting of the projection in (4.33) in terms of the monodromy matrix elements relies on the fact that projections of the composed currents $j\,_{\jmath}^{\,+}\!+\!\epsilon_{\,xt}\!+\!U,$, coincide with the Gauss coordinates $\mathrm{t}_{\,xt}\!+\!U,$ (see appendix A of the paper [19])

$$j\,_{\jmath}^{\,+}\!+\!\epsilon_{\,xt}\!+\!U,\, \mathrm{e}\ \ \delta^{x-t-2}\mathrm{t}^{\,+}_{\,xt}\!+\!U,\ \mathrm{e}\ \ \delta^{x-t-2}\mathrm{t}_{\,xt}\!+\!U,[\tag{4.34}$$

In order to prove the statement (4.2) let us consider the rhs of this equality using the expression (4.33). We have

$$\frac{+\!-\!0,^{=1}}{\prod_{p\#2}^{\ell-\hat{}} P\!+\!\bar{L}^{p+2}\!=\!\bar{L}^p,}\ \widehat{\mathbb{B}}\ u\!+\!\bar{L},($$

$$\mathrm{e}\ \frac{+\!-\!0,^{=1}\prod_{p\#2}^{\ell-2}(\,_{\jmath}+\!\bar{L}^{\ell-p}-\bar{F}\delta,}{\prod_{p\#2}^{\ell-\hat{}} P\!+\!\bar{L}^{p+2}\!=\!\bar{L}^p,\ P\!+\!\bar{L}^{\ell-p-2}-\!+\!\bar{F}.\ \ 0,\delta\!=\!\bar{L}^{\ell-p}-\bar{F}\delta,}$$

$$\mathcal{E}\ j\,_{\jmath}^{\,+}\ \mathcal{H}_{\ell-2}\!+\!\bar{L}^{2}-\!+\!\!\!>-0,\delta,\mathcal{H}_{\ell-\hat{}}\!+\!\hat{\bar{L}}-\!+\!\!\!>-1,\delta,\otimes\!\otimes\!\otimes\mathcal{H}_2\!+\!\bar{L}^{\ell-2}-\delta,(\ ]$$

$$\mathrm{e}\ \prod_{p\#2}^{\ell-2}(\,_{\jmath}+\!\bar{L}^p,j\,_{\jmath}^{\,+}\sum\bar{\mathcal{H}}_2\!+\!\bar{L}^2,\bar{\mathcal{H}}_{\hat{}}\!+\!\hat{\bar{L}},\otimes\!\otimes\!\otimes\bar{\mathcal{H}}_{\ell-\hat{}}\!+\!\bar{L}^{\ell-\hat{}},\bar{\mathcal{H}}_{\ell-2}\!+\!\bar{L}^{\ell-2},(\ ]$$

$$\mathrm{e}\ \frac{\prod_{p\#2}^{\ell-2}(\,_{\jmath}+\!\bar{L}^p,}{\prod_{r\#2}^{\ell-\hat{}} P\!+\!\bar{L}^{p+2}\!=\!\bar{L}^p,}\,j\,_{\jmath}^{\,+}\sum\bar{\mathcal{H}}_{\ell-2}\!+\!\bar{L}^{\ell-2},\bar{\mathcal{H}}_{\ell-\hat{}}\!+\!\bar{L}^{\ell-\hat{}},\otimes\!\otimes\!\otimes\bar{\mathcal{H}}_{\hat{}}\!+\!\hat{\bar{L}},\bar{\mathcal{H}}_2\!+\!\bar{L}^2,(\ ]\ [\tag{4.35}$$

Here we have introduced the ordered product $\bar{\mathcal{H}}_t\!+\!\bar{L},$ of the shifted currents given by the product (4.32) with the currents $F_i(u)$ replaced by the shifted currents $\bar{\mathcal{E}}_t\!+\!U,$

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$$\hat{e}^{S}_{t}+\!U, \text{ e } -\epsilon_{\ell\ -t}+\!U - \Rightarrow - \psi\delta, [\tag{4.36}$$

In (4.35), we also used the identity $P+ = U, P+U - \delta = ,$ e 0 and the fact that the function $P+U = ,$ is translation invariant which implies $(\ _{J} +\!\psi - \lambda,$ e $(\ _{J} +\!\psi,.$ We also used the commutation relations between currents $\hat{e}^{S}_{t}+\!U,$ and $\hat{e}^{S}_{t+2}+,$ which follow from (4.26). The fact that one can use these commutation relations under the action of the projection was proved in paper [14].

The assertion (4.2) of theorem 4.1 now follows from two lemmas.

Lemma 4.3. *The mapping*

$$\epsilon_{t}+\!U, \equiv \hat{e}^{S}_{t}+\!U, \text{ e } -\epsilon_{\ell\ -t}+\!U - \Rightarrow - \psi\delta, = \ \psi\text{e } 0 = ppp \Rightarrow - 0 =$$

$$f_{t}+\!U, \equiv \hat{f}^{S}_{t}+\!U, \text{ e } -f_{\ell\ -t}+\!U - \Rightarrow - \psi\delta, = \ \psi\text{e } 0 = ppp \Rightarrow - 0 =$$

$$B^{\rightarrow}_{x}+\!U, \equiv \hat{B}^{\rightarrow}_{x}+\!U, \text{ e } \frac{0}{B^{\rightarrow}_{\ell\ +2-x}+\!U - \Rightarrow - A\delta,} \prod_{r\#2}^{\ell\ -x} \frac{B^{\rightarrow}_{p}+\!U - F\delta,}{B^{\rightarrow}_{p}+\!U - +F - 0, \delta,} = \ A\text{e } 0 = ppp \Rightarrow \tag{4.37}$$

is an automorphism of the Yangian double given by the commutation relations (4.22)–(4.28).

