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Integrable structures of the affine Yangian

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I hereby declare that the work presented in this thesis was carried out by myself at Skolkovo Institute of Science and Technology, Moscow and National Research University 'Higher School of Economics', Moscow, except where due acknowledgement is made, and has not been submitted for any other degree.

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Abstract

This thesis is devoted to the study of integrable structures of the conformal field theories by the methods of the hidden affine Yangian symmetry. We introduce the RLL algebra generated by Liouville reflection operator/Maulik Okounkov R-matrix, and discuss its relation to the current realization of the affine Yangian of $\mathfrak{gl}(1)$. We observe that the Integrals of Motion of the W algebras of type A coincide with the ones associated to an affine Yangian "spin chain" with periodic boundary conditions. The integrable structures of W algebras of types BCD are identified with affine Yangian "spin chains" with boundaries. We derive the corresponding Bethe ansatz equations and Bethe vectors for the spectrum of the IOMs. We also construct the q-deformed versions of the reflection matrices and local Integrals of Motion.

Publications

1. Alexey Litvinov; Ilya Vilkoviskiy. *Liouville reflection operator, affine Yangian and Bethe ansatz.* JHEP, 12:100, 2020.

2. Alexey Litvinov; Ilya Vilkoviskiy. *Integrable structure of* BCD *conformal field theory and boundary Bethe ansatz for affine Yangian.* JHEP, 141 (2021).

3. B. Feigin, M. Jimbo, E. Mukhin and I. Vilkoviskiy. *Deformation of W algebras via quantum toroidal algebra* Selecta Mathematica, 27(4):1-62, 2021

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Contents

Introduction

0.1 Integrable field theories, integrable structures of CFT

As was pointed out by Zamolodchikov [Zam89] there is a natural relation between integrable and conformal field theories. Namely having an integrable field theory it is always possible to consider its ultraviolet (UV) limit which is controlled by conformal field theory (CFT). The infinite tower of Integrals of Motion $\mathbf{I_s}(\lambda)$ in this limit splits into two independent family of Integrals of Motion defined in a purely CFT terms.

$$\mathbf{I}_s(\lambda) = \mathbf{I}_s + O(\lambda), \qquad \mathbf{I}_{-s}(\lambda) = \bar{\mathbf{I}}_s + O(\lambda),$$

here λ is a scale parameter and turns to zero in a UV limit, $\mathbf{I}_s$ and $\bar{\mathbf{I}}_s$ are two decoupled integrable systems acting in the space of holomorphic and antiholomorphic fields correspondingly. More importantly, as explained in [Zam89] it is often possible to recover the massive integrable field theory out of integrable structure of CFT. The integrable systems in CFTs is much more simple than the ones in massive integrable field theories, and so, the study of integrable structures in CFT serves as a good playground to understand the space of Integrable quantum field theories (IQFT). In particular the integrable structures of CFTs plays an important role in [Lit19], [LV20] and allows to guess new integrable Toda field theories, and provide a duality between them and Integrable sigma models.

Despite the great simplifications complete diagonalization of chiral Integrals of Motion (IOMs) is yet a nontrivial problem. The study of integrable structure of conformal field theory began with the seminal series of papers of Bazhanov, Lukyanov and Zamolodchikov (BLZ) [BLZ96, BLZ97, BLZ99] devoted to study of quantum KDV integrable system, which appears in the UV limit of sine-Gordon theory. In particular, the set of generating functions for local and non-local Integrals of Motion has been explicitly constructed. Unfortunately the construction of [BLZ96,BLZ97,BLZ99] does not known to provide by itself any equations for the spectrum of the Integrals of Motion.

New ideas appear since the discovery of Ordinary Differential Equation/Integrable Model (ODE/IM) correspondence [DT99a,BLZ01,DT99b]. Using this approach and bunch of analytic intuition, Bazhanov, Lukyanov and Zamolodchikov [BLZ04] were able to express the spectrum of the local IOMs in terms of solutions of certain algebraic system of equations. Later these equations were generalized for some other integrable structures, such as Fateev models or quantum AKNS model (see [KL20] for the list of all known cases). Despite the obvious success of BLZ program, it is still unclear where the algebraic equations of [BLZ04] come from, and whether they can be easily generalized for other models of CFT.

