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RE: Report on the Doctor of Philosophy in Mathematics thesis:

"Integrable structures of the affine Yangian"

by Ilya Vilkoviskiy

Driven by important applications including to low dimensional condensed matter systems and non-Abelian gauge theories, the study of quantum integrable models is currently among the major avenues of research in modern theoretical physics. They have attracted a lot of attention in pure mathematics as well, most prominently, having served as the inspiration for the development of quantum algebras and their representation theory. An important class of quantum integrable models are the Toda type theories associated with a simple Lie algebra. They were originally formulated as classical/quantum mechanical systems but were then generalized to 2D quantum field theories. On top of this, deformations exist that go beyond the framework of a local field theory but are integrable and still of certain physical interest. The subject matter of the dissertation is the study of the integrable structures that underlie these models. This includes the construction of the family of mutually commuting operators and their simultaneous diagonalization.

In the simplest setup, where the rôle of the Lie algebra is played by $\mathfrak{sl}(2)$, the integrable hierarchy of the Toda theories is the KdV/sine-Gordon hierarchy. In this case the simultaneous diagonalization of the family of commuting operators was originally performed within the ODE/IQFT approach. For the deformed hierarchy commonly known as the Intermediate Long Wave (ILW), the ODE/IQFT correspondence has not been developed yet. Nevertheless the set of Bethe ansatz equations describing the spectrum for the commuting family was proposed by Nekrasov & Okounkov and independently by Litivnov in 2013. The ILW hierarchy admits a generalization to $\mathfrak{sl}(n+1)$ (A_n) (the so-called ILW_n hierarchy) and the corresponding Bethe ansatz equations have also been obtained. One of the important results presented in the dissertation is a novel derivation of the Bethe ansatz equations for ILW_n described in chapter 1. It was carried out using what is referred to as the method based on the affine Yangian symmetry.

The advantage of the affine Yangian approach is that it admits a generalization to the (deformed) Toda-type hierarchies associated with the B, C, D series of Lie algebras. Their diagonalization problem is solved in chapter 2 of the dissertation and constitutes its main result (see eqs. 2.4.7, 2.6.5). I believe this to be among the most important recent advances in the area of integrable quantum field theory.

Along with the ILW like deformations, the commuting family of operators possesses what is referred

to in the dissertation as a $\mathfrak{q}$ deformation. The latter originally occurred in the context of integrable lattice systems with $\mathfrak{q}$ being the elliptic parameter in the R matrix and its full physical significance is not well understood yet. The $\mathfrak{q}$ deformation admits a remarkable feature. While the local integrals of motion in the undeformed family of commuting operators do not possess a simple closed expression, turning on the $\mathfrak{q}$ deformation allows one to give an analytical formula for these operators. In chapter 3 of the dissertation such a formula for the local integrals of motion of the $\mathfrak{q}$ deformed integrable hierarchies associated to the Lie algebras of BCD type is presented. This is an interesting result that certainly deserves special mention.

To summarize, the research performed in the dissertation is of significant interest to modern theoretical physics and mathematics. The results obtained therein are original and constitute an important contribution to the current state of the art. It goes without saying that the candidate is worthy of the degree of Doctor of Philosophy in Mathematics.

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