Proof is based on a direct verification. It is clear that the automorphism (4.37) is induced by the corresponding automorphism (2.11) of the RTT-algebra. ∎

Lemma 4.4. *The projections of the composed currents* $j^{\ +}_{J} +\hat{e}^{S}_{xt}+\!U,,$, $i < j$ *which appear in the commutation relations of the currents* $\hat{e}^{S}_{x(c+2}+\!U,$ *and* $\hat{e}^{S}_{ct}+\!U,$ *for* $($ e $\imath.\ 0 = pp\mathfrak{l} = \prime - 0$ *coincide with the shifted Gauss coordinates of the 'transpose-inverse' monodromy matrix* $k_{\mathsf{I}}+\!U,$

$$\begin{aligned} j^{\ +}_{J} +\hat{e}^{S}_{xt}+\!U,, \text{ e } & \delta^{x-t-2}\hat{F}^{+}_{\ell\ +2-t(\ell\ +2-x}+\!U - \Rightarrow .\ 0 - A\delta, \\ \text{e } & \delta^{x-t-2}\hat{F}_{\ell\ +2-t(\ell\ +2-x}+\!U - \Rightarrow .\ 0 - A\delta, \end{aligned} \tag{4.38}$$

given by the multiple commutators (4.12).

Proof is given in appendix A. ∎

Proof of theorem 4.1. As we can see from the equation (4.35) the Bethe vector $\mathbb{B}+\!L,$ for the generalized quantum integrable models built from the 'transpose-inverse' monodromy matrix is given by the same formula as in (4.33) with currents $F_i(u)$ replaced by the currents $\hat{e}^{S}_{t}+\!U,.$ They satisfy the same commutation relations (4.22)–(4.28) with the currents $\hat{f}_{t}+\!U,$ and $\hat{B}^{\rightarrow}_{x}+\!U,$ due to lemma 4.3. Now using the statement of lemma 4.4 we can apply all the techniques developed in the papers [14, 19] and prove that $\mathbb{B}+\!L,$ is the off-shell Bethe vector constructed from the monodromy matrix elements $k_{\mathsf{I}x}+\!U,$ (2.10). Then, this proves the statement of theorem 4.1. ∎

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5. Symmetry of the highest coefficients

As a direct application of equation (4.2), we study symmetry properties of the scalar products. For this, we should introduce dual Bethe vectors.

5.1. Dual Bethe vectors

Dual Bethe vectors belong to the dual space $\mathcal{H}^*$ and can be obtained by the successive action of T_{ji} with $i<j$ from the right onto a dual pseudovacuum $\langle 0| \in \mathcal{H}^*$. They also depend on $N-1$ sets of complex numbers $\bar{s} = \{\bar{s}^1, \dots, \bar{s}^{N-1}\}$. Dual Bethe vectors become dual eigenstates of the transfer matrix, if these parameters enjoy the system of Bethe equations. For more details about these vectors, we refer the reader to the works [19, 24].

For the moment, it is important for us that the dual Bethe vectors can be obtained by a transposition of ordinary Bethe vectors. Namely, a mapping $\psi: T_{ij}(u) \mapsto T_{ji}(u)$ defines an anti-automorphism of the RTT-algebra [23]:

$$\psi(AB) = \psi(B)\psi(A). \tag{5.1}$$

Here A and B are arbitrary products of the monodromy matrix entries T_{ij}. Extending this mapping to the Bethe vectors by $\psi(|0\rangle) = \langle 0|$, one can prove that [18, 19]

$$\mathbb{C}(\bar{s}) = \psi\big(\mathbb{B}(\bar{s})\big), \tag{5.2}$$

where $\mathbb{C}(\bar{s})$ is the dual Bethe vector. Using this formula one can prove that the dual Bethe vectors also satisfy a property similar to (4.2). Namely, let $\mathbb{C}(\bar{s})$ and $\widetilde{\mathbb{C}}(\bar{s})$ be dual Bethe vectors respectively associated to the monodromy matrices $T(u)$ and $\widetilde{T}(u)$. Then

$$\widetilde{\mathbb{C}}(\bar{s}) = (-1)^{\#\bar{s}} \prod_{k=1}^{N-1} \alpha_k(\bar{s}^k)^{-1}\, \mathbb{C}(\bar{u}). \tag{5.3}$$

Here the notation is the same as in (4.2).

5.2. Symmetries of the scalar products

The scalar products of the Bethe vectors are defined as

$$S(\bar{s}|\bar{t}) = \mathbb{C}(\bar{s})\mathbb{B}(\bar{t}). \tag{5.4}$$

The sets $\bar{s}$ and $\bar{t}$ are generic complex numbers such that $\#\bar{s}^i = \#\bar{t}^i$ for $i = 1, \dots, N-1$. If the latter condition does not hold, then the scalar product vanishes.

The scalar product of generic Bethe vectors can be described by a *sum formula* [24]

$$\mathbb{C}(\bar{s})\mathbb{B}(\bar{t}) = \sum W_{\rm part}(\bar{s}_{\rm I}, \bar{s}_{\rm II}|\bar{t}_{\rm I}, \bar{t}_{\rm II}) \prod_{k=1}^{N-1} \alpha_k(\bar{s}^k_{\rm I})\,\alpha_k(\bar{t}^k_{\rm II}). \tag{5.5}$$

Here all the sets of Bethe parameters $\bar{t}^k$ and $\bar{s}^k$ are divided into two subsets $\{\bar{t}^k_{\rm I}, \bar{t}^k_{\rm II}\} \Rightarrow \bar{t}^k$ and $\{\bar{s}^k_{\rm I}, \bar{s}^k_{\rm II}\} \Rightarrow \bar{s}^k$, such that $\#\bar{t}^k_{\rm I} = \#\bar{s}^k_{\rm I}$. The sum is taken over all possible partitions of this type. The coefficients $W_{\rm part}$ are rational functions completely determined by the

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R-matrix. They do not depend on the ratios of the vacuum eigenvalues λ_i. Using the results of section 4 we can easily find symmetry properties of these coefficients.

Proposition 5.1. *For arbitrary partitions $\bar t^i \Rightarrow \{\bar t^i_{\rm I}, \bar t^i_{\rm II}\}$ and $\bar s^i \Rightarrow \{\bar s^i_{\rm I}, \bar s^i_{\rm II}\}$, such that $\#\bar t^i_{\rm I} = \#\bar s^i_{\rm I}$, the corresponding coefficient $Z_{\rm part}$ satisfies the following property:*

$$Z_{\rm part}(\bar s_{\rm I},\bar s_{\rm II}|\bar t_{\rm I},\bar t_{\rm II})\prod_{i=1}^{N-1} f(\bar s^{i+1},\bar s^i)\, f(\bar t^{i+1},\bar t^i) = \widehat{Z}_{\rm part}\big(\mu(\bar s)_{\rm I},\mu(\bar s)_{\rm II}\big|\mu(\bar t)_{\rm I},\mu(\bar t)_{\rm II}\big) \tag{5.6}$$

where $\mu(\bar t)$ is defined in (4.1).