In this thesis we develop a parallel approach based on the affine Yangian symmetry. The advantage of this approach is that it fits in general framework of the quantum inverse scattering method, provides Bethe ansatz equations for the spectrum and allows to treat a lot of integrable structures in a unified way. Being originally formulated geometrically [Var00, Nak01, MO19], it can be rephrased entirely algebraically in CFT terms[1]. In [LV20], using this algebraic approach, we studied the integrable structures in CFT related to $\mathrm{Y}\big(\widehat{\mathfrak{gl}}(1)\big)$, the affine Yangian of $\mathfrak{gl}(1)$ [Tsy17]. These integrable structures describe W algebras of A_n type and its super-algebra generalizations and can be viewed as twist

[1] For the modern review of the geometric approach and more advanced topics see Andrei Okounkov's summer lecture course `sites.google.com/view/andrei-okounkov-lecture-course/home`.

deformations of the quantum Gelfand-Dikii hierarchies (quantum ILWtype integrable systems). We also were able to study integrable structures of W algebras of BCD type, by realising corresponding integrable systems as an affine Yangian "spin chain" with boundaries [LV21].

Affine Yangian of $\mathfrak{gl}(1)$ admits two different descriptions: the current realisation which is useful in studying the spectrum and Bethe eigenfunctions, and the so called Chevalley description in terms of generators of $W_{1+\infty}$ algebra. The second description is more useful in study of the local Integrals of Motion. In order to clarify the structure of W algebra, it may be useful to study its q-deformation. The q-deformations of W algebras have been provided in [AKOS96] for type A, and in [FR97] for simple Lie algebras. The deformations of the local IOMs associated to W algebras of type A were constructed in [KOJ06], [FJM17]. In the third chapter we review the q-deformation of W algebras defined as a commutant of screenings and provide a construction for a q-deformation of local integrals of motion of arbitrary high spin for W algebras of type $\mathrm{B}, \mathrm{C}, \mathrm{D}$.

Chiral Integrals of Motion, example. In order to clarify the ideas above, let us consider an example of classical Sinh-Gordon model living on a cylinder of length $L = 2\pi$ and defined by the action:

$$S = \int \left(\frac{1}{\pi} (\partial_z \varphi \partial_{\bar{z}} \varphi) + \lambda \cosh\left(2b\varphi\right) \right) d^2 z, \tag{1}$$

where $z = x + iy$, $\bar{z} = x - iy$ are the complex coordinates.
The theory is known to contain an infinite tower of Integrals of Motions:

$$\partial_{\bar{z}} T_{s+1} = \lambda \partial_z \Theta_{s-1}, \quad \partial_z T_{-s-1} = \lambda \partial_{\bar{z}} \Theta_{-s+1}, \quad s \geq 1$$

$$\mathbf{I}_s(\lambda) = \int \frac{dx}{2\pi} \Big(T_{s+1} - \lambda \Theta_{s-1} \Big).$$

In the classical limit the first few IOMs are given by the following formulas:

$$T_2 = (\partial_z \varphi)^2 \,, \quad \Theta_0 = 2\pi\lambda \cosh(2b\varphi) \tag{2}$$

$$T_4 = (\partial_z \varphi)^4 + b^{-2} (\partial_z^2 \varphi)^2 \,, \quad \Theta_2 = 4\pi\lambda \, (\partial_z \varphi)^2 \cosh(2b\varphi), \tag{3}$$

which should be corrected at the quantum level. One may already see that in the UV limit ($\lambda \to 0$) Θ_s vanishes, and we are left with the chiral mutually commuting Integrals of Motion $\mathbf{I}_s \overset{\text{def}}{=} \mathbf{I}_s(0)$. It may also be shown (see for eg [FF95]) that that the classical chiral Integrals of Motion may be selected by the condition of the Poisson commutativity with the screenings

$$\{\mathbf{I}_s, \mathcal{S}_i\} = 0,$$

where

$$\mathcal{S}_1 = \oint e^{2b\varphi(z)} \frac{dz}{2\pi} \,, \quad \mathcal{S}_2 = \oint e^{-2b\varphi(z)} \frac{dz}{2\pi}.$$