Proof. We compute the scalar product in two different ways. First, performing in (5.5) the replacements $s^i \equiv s^{N-i} - c$ and $t^i \equiv t^{N-i} - c$, we arrive at

$$\mathbb{C}\big(\mu(\bar s)\big)\mathbb{B}\big(\mu(\bar t)\big) = \sum Z_{\rm part}\big(\mu(\bar s)_{\rm I},\mu(\bar s)_{\rm II}\big|\mu(\bar t)_{\rm I},\mu(\bar t)_{\rm II}\big) \times \prod_{i=1}^{N-1}\alpha_i(\bar s^{N-i}_{\rm I}-c)\,\alpha_i(\bar t^{N-i}_{\rm II}-c). \tag{5.7}$$

Due to (3.4) we obtain

$$\widehat{\mathbb{C}}\big(\mu(\bar s)\big)\widehat{\mathbb{B}}\big(\mu(\bar t)\big) = \sum \widehat{Z}_{\rm part}\big(\mu(\bar s)_{\rm I},\mu(\bar s)_{\rm II}\big|\mu(\bar t)_{\rm I},\mu(\bar t)_{\rm II}\big)\prod_{i=1}^{N-1}\hat\alpha_i(\bar s^i_{\rm I})\,\hat\alpha_i(\bar t^i_{\rm II}). \tag{5.8}$$

Finally, using (4.2) and (5.3) we transform the lhs as follows:

$$\mathbb{C}(\bar s)\mathbb{B}(\bar t)\prod_{i=1}^{N-1} f(\bar s^{i+1},\bar s^i)\, f(\bar t^{i+1},\bar t^i) = \sum \widehat{Z}_{\rm part}\big(\mu(\bar s)_{\rm I},\mu(\bar s)_{\rm II}\big|\mu(\bar t)_{\rm I},\mu(\bar t)_{\rm II}\big)\prod_{i=1}^{N-1}\hat\alpha_i(\bar s^i_{\rm I})\,\hat\alpha_i(\bar t^i_{\rm II}). \tag{5.9}$$

On the other hand, the scalar product of the Bethe vectors $\mathbb{C}(\bar s)$ and $\mathbb{B}(\bar t)$ is given by the sum formula

$$\mathbb{C}(\bar s)\mathbb{B}(\bar t) = \sum Z_{\rm part}(\bar s_{\rm I},\bar s_{\rm II}|\bar t_{\rm I},\bar t_{\rm II})\prod_{i=1}^{N-1}\hat\alpha_i(\bar s^i_{\rm I})\,\hat\alpha_i(\bar t^i_{\rm II}). \tag{5.10}$$

Since the functions $\hat\alpha_i(u)$ are free functional parameters, the equations (5.9) and (5.10) can give the same result if and only if the coefficients of every product of $\hat\alpha_i$ coincide. Thus, we arrive at (5.6). □

In particular, we can consider a partition such that $\bar s_{\rm I} = \bar s$ and $\bar t_{\rm I} = \bar t$. Then respectively $\bar s_{\rm II} = \bar t_{\rm II} = \emptyset$. The corresponding coefficient $Z_{\rm part}$ is called the highest coefficient. We denote it by $Z(\bar s|\bar t)$:

$$Z(\bar s|\bar t) = Z_{\rm part}(\bar s,\emptyset|\emptyset,\bar t). \tag{5.11}$$

Then it follows immediately from (5.6) that

$$\widehat{E}\ u+), \ u+L, (\ \mathrm{e}\ \ E+)\ L, \prod_{<\#2}^{\ell\,-\,\hat{}} P+)^{<+2}=)^{<}, P+L^{<+2}=L^{<}, [\qquad (5.12)$$

Conclusion

In this paper we have found a new symmetry of Bethe vectors. As we have mentioned, an off-shell Bethe vector is a polynomial in the monodromy matrix entries T_{ij} applied to the pseudovacuum. The new symmetry gives a description of the Bethe vector in terms of the entries of the monodromy matrix $k_{łx}$ (2.10). As we have already mentioned, this symmetry is specific to the algebras with the rank higher than 1. It cannot be seen on the Bethe vectors corresponding to the $\mathfrak{gl}$+1, case, as it becomes trivial.

In paper [27], we have used already the symmetry of the Bethe vectors in the models with $\mathfrak{gl}$+d,-invariant R-matrix. In that paper the equivalence of the two representations was proved by the use of a recursion for the Bethe vectors. Generalization of this method to the case of higher rank algebras is possible, but is technically very complex. Therefore, our proof is based on the Gauss decomposition of the monodromy matrix and the underlying current algebra. This approach was found to be very powerful in the study of the Bethe vectors for the models with high rank of symmetry [19].

As a direct application of the new symmetry, we proved the identity for the highest coefficients of the scalar product (5.12). However, this is not the only possible application. The new representation allows one to study the properties of combined operators that arise from the original monodromy matrix T_{ij} and from the monodromy matrix $k_{łx}$. Recently this type of operators was considered in [28]. There, in particular, it was conjectured that in $\mathfrak{gl}$+d,-invariant spin chains the operator

$$t\ \ +U,\ \mathrm{e}\ \ k_{\hat{}3}+U, k_{\mathfrak{p}3}+U, - k_{23}+U, k_{\mathfrak{p}\hat{}}+U, \qquad (5.13)$$

can be used for generating on-shell Bethe vectors. Our result allows us to obtain explicit formulas for the action of $B^g(u)$ onto the Bethe vectors using known action formulas of the operators $T_{ij}(u)$ [18]. This allowed us to prove the conjecture of [28] and show that it is valid only for special (symmetric) representations of the Yangian [27].

Concluding, we would like to mention that symmetries of the RTT-algebra, analogous to those considered in this paper, also exist for the RTT-relations associated to the $_{m}$+$\mathfrak{gl}_f$, algebras and $\mathfrak{gl}$+WZ, superalgebras. As in the case discussed above, these symmetries generate new representations for the Bethe vectors associated with the inverse monodromy matrix. In turn, these representations imply symmetries of scalar products, in particular, symmetries of the highest coefficients. For the sake of completeness, we present the latter in the case of $_{m}$+$\mathfrak{gl}_f$, and $\mathfrak{gl}$+WZ, algebras.

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For q-deformed algebra case $_{m}\!\mathfrak{gl}_{f},$ the highest coefficient $E^{n}+) \; L,$ was introduced in [29]. Its symmetric property formally coincides with (5.12):

$$\widehat{E^{m}} u+), \; u+L, (\text{ e } E^{m}+) \; L, \prod_{<\#2}^{f-\hat{}} P^{n}+)^{<+2} =)^{<}, P^{n}+L^{<+2}=L^{<}, = \tag{5.14}$$

where

$$u+L, \text{ e } \text{c}l^{-\hat{}} L^{f-2} = l^{-\bar{}} L^{f-\hat{}} = ppp = l^{-\hat{}, f-20} L^{2} \tag{5.15}$$

and

$$F^{m}+ = L, \text{ e } \frac{l \quad - l^{-2} L}{- l} [\tag{5.16}$$

Relation (5.14) for the models described by $_{m}\!\mathfrak{gl}_{3},$ algebra was proven in [30] via explicit representations for the highest coefficient.

For the superalgebra case $\mathfrak{gl}+W Z,$ (with $W\!: Z <]$ and the grading $[i] = 0$ for $\imath \leqslant W$ and $[i] = 1$ for $i > m$), the highest coefficient $E^{f\ n}+) \; L,$ was introduced in [24]. The relations between highest coefficients have slightly more complex form:

$$E^{f\ n} \; u+), \; u+L, (\text{ e } +-0,^{=1\kappa} \; E^{n\ f}+) \; L'_{b| \; -b}$$
$$\prod_{<\#2}^{n\ -2} P+)^{<} =)^{<+2}, P+L^{<} = L^{<+2}, \prod_{<\#n}^{f+n\ -\hat{}} P+)^{<+2} =)^{<}, P+L^{<+2} = L^{<}, = \tag{5.17}$$

with

$$u+L, \text{ e } \text{c}L^{n\ +f-2} \, . \; +Z-0, \delta = L^{f+n\ -\hat{}} \, . \; +Z-1, \delta = ppp = L^{n\ +2} \, . \; \delta = L^{n} = L^{n\ -2} \, . \; \delta = ppp = L^{2} \, . \; +W-0, \delta \; [\tag{5.18}$$

Note that equation (5.17) maps the highest coefficient of the scalar product in the $\mathfrak{gl}+W Z,$ superalgebra to that of the scalar product in the $\mathfrak{gl}+Z W,$ superalgebra. The map $\delta \equiv -\delta$ is specific to the superalgebra case (see [24] for more details).

Let us stress once more that equations (5.14) and (5.17) are direct consequences of the symmetries of the Bethe vectors. The latter can be proved exactly by the same method used in this paper.

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Appendix A. Proof of lemmas 4.2 and 4.4

We prove the statement of lemma 4.2 using the commutation relations between Gauss coordinates. In order to obtain these commutation relations from the RTT-relation (2.3) we use the approach of paper [25]. We also use the fact that we consider the generalized model, and hence, eigenvalues of the diagonal monodromy matrix elements are arbitrary functional parameters. This means that after substitution of the Gauss decomposition formulas into commutation relations (2.3), we obtain equations for all possible products of the currents B_i+U, B_j+, after normal ordering of the Gauss coordinates according to the rules described before theorem 4.2. In particular, we obtain

$$E_t\text{+U, t}_{t+2(t}\text{+ , }B_t\text{+U,}^{-2}\text{ e } P\text{+ =U, t}_{t+2(t}\text{+ , . } s\text{+U= , t}_{t+2(t}\text{+U, =} \tag{A.1}$$

$$E_t\text{+U,}^{-2}\text{q}_{t(t+2}\text{+ , }B_t\text{+U, e } P\text{+ =U, q}_{t(t+2}\text{+ , . } s\text{+U= , q}_{t(t+2}\text{+U, =} \tag{A.2}$$

$$\text{gq}_{t(t+2}\text{+ , =t}_{x+2(x}\text{+U, ne } \alpha_{t(x}\, s\text{+ =U, } B_t\text{+U, }B_{t+2}\text{+U,}^{-2} - B_t\text{+ , }B_{t+2}\text{+ ,}^{-2}\text{(=} \tag{A.3}$$

$$\begin{aligned}&\text{t}_{x(x-2}\text{+ , t}_{x-2(t}\text{+U, e } P\text{+ =U, t}_{x-2(t}\text{+U, t}_{x(x-2}\text{+ ,}\\ &\qquad\text{. } s\text{+U= , }\textstyle\sum\text{t}_{xt}\text{+ , } - \text{t}_{xt}\text{+U, . t}_{x-2(t}\text{+U, t}_{x(x-2}\text{+U, (=}\end{aligned} \tag{A.4}$$

$$\begin{aligned}&\text{q}_{t(x-2}\text{+U, q}_{x-2(x}\text{+ , e } P\text{+ =U, q}_{x-2(x}\text{+ , q}_{t(x-2}\text{+U,}\\ &\qquad\text{. } s\text{+U= , }\textstyle\sum\text{q}_{tx}\text{+ , } - \text{q}_{tx}\text{+U, . q}_{x-2(x}\text{+U, q}_{t(x-2}\text{+U, ([}\end{aligned} \tag{A.5}$$

These equalities can be used to prove (4.12) and (4.13). Since both proofs are identical we consider only (4.12). Using the dependence of the Gauss coordinates on the spectral parameter (4.30) we can send $U \equiv$ ⟩ or $\equiv$ ⟩ and consider the coefficients of the leading terms in (A.4) at u^{-1} or $^{-2}$ respectively. We obtain

$$\text{t}_{xt}\text{+ , e } \delta^{-2}\text{gt}_{x(x-2}\text{+ , =t}_{x-2(t}\text{g] n} \tag{A.6}$$

and

$$\text{t}_{xt}\text{+U, } - \text{t}_{x-2(t}\text{+U, t}_{x(x-2}\text{+U, e } \delta^{-2}\text{gt}_{x(x-2}\text{g] n=t}_{x-2(t}\text{+U, n[} \tag{A.7}$$