It turns out that the quantization of chiral integrable system may be defined very directly. Namely, following the ideas of Zamolodchikov [Zam89] developed in [LF91] and also [FF96] we will postulate the following formula for the chiral Integrals of Motion in the quantum case[2]:

$$[\mathbf{I}_s, \mathcal{S}_i] = 0, \tag{4}$$

and

$$\mathcal{S}_1 = \oint e^{2b\varphi(z)} \frac{dz}{2\pi} \,, \quad \mathcal{S}_2 = \oint e^{-2b\varphi(z)} \frac{dz}{2\pi}.$$

[2]Strictly speaking, the commutator in the LHS is not well defined, as the contour of integration is not closed for the general values of the zero mode of the field $\varphi(z)$. We, nevertheless, make this inaccuracy, the proper definition is explained below (see (6), (7)).

We would like to stress out that the rigorous quantization and definition of the IOMs for the full massive integrable model with non zero λ is far more non trivial problem, which we don't even touch in this thesis.
Let us be more precise, we are working in a second quantisation picture, $\varphi(z)$ is the free bosonic field:

$$\partial\varphi(z) = u + \sum_{n\neq 0} a_n e^{inz}, \quad [a_n, a_m] = \frac{m}{2}\delta_{m,-n}. \tag{5}$$

The field $\varphi(z)$ acts in the standard Fock space $\mathcal{F}_u$:

$$F_u = \{\mathbf{C}[a_{-1}, a_{-2}, \dots]|\varnothing\rangle\},$$
$$a_n|\varnothing\rangle = 0, \text{ for } n > 0,$$
$$a_0|\varnothing\rangle = u.$$

We will search for the Integrals of Motion of fixed spin s, as an integrals of local densities $\mathbf{I}_s = \int_0^{2\pi} G_{s+1}(\partial\varphi(z), \partial^2\varphi(z), \dots)\frac{dz}{2\pi}$, which are polynomials in $\partial\varphi$ and its derivatives. We further introduce two vertex operators $V_\pm(z) = e^{\pm 2b\varphi(z)}$, the equations (4) then reads as a conditions on the coefficients in the operator product expansion:

$$V_\pm(w)G_{s+1}(z) = \text{reg} + \frac{\partial X_s^{(1)}(z)}{z-w} + \sum_{k\geq 2}\frac{X_s^{(k)}(z)}{(z-w)^k}, \tag{6}$$

or equivalently:

$$\oint_z V_\pm(w)G_{s+1}(z)\frac{dw}{2\pi} = \partial X_s^{(1)}(z), \tag{7}$$

where $X_s^{(k)}(z)$ are some local fields. Equations (4) then is nothing but a system of a linear equations on a coefficients of density G_{s+1}. Direct computation provides for the first few Integrals of Motion:

$$G_2 =: (\partial_z\varphi(z))^2 :$$
$$G_4 =: (\partial_z\varphi)^4 : + (Q^2+1) : (\partial_z^2\varphi)^2 :$$
$$G_6 =: (\partial_z\varphi)^6 : -\frac{5}{8} : (\partial\varphi)^4 : +5(Q^2+2)\Big(: (\partial_z^2\varphi)^2\partial_z\varphi^2 : -\frac{1}{24} : (\partial^2\varphi)^2 : \Big) + \Big(Q^4 + \frac{8}{3}Q^2 + \frac{19}{12}\Big) : (\partial^2\varphi)^2 :$$

$$G_8 = \Big(: (\partial_z\varphi)^8 : + \dots\Big),$$

here $Q = b + \frac{1}{b}$, and " : : " denotes the Wick normal ordering.

While I_3 and I_5 obviously commute with the I_1 which plays the role of grading operator, the commutativity of I_3 and I_5 is not obvious but straightforward to check. Note that this densities coincide with densities T_s (2),(3) in the semiclassical limit $b \to \infty$ as it should be.