Now the first equation in (4.12) follows from a trivial induction of the relation (A.6). By the induction over j, one can prove from (A.7) that following relation is valid

$$\begin{aligned}&\delta^{c-x}\ \text{t}_{x(x-2}\text{g] n= t}_{x-2(x-\hat{}}\text{g] n=⊗⊗⊗= t}_{c+\hat{}(c+2}\text{g] n= t}_{c+2(c}\text{g] n=t}_{ct}\text{+U, ⊗⊗⊗}\\ &\qquad \text{e} \int_{p\#)}^{c-x} \text{+ ,}^{p} \int_{xl\ t_\ell l\ \cdots l\ t_1 \geqslant c} \text{t}_{t_1(t}\text{+U, t}_{t_2(t_1}\text{+U, ⊗⊗⊗t}_{t_\ell(t_{\ell-1}}\text{+U, t}_{x(t_\ell}\text{+U, =}\end{aligned} \tag{A.8}$$

for any s satisfying $i < s < j$. The second equality in (4.12) is a particular case of (A.8) at $s = i + 1$. This ends the proof of lemma 4.2. ∎

In order to prove the statement of lemma 4.4 we use the results of the appendix A of paper [19]. We consider the shifted currents $\mathcal{E}_t$+U, (4.36) and the corresponding composed currents $\mathcal{E}_{xt}$+U, defined in this appendix by the formulas (A.3) and (A.7). These composed currents satisfy a relation identical to (A.17) in the same appendix of [19], which implies

$$j_j^{\ +}\textstyle\sum \mathcal{E}_{xt}\text{+U, (e } \ j_j^{\ +}\textstyle\sum \mathcal{E}_{x(t+2}\text{+U, (=}\mathcal{E}_t\text{g] n [} \tag{A.9}$$

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The commutativity between the projections and commutation relations with zero modes was proved in appendix B of [19]. Now the chain of equations (ψ e $>$. $0 - A$ and A e $>$. $0 - \psi$)

$$
\begin{aligned}
& j_{\jmath}^{+}\textstyle\sum \epsilon_{xt} \mathcal{U}, (\; \text{e} \quad \otimes\otimes\otimes \quad j_{\jmath}^{+}\textstyle\sum \epsilon_{x-2} \mathcal{U}, (= \epsilon_{x-\hat{}} \; \text{g n} = \epsilon_{x-3} \; \text{g n} = \otimes\otimes\otimes = \epsilon_{t+2} \; \text{g n} = \epsilon_{t} \; \text{g n} \\
& \quad \text{e} \; - \; \epsilon_{\ell\; -t} \text{g n} = \epsilon_{\ell\; -t-2} \text{g n} = \otimes\otimes\otimes = \epsilon_{\ell\; +\hat{}-x} \text{g n} = j_{\jmath}^{+}\textstyle\sum \epsilon_{\ell\; +2-x} \mathcal{U} - \Rightarrow . \; 0 - A\delta, (\quad \otimes\otimes\otimes \\
& \quad \text{e} \; - \; \text{t}_{x\,(x-2} \text{g n} = \text{t}_{x-2(x-\hat{}} \text{g n} = \otimes\otimes\otimes = \text{t}_{t\,+\hat{}(t\,+2} \text{g n} = \text{t}_{t\,+2(t} \mathcal{U} - \psi\delta, \quad \otimes\otimes\otimes \\
& \quad \text{e} \;\; \delta^{x-t-2} \tilde{\text{t}}_{x\,t} \mathcal{U} - \psi\delta, \; \text{e} \;\; \delta^{x-t-2} \tilde{\text{t}}_{\ell\; +2-t(\ell\; +2-x} \mathcal{U} - \Rightarrow . \; 0 - A\delta,
\end{aligned}
\tag{A.10}
$$

proves relation (4.38). This ends the proof of lemma 4.4. ∎

Appendix B. Gauss coordinates and proof of theorem 4.2

Before starting the proof of theorem 4.2 we provide explicit formulas for the Gauss decomposition used in this paper in the simplest nontrivial case $N = 3$. The monodromy matrix reads

$$
\begin{aligned}
k\mathcal{U}, \text{ e } & \begin{pmatrix} E_2 . \; \text{t}_{\hat{}2} E_{\hat{}} \text{q}_{2\hat{}} . \; \text{t}_{32} E_3 \text{q}_{23} & \text{t}_{\hat{}2} E_{\hat{}} . \; \text{t}_{32} E_3 \text{q}_{\hat{}3} & \text{t}_{32} E_3 \\ B\text{q}_{2\hat{}} . \; \text{t}_{3\hat{}} B_3 \text{q}_{23} & B . \; \text{t}_{3\hat{}} B_3 \text{q}_{\hat{}3} & \text{t}_{3\hat{}} B_3 \\ B_3 \text{q}_{23} & B_3 \text{q}_{\hat{}3} & B_3 \end{pmatrix} \\
\text{e } & \begin{pmatrix} 0 & \text{t}_{\hat{}2} & \text{t}_{32} \\ ] & 0 & \text{t}_{3\hat{}} \\ ] &] & 0 \end{pmatrix} \begin{pmatrix} B_2 &] &] \\ ] & B &] \\ ] &] & B_3 \end{pmatrix} \begin{pmatrix} 0 &] &] \\ \text{q}_{2\hat{}} & 0 &] \\ \text{q}_{23} & \text{q}_{\hat{}3} & 0 \end{pmatrix} [
\end{aligned}
\tag{B.1}
$$

For brevity, we omitted in (B.1) the dependence on the spectral parameter u for all Gauss coordinates $\text{q}_{tx}\mathcal{U}$,, $\text{t}_{xt}\mathcal{U}$,, and $k_i(u)$.

The Gauss decomposition (B.1) allows one to find easily the inverse monodromy matrix

$$
\begin{aligned}
\tilde{k}\mathcal{U}, \text{ e } k\mathcal{U},^{-2} \text{ e } & \begin{pmatrix} 0 &] &] \\ \tilde{\text{q}}_{2\hat{}} & 0 &] \\ \tilde{\text{q}}_{23} & \tilde{\text{q}}_{\hat{}3} & 0 \end{pmatrix} \begin{pmatrix} B_2^{-2} &] &] \\ ] & B^{-2} &] \\ ] &] & B_3^{-2} \end{pmatrix} \begin{pmatrix} 0 & \tilde{\text{t}}_{\hat{}2} & \tilde{\text{t}}_{32} \\ ] & 0 & \tilde{\text{t}}_{3\hat{}} \\ ] &] & 0 \end{pmatrix} \\
\text{e } & \begin{pmatrix} B_2^{-2} & B_2^{-2}\tilde{\text{t}}_{\hat{}2} & B_2^{-2}\tilde{\text{t}}_{32} \\ \tilde{\text{q}}_{2\hat{}} B_2^{-2} & B^{-2} . \; \tilde{\text{q}}_{2\hat{}} B_2^{-2} \tilde{\text{t}}_{\hat{}2} & B^{-2}\tilde{\text{t}}_{3\hat{}} . \; \tilde{\text{q}}_{2\hat{}} B_2^{-2} \tilde{\text{t}}_{32} \\ \tilde{\text{q}}_{23} B_2^{-2} & \tilde{\text{q}}_{\hat{}3} B^{-2} . \; \tilde{\text{q}}_{23} B_2^{-2} \tilde{\text{t}}_{\hat{}2} & B_3^{-2} . \; \tilde{\text{q}}_{\hat{}3} B^{-2} \tilde{\text{t}}_{3\hat{}} . \; \tilde{\text{q}}_{23} B_2^{-2} \tilde{\text{t}}_{32} \end{pmatrix}
\end{aligned}
\tag{B.2}
$$

where

$$
\begin{aligned}
& \tilde{\text{t}}_{2\hat{}}\mathcal{U}, \text{ e } -\text{t}_{2\hat{}}\mathcal{U}, = \; \tilde{\text{t}}_{\hat{}3}\mathcal{U}, \text{ e } -\text{t}_{\hat{}3}\mathcal{U}, = \; \tilde{\text{t}}_{32}\mathcal{U}, \text{ e } -\text{t}_{32}\mathcal{U}, . \;\; \text{t}_{\hat{}2}\mathcal{U}, \text{t}_{3\hat{}}\mathcal{U}, = \\
& \tilde{\text{q}}_{2\hat{}}\mathcal{U}, \text{ e } -\text{q}_{2\hat{}}\mathcal{U}, = \; \tilde{\text{q}}_{\hat{}3}\mathcal{U}, \text{ e } -\text{q}_{\hat{}3}\mathcal{U}, = \; \tilde{\text{q}}_{23}\mathcal{U}, \text{ e } -\text{q}_{23}\mathcal{U}, . \;\; \text{q}_{\hat{}3}\mathcal{U}, \text{q}_{2\hat{}}\mathcal{U}, [
\end{aligned}
\tag{B.3}
$$

Now the monodromy matrix $k_{\Gamma}\mathcal{U}$, given by the relation (2.10) has the following structure:

$$k_{\lceil}\!+\!U, \text{ e } \begin{pmatrix} B_3^{-2} . & \hat{q}_{\hat{}3}B^{-2}\hat{t}_{3\hat{}} . & \hat{q}_{23}B_2^{-2}\hat{t}_{32} & B^{-2}\hat{t}_{3\hat{}} . & \hat{q}_{2\hat{}}B_2^{-2}\hat{t}_{32} & B_2^{-2}\hat{t}_{32} \\ & \hat{q}_{\hat{}3}B^{-2} . & \hat{q}_{23}B_2^{-2}\hat{t}_{\hat{}2} & B^{-2} . & \hat{q}_{2\hat{}}B_2^{-2}\hat{t}_{\hat{}2} & B_2^{-2}\hat{t}_{\hat{}2} \\ & & \hat{q}_{23}B_2^{-2} & & \hat{q}_{2\hat{}}B_2^{-2} & B_2^{-2} \end{pmatrix} [\tag{B.4}$$

It is similar to the structure of the original monodromy matrix $k\,+\!U,$ (B.1).

We prove theorem 4.2 by induction starting from the lower-right corner of the monodromy matrix $k_{\lceil}\!+\!U,$. Due to the formulas (4.14)–(4.16) the matrix elements from the lower-right corner $k_{\ell\,\ell}+\!U,$, $k_{\ell\,-2(\ell}+\!U,$ and $k_{\ell\,(\ell\,-2}+\!U,$ have following form:

$$k_{\ell\,\ell}+\!U, \text{ e } B_2\!+\!U,^{-2}= \quad k_{\ell\,-2(\ell}+\!U, \text{ e } B_2\!+\!U,^{-2}\hat{t}_{\hat{}2}\!+\!U,= \quad k_{\ell\,(\ell\,-2}+\!U, \text{ e } \hat{q}_{2\hat{}}\!+\!U,B_2\!+\!U,^{-2}[\tag{B.5}$$

In order to normal order these matrix elements we can use the commutation relations (A.1) and (A.2) specialised to $i = 1$ and e $\upsilon-\textit{o}$. This yields

$$k_{\ell\,-2(\ell}+\!U, \text{ e } B_2\!+\!U,^{-2}\mathrm{t}_{\hat{}2}\!+\!U, \text{ e } \mathrm{t}_{\hat{}2}\!+\!U-\delta, B_2\!+\!U,^{-2}= \tag{B.6}$$

$$k_{\ell\,(\ell\,-2}+\!U, \text{ e } \mathrm{q}_{2\hat{}}\!+\!U,B_2\!+\!U,^{-2} \text{ e } B_2\!+\!U,^{-2}\mathrm{q}_{2\hat{}}\!+\!U-\delta,= \tag{B.7}$$

and proves formulas (4.17) and (4.19) in the particular case $i = N - 1$ and $j = N$. Now using (A.3) at $i = 1$ and (B.6), (B.7) we can normal order the monodromy matrix element

$$k_{\ell\,-2(\ell\,-2}+, \text{ e } B+,^{-2} . \ \hat{q}_{2\hat{}}+,B_2+,^{-2}\hat{t}_{\hat{}2}+,$$

to obtain

$$\begin{aligned} &\mathrm{q}_{2\hat{}}+,B_2+,^{-2}\mathrm{t}_{\hat{}2}+, \text{ e } \mathrm{q}_{2\hat{}}+,\mathrm{t}_{\hat{}2}+-\delta,B_2+,^{-2} \\ &\text{ e } \mathrm{t}_{\hat{}2}+-\delta,B_2+,^{-2}\mathrm{q}_{2\hat{}}+-\delta, . \ \frac{B_2+-\delta,}{B+-\delta,B_2+,} - B+,^{-2}[\end{aligned}$$