More generally, one can consider the tensor product of n Fock spaces $\mathcal{F}_{u_1} \otimes \dots \otimes \mathcal{F}_{u_n}$ and the corresponding n-component bosonic field $\boldsymbol{\varphi}(z) = (\varphi_1, \dots, \varphi_n)$[3]:

$$\partial\varphi_j(z) = u_j + \sum_{n\neq 0} a_n^{(j)} e^{inz}, \quad [a_n^i, a_m^j] = m\delta_{i,j}\delta_{m,-n},$$

and affine set of screenings corresponding to an affine Lie algebra $\hat{\mathfrak{g}}$.

$$\mathcal{S}_r = \oint e^{b(\boldsymbol{\alpha}_r\cdot\boldsymbol{\varphi}(z))}\frac{dz}{2\pi},$$

[3] Note that commutation relation of bosonic modes $a_n^{(i)}$ are different from ones defined previously in case of a single field (5). This is because the Sinh-Gordon model is an $\hat{\mathrm{A}}_1$ Toda and in (1) we already decouple the $U(1)$ center of mass $U = \frac{\varphi_1+\varphi_2}{2}$, and left with a single bosonic field $\varphi = \frac{\varphi_1-\varphi_2}{2}$.

where $\boldsymbol{\alpha}_r$ have scalar products corresponding to the Dynkin diagram of an affine Lie algebra $\hat{\mathfrak{g}}$: $(\alpha_r \cdot \alpha_s) = c_{r,s}$. The Integrals of Motion can be again found as the intersection of kernels of all the screenings [LF91], [FF96]:

$$[\mathbf{I}_s, \mathcal{S}_r] = 0.$$

In this thesis we will consider in details the cases of $\hat{\mathfrak{g}} = \hat{\mathbf{A}}_n$ and $\hat{\mathfrak{g}} = \hat{\mathbf{B}}_\mathbf{n}, \hat{\mathbf{C}}_\mathbf{n}, \hat{\mathbf{D}}_\mathbf{n}$. The existence of a grading operator $\mathbf{I}_1$ among the Integrals of Motion allows to restrict the IOMs on a finite dimensional space $\mathbf{I}_1 = N$, such that they becomes a finite dimensional matrices. Nonetheless their exact diagonalization is by far a non trivial problem. Our strategy in analysing this problem is to identify corresponding integrable systems with integrable "spin chains" with the symmetry of affine Yangian, and then apply to them a machinery of algebraic Bethe ansatz.

0.2 Thesis results

The main results of chapters 1 and 2 are the Bethe ansatz equations and the Bethe eigenvectors, which provide a diagonalization of the chiral integrals of motion obtained as a UV limit of the Toda integrable system.

- For the A_n case we derive the Bethe ansatz equations for the spectrum of the local (1.13) and KZ (1.66) Integrals of Motion:

$$q \prod_{j \neq i} \prod_{\alpha=1}^{3} \frac{x_i - x_j - \epsilon_\alpha}{x_i - x_j + \epsilon_\alpha} \prod_{k=1}^{n} \frac{x_i - u_k + \epsilon_3}{x_i - u_k} = 1 \quad \text{for all} \quad i = 1, \dots, N, \tag{8}$$

  here we used Nekrasov epsilon notations $\epsilon_1 \sim b^{-1}, \epsilon_2 \sim b, \epsilon_3 \sim -Q$, see formula (1.38) for details. Corresponding Bethe vectors are given by the formula (1.83).

- For the BCD case we derive the boundary Bethe ansatz equations for the spectrum of the local (2.1),(2.5) and KZ (2.14) Integrals of Motion:

$$\begin{gathered} r^\alpha(x_i) r^\beta(x_i) A(x_i) A^{-1}(-x_i) \prod_{j \neq i} G(x_i - x_j) G^{-1}(-x_i - x_j) = 1, \\ G(x) = \frac{(x - \epsilon_1)(x - \epsilon_2)(x - \epsilon_3)}{(x + \epsilon_1)(x + \epsilon_2)(x + \epsilon_3)}, \qquad A(x) = \prod_{k=1}^{n} \frac{x - u_k + \frac{\epsilon_3}{2}}{x - u_k - \frac{\epsilon_3}{2}}, \quad r^\alpha(x) = -\frac{x + \epsilon_\alpha/2}{x - \epsilon_\alpha/2}. \end{gathered} \tag{9}$$

  And the Bethe vectors are defined in (2.25).

Another important results include:

- explicit computation of the current realisation (1.40) of the RLL algebra with Maulik-Okounkov [MO19] $R-$matrix.

- three different solutions $\mathcal{K}^i$ (2.11)-(2.12) of the Sklyanin's KRKR relation with the Maulik-Okounkov [MO19] $R-$matrix.

In chapter 3 we studied Integrals of Motion for the q-deformed W algebras.

- We provide explicit formulas for the Integrals of Motion of the q-deformed W algebras of BCD type (3.59).

- We construct the q-deformed versions of the reflection R and K operators (3.62).

0.3 Thesis review

This section is a short guide through the thesis, which contains main statements and ideas. The thesis consists of three chapters. The chapter 1 of the thesis is devoted to the study of $\hat{\mathfrak{gl}}_1$ affine Yangian and related integrable systems. We studied in details the connection between the RLL algebra and its current realisation. We derive the local Integrals of Motion for W algebras of type A (1.13) and corresponding Bethe ansatz equations (1.81) for their spectrum. In the chapter 2 we introduce the Integrals of Motion of BCD type (2.1),(2.5), and studied their spectrum by means of the boundary Bethe ansatz of the affine Yangian. We provide three different solutions $K^{1,2,3}$ of the Sklyanin's KRKR equation (2.11)-(2.12), and the Bethe ansatz equations (2.27) for the spectrum of the local Integrals of Motion. In the chapter 3 we studied the q-deformation of the Local and KZ Integrals of Motion. We provide explicit formulas for the q-deformed versions of the local Integrals of Motion of arbitrary high spin (3.59) for the q-deformed W algebras of type BCD.

W algebras and Maulik-Okounkov $R-$matrix. In section 1.2 we recall the definition of our main tool the Maulik-Okounkov $R-$matrix [MO19] as a unique (up to a normalisation factor) solution of the intertwining relation:

$$\mathcal{R}_{i,j}\big(Q\partial-\partial\varphi_i\big)\big(Q\partial-\partial\varphi_j\big)=\big(Q\partial-\partial\varphi_j\big)\big(Q\partial-\partial\varphi_i\big)\mathcal{R}_{i,j}, \tag{10}$$

where the product of two brackets is a Miura-Gelfand-Dikii transformation [FL88,Luk88] which defines generators of W algebra. Multiplying the brackets in different orders we obtain two isomorphic but not identical W algebras

$$\big(Q\partial-\partial\varphi_j\big)\big(Q\partial-\partial\varphi_i\big)=(Q\partial)^2+W^{(1)}(z)(Q\partial)+W^{(2)}(z),$$
$$\big(Q\partial-\partial\varphi_i\big)\big(Q\partial-\partial\varphi_j\big)=(Q\partial)^2+\tilde{W}^{(1)}(z)(Q\partial)+\tilde{W}^{(2)}(z)$$

The operator $\mathcal{R}_{i,j}$ then intertwines the two W algebras and acts in the tensor product of two Fock representations of Heisenberg algebra with the highest weight parameters u_i and u_j

$$\mathcal{F}_{u_i}\otimes\mathcal{F}_{u_j}\xrightarrow{\mathcal{R}_{i,j}}\mathcal{F}_{u_i}\otimes\mathcal{F}_{u_j}$$

and its matrix depends on the difference u_i-u_j. Then, by considering W_3 algebra generated by the product of three terms $\big(Q\partial-\partial\varphi_1\big)\big(Q\partial-\partial\varphi_2\big)\big(Q\partial-\partial\varphi_3\big)$, we immediately obtain from the definition (10) that the $\mathcal{R}_{i,j}(u_i-u_j)$ matrix satisfies the Yang-Baxter equation

$$\mathcal{R}_{1,2}(u_1-u_2)\mathcal{R}_{1,3}(u_1-u_3)\mathcal{R}_{2,3}(u_2-u_3)=\mathcal{R}_{2,3}(u_2-u_3)\mathcal{R}_{1,3}(u_1-u_3)\mathcal{R}_{1,2}(u_1-u_2),$$

and hence the whole machinery of quantum inverse scattering method can be applied.