As a result, the element $k_{\ell\,-2(\ell\,-2}+,$ in the normal ordered form is equal to

$$k_{\ell\,-2(\ell\,-2}+, \text{ e } \frac{B_2+-\delta,}{B+-\delta,B_2+,} . \ \hat{t}_{\hat{}2}+-\delta,B_2+,^{-2}\hat{q}_{2\hat{}}+-\delta,= \tag{B.8}$$

thus proving (4.18) for $j = N - 1$.

Formulas (B.6), (B.7), and (B.8) are the base of the induction. Let us assume that the statement of theorem 4.2 is valid for $I \leqslant \psi v \ \acute{} \leqslant >$ in (4.17), (4.19) and for $I \leqslant \acute{} \leqslant >$ in (4.18). By exploring the commutation relations between the Gauss coordinates and lemma 4.2 we will prove that these formulas are valid for $I \equiv I - 0$.

Let us consider the commutation relation (2.3) for the monodromy matrix elements $k_{\mathfrak{l}x}\!+\!U,$ at the values of indices $\text{-}\psi\!=\!A\!=\!B\!=\!S \equiv +\!F-0\!=\!A\!=\!A\!=\!A$ and send $\upsilon \equiv \wr$. Then the coefficient of u^{-1} gives (for $\acute{}$ e $I\!:\!pp\eta\!:\!>$)

$$k_{\mathfrak{p}-2(x}\!+\!U, \text{ e } \delta^{-2} \ k_{\mathfrak{l}x}\!+\!U,\!=\!k_{\mathfrak{p}-2(x}\!]\!\mathrm{n}\,[\tag{B.9}$$

The zero mode of the monodromy matrix element $k_{\mathfrak{p}-2(x}\!]\,\mathrm{n}$ can be obtained from the relation (4.14) and is equal to

$$k_{\mathfrak{p}-2(x} g] \text{ ne } \overset{3}{\mathfrak{t}}_{p+2(x} g] \text{ n=} \tag{B.10}$$

where here and below the prime on the index j mean A e $>$. $0 - A$for any index j.

According to the induction assumption, the monodromy matrix elements $k_{\mathfrak{p}x}$ ($\not$ e $I\!\!:pp\!\!\!\!/\,\!:>$) have the normal ordered form

$$k_{\mathfrak{p}x}+U, \text{ e } \mathfrak{B}_x+U, \,.\, \int_{xk\, c\leqslant \ell} \mathfrak{t}_{cx}+U, \mathfrak{B}_c+U, \mathfrak{q}_{xc}+U, = \tag{B.11}$$

where the Gauss coordinates $\mathfrak{t}_{cx}+U,$, $\mathfrak{B}_c+U,$, and $\mathfrak{q}_{xc}+U,$ respectively are given by equations (4.17)–(4.19). One can prove from the commutation relations between the Gauss coordinates that the zero mode $k_{\mathfrak{p}-2(x} g]$ n (B.10) commutes with $\mathfrak{B}_{}+U,$ and $\mathfrak{t}_{ct}+U,$ $\imath$, except for $\mathfrak{B}_x+U,$ and $\mathfrak{t}_{cx}+U,$ e $\overset{3}{\mathfrak{t}}_{x\,c}+U - G\delta,$. These commutation relations are

$$\delta^{-2}\; \mathfrak{B}_x+U, = \overset{3}{\mathfrak{t}}_{p+2(x} g] \text{ n e } \mathfrak{t}_{x(p-2}+U, \mathfrak{B}_x+U, \tag{B.12}$$

and

$$\delta^{-2}\; \mathfrak{t}_{cx}+U, = \overset{3}{\mathfrak{t}}_{p+2(x} g] \text{ n e } \overset{3}{\mathfrak{t}}_{p+2(c}+U - G\delta, \text{ e } \mathfrak{t}_{c(p-2}+U, [\tag{B.13}$$

To obtain (B.12) we used the second relation in (4.12), the commutation relation

$$\xi B_t+ ,^{-2} = \mathfrak{t}_{t+2(t} g] \text{ me } \delta B_t+ ,^{-2} \mathfrak{t}_{t+2(t}+ , \text{ e } \delta\, \mathfrak{t}_{t+2(t}+ - \delta, B_t+ ,^{-2} =$$

which follows from (A.1), and the commutativity $\xi B_t+ , = \mathfrak{t}_{x+2(x}+U,$ne $]$ for $j > i$. Equalities (B.12) and (B.13) imply that the rhs of (B.9) is (for $\not$ e $I\!\!:pp\!\!\!\!/\,\!:>$)

$$k_{\mathfrak{p}-2(x}+U, \text{ e } \mathfrak{t}_{x(p-2}+U, \mathfrak{B}_x+U, \,.\, \int_{xk\, c\leqslant \ell} \mathfrak{t}_{c(p-2}+U, \mathfrak{B}_c+U, \mathfrak{q}_{xc}+U, [\tag{B.14}$$

Similarly we can prove that the commutation relations between the Gauss coordinates yield

$$k_{\mathfrak{p}(p-2}+U, \text{ e } \mathfrak{B}_x+U, \mathfrak{q}_{p-2(x}+U, \,.\, \int_{xk\, c\leqslant \ell} \mathfrak{t}_{cx}+U, \mathfrak{B}_c+U, \mathfrak{q}_{p-2(c}+U, = \tag{B.15}$$

where the Gauss coordinates $\mathfrak{t}_{c(p-2}+U,$ and $\mathfrak{q}_{p-2(c}+U,$ are given by (4.17) and (4.19) for (e $I\!\!:pp\!\!\!\!/\,\!:>$.