RLL algebra and its current realisation. In section 1.3 we introduce an RLL algebra:

$$\mathcal{R}_{ij}(u-v)\mathcal{L}_i(u)\mathcal{L}_j(v)=\mathcal{L}_j(v)\mathcal{L}_i(u)\mathcal{R}_{ij}(u-v). \tag{11}$$

The left and right hand sides of this equation both act in the tensor product of three Fock spaces $\mathcal{F}_{u_i}\otimes\mathcal{F}_{u_j}\otimes\mathcal{F}_q$. The $\mathcal{R}_{ij}(u_i-u_j)$ matrix acts in the product of two Fock spaces $\mathcal{F}_{u_i}\otimes\mathcal{F}_{u_j}$, and $\mathcal{L}_i(u_i)$ operator acts in $\mathcal{F}_{u_i}\otimes\mathcal{F}_q$. Hence the RLL algebra (11) may be considered as an infinite set of quadratic relations between the matrix elements of $L-$operator, labeled by two partitions

$$\mathcal{L}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(u)\overset{\text{def}}{=}\langle u|a_{\boldsymbol{\lambda}}\mathcal{L}(u)a_{-\boldsymbol{\mu}}|u\rangle\quad\text{where}\quad a_{-\boldsymbol{\mu}}|u\rangle=a_{-\mu_1}a_{-\mu_2}\dots|u\rangle.$$

It is well known that the commutation relations of RLL algebras could be rewritten in an equivalent current form, see [DF93] where such an analysis was performed for $U_q(gl(n))$. In this thesis we provide similar analysis for the case of Maulik-Okounkov $R-$matrix. We conjecture that the *RLL* algebra (11) factorized over its center is related to the Yangian of $\widehat{\mathfrak{gl}}(1)$ considered by Tsymbaliuk in [Tsy17]. This is similar to the well known fact that the Yangians of $\mathfrak{gl}(n)$ and of $\mathfrak{sl}(n)$ are differ by central elements [KS82]. We will usually refer to the RLL algebra as Yang-Baxter algebra and denote as $\mathrm{YB}\big(\widehat{\mathfrak{gl}}(1)\big)$, reserving the notation $\mathrm{Y}\big(\widehat{\mathfrak{gl}}(1)\big)$ for Tsymbaliuk's algebra.

Our methods are similar to the analysis performed in [DF93]. We introduce three basic currents of degree 0, 1 and -1 (see appendix A.2 for more details)

$$h(u) \stackrel{\text{def}}{=} \mathcal{L}_{\varnothing,\varnothing}(u), \qquad e(u) \stackrel{\text{def}}{=} h^{-1}(u)\cdot \mathcal{L}_{\varnothing,\square}(u) \quad \text{and} \quad f(u) \stackrel{\text{def}}{=} \mathcal{L}_{\square,\varnothing}(u)\cdot h^{-1}(u),$$

as well as an auxiliary current (as we will see (1.40a) it also belongs to the Cartan subalgebra of $\mathrm{YB}\big(\widehat{\mathfrak{gl}}(1)\big)$)

$$\psi(u) \stackrel{\text{def}}{=} \Big(\mathcal{L}_{\square,\square}(u-Q) - \mathcal{L}_{\varnothing,\square}(u-Q)h^{-1}(u-Q)\mathcal{L}_{\square,\varnothing}(u-Q)\Big)h^{-1}(u-Q). \tag{12}$$

The direct computation (provided in the appendix A.2) allows to rewrite the RLL commutation relations (11) in terms of e, f, h currents. The results are presented at the beginning of section 1.3.1. There also exists an inverse mapping which allows to express $\mathcal{L}_{\lambda,\mu}(u)$ operators in terms of e, h, f, ψ currents. In particular there is an important for us formula

$$\mathcal{L}_{\boldsymbol{\lambda},\varnothing}(u) = \frac{1}{(2\pi i)^{|\boldsymbol{\lambda}|}} \oint \cdots \oint F_{\lambda}(\boldsymbol{z}|u)\, h(u) f(z_{|\boldsymbol{\lambda}|})\dots f(z_1) dz_1 \dots dz_{|\boldsymbol{\lambda}|} \tag{13}$$

where $F_{\lambda}(\boldsymbol{z})$ is a concrete function and contours go clockwise around ∞ and all poles of $F_{\lambda}(\boldsymbol{z})$. This formula and recurrent definition of function $F_{\lambda}(\boldsymbol{z})$ is explained in the appendix A.3, see formulas (A.25),(A.27).