To finish the proof of the theorem we have to prove that the Gauss coordinates $\mathfrak{t}_{c(p-2}+U,$, $\mathfrak{q}_{p-2(c}+U,$ and $\mathfrak{B}_c+U,$ given by the equalities (4.17)–(4.19) for (e $I\!\!:pp\!\!\!\!/\,\!:>$ will imply the same structure of the Gauss coordinate $\mathfrak{B}_{p-2}+U,$.

To do this we can use again the commutation relations (2.3) for -$\psi\!\!=\!\!A\!\!-\!\!B\!\!-\!\!S \equiv +F - 0\!\!=\!\!F\!\!-\!\!F\!\!-\!\!F - 0$, to obtain in the limit $\equiv$ ⅊

$$k_{\mathfrak{p}-2(p}+U, = k_{\mathfrak{p}(p-2} g] \text{ n e } \delta \sum k_{\mathfrak{p}p}+U, - k_{\mathfrak{p}-2(p-2}+U, (= \tag{B.16}$$

where the zero mode operator $k_{\mathfrak{p}(p-2} g]$ ncan be deduced from (4.15)

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$$k_{p(p-2} \mathrm{g}] \,\mathrm{ne} \;-\mathrm{q}_{\ell\ +2-p(\ell\ +\hat{}-p} \mathrm{g}] \,\mathrm{ne} \;-\mathrm{q}_{p\ (p\ +2} \mathrm{g}] \,\mathrm{n}[$$

Now the proof of (4.18) for $\mathbb{B}_{\ell-2}+U,$ follows from the inductive assumption (B.11) and the commutation relations

$$\mathrm{q}_{x(x+2} \mathrm{g}] \,\mathrm{n} \mathrm{t}_{x+2(x}+U, \quad \mathrm{e} \quad \delta+B_x+U, B_{x+2}+U,^{-2} - 0, =$$

$$\mathrm{q}_{x(x+2} \mathrm{g}] \,\mathrm{n} \mathrm{t}_{x+2(x} \mathrm{g}] \,\mathrm{n} \quad \mathrm{e} \quad \delta+B_x \mathrm{g}] \,\mathrm{n}- B_{x+2} \mathrm{g}] \,\mathrm{n}, =$$

$$\mathrm{q}_{x(x+2} \mathrm{g}] \,\mathrm{n} B_x+U,^{-2} \quad \mathrm{e} \quad \delta B_x+U,^{-2} \mathrm{q}_{x(x+2}+U - \delta, =$$

$$\mathrm{q}_{x(x+2} \mathrm{g}] \,\mathrm{n} \mathrm{t}^{3}_{x+2(p}+U, \quad \mathrm{e} \quad \delta \mathrm{t}^{3}_{xp}+U, =$$

and

$$\mathrm{q}_{x(x+2} \mathrm{g}] \,\mathrm{n} \mathrm{e}^{3}_{px}+U, \quad \mathrm{e} \quad -\delta \mathrm{e}^{3}_{p(x+2}+U, [$$

This finishes the proof of theorem 4.2. ∎

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Conclusions

In the thesis I consider description of Bethe vectors for quantum integrable models with super-Yangian $Y(\mathfrak{gl}_{n|m})$ and quantum affine $U_q(\hat{\mathfrak{gl}}_n)$ symmetries. We use the "current approach" for the description of Bethe vectors based on the Ding-Frenkel isomorphism between RTT and current realizations of quantum algebras. This approach allows to obtain many useful properties of Bethe vectors that are used to study their scalar product.

One can summarize our results:

- In the case of $Y(\mathfrak{gl}_{n|m})$ using Gaussian decompositions of the monodromy matrix, the vector $\mathbb{B}(\bar{t})$ was constructed in terms of the total currents associated with simple roots of the $\mathfrak{gl}_{n|m}$ algebra. It has been shown that with both Gaussian expansions get the same Bethe vectors.
- In the case of $Y(\mathfrak{gl}_{n|m})$ action formulas of upper-triangular and diagonal monodromy matrix elements $T_{ij}(u)$ (with $i \leq j$) onto Bethe vector was obtained in terms of Bethe vectors decomposition. Action of transfer matrix $t(u)$ and conditions for eigenvectors (system of Bethe equations) were also obtained.
- In the case of $Y(\mathfrak{gl}_{n|m})$ co-product formula for Bethe vectors was found using Drinfeld co-product properties of currents.
- In the cases of $Y(\mathfrak{gl}_{n|m})$ and $U_q(\hat{\mathfrak{gl}}_n)$ bilinear sum formula for scalar product was found using co-product formula for Bethe vectors. This result is generalization of Izergin-Korepin formula in $\mathfrak{gl}_2$ case and Reshetikhin formula in $\mathfrak{gl}_3$ case to the higher rank cases.
- In the cases of $Y(\mathfrak{gl}_{n|m})$ and $U_q(\hat{\mathfrak{gl}}_n)$ recursion equations for Bethe vectors and the highest coefficient in the sum formula were found using action formulas of monodromy matrix entries onto Bethe vectors.
- In the cases of $Y(\mathfrak{gl}_{n|m})$ and $U_q(\hat{\mathfrak{gl}}_n)$ it was proven that the norm of eigenvector is proportional to the Jacobian of Bethe equations. This statement was first proposed by Gaudin for the $\mathfrak{gl}_2$ case.

- In the case of $Y(\mathfrak{gl}_n)$ it was shown how to build Bethe vectors in terms of inverse monodromy matrix entries. The connection of this representation of Bethe vector with usual one was determined.

The results obtained above are extremely important in context of calculation of the correlation functions of quantum integrable models with higher rank algebras symmetry. The sum formula is milestone on this way. The next step is to obtain a determinant representation of scalar products in the case when one of the Bethe vectors is an eigenvector of transfer matrix. An application of zero modes allows us to derive form-factors of local operators from the form-factors of the monodromy matrix entries. And the multi-point correlation function can be expanded in the form-factors of local operators. We will consider these problems in our further work.

The current approach has proven itself as powerful and agile instrument in algebraic Bethe ansatz. It provides a deep understanding of the symmetries and properties that underlie integrability, and allows us to simplify and unify the proofs of the properties of a scalar product of Bethe vectors. Thus, all results of the thesis can be generalized to a wide class of integrable models that are solved by algebraic Bethe ansatz. This class includes models related to Yangian and quantum affine algebras of types A, B, C, and D and their super generalization.