ϵ- notations. It is easy to note that quantum Integrals of Motion depends only on combination $Q = b + \frac{1}{b}$ and not b, b^{-1} themselves. Which results in a very well known symmetry $b \to b^{-1}$. As can be seen for example in [Tsy14], defining relations of affine Yangian algebra are symmetric under all three parameters b, b^{-1} and Q parameters[4]. For this reason it will be more convenient to use Nekrasov epsilon notations rather than Liouville notations. Formally, they are obtained by replacing central charge parameter

$$b \to \frac{\epsilon_2}{\sqrt{\epsilon_1\epsilon_2}}, \quad b^{-1} \to \frac{\epsilon_1}{\sqrt{\epsilon_1\epsilon_2}}, \quad Q \to -\frac{\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}} \implies \epsilon_1+\epsilon_2+\epsilon_3 = 0.$$

Note that without loss of generality it is always possible to put $\epsilon_1\epsilon_2 = 1$.

Center of $\mathbf{YB}(\widehat{\mathfrak{gl}}(1))$ The section 1.3.2 is insufficient for the understanding of the main results of the thesis. In this section we show that the algebra $\mathrm{YB}(\widehat{\mathfrak{gl}}(1))$ contains an infinite dimensional center. Namely for any singular vector $|s\rangle$ of W_n algebra acting in the space of n bosons we assign a central element D_s (1.52). First element of this series is related to the operator $\psi(u)$ (12) as

$$D_{1,1}(u) = \psi(u)\frac{h(u)h(u+\epsilon_3)}{h(u-\epsilon_1)h(u-\epsilon_2)}.$$

$$\psi(u) = \frac{\langle s_{1,1}|\mathcal{L}^1(u)\mathcal{L}^2(u+\epsilon_3)|s_{1,1}\rangle}{h(u)h(u+\epsilon_3)},$$

[4]For the case of KDV and ILW integrable systems this symmetry is broken by a particular choice of Fock representation.

where

$$|s_{1,1}\rangle_u \stackrel{\text{def}}{=} \left(a^{(1)}_{-1} - a^{(2)}_{-1}\right)|\varnothing\rangle_u \otimes |\varnothing\rangle_{u+\epsilon_3}$$

is a singular vector of a W algebra which appears in the tensor product of two Fock spaces $\mathcal{F}_{u_1} \otimes \mathcal{F}_{u_2}$ at special value of spectral parameters $u_2 = u_1 + \epsilon_3$.
In general, for the singular vector $|s\rangle$ of W_n algebra acting in the space of n Fock spaces $\mathcal{F}_1(u_1)\dots\mathcal{F}_n(u_n)$[5] we may define a Cartan current acting on a quantum space as

$$h_s = \langle s|\mathcal{L}^1(u-u_1)\dots\mathcal{L}^n(u-u_n)|s\rangle.$$

And the operator:

$$D_s = \frac{h_s(u)}{\prod\limits_{i=1}^{n} h(u-v_i)} \tag{14}$$

is central.

Zero twist integrable system. In section 1.3.3 we considered the integrable system with zero twist $q = 0$. In this case twist deformed transfer matrix $\mathbf{T}_q$ turns to the $h(u)$ current introduced in previous section. The spectrum and eigenbasis of $h(u)$ is very simple and may be written explicitly. For example for a representation in the tensor product of n Fock spaces: $\mathcal{F}_{x_1} \otimes \dots \otimes \mathcal{F}_{x_n}$ the eigenbasis is enumerated by the collection of n Young diagrams $\vec{\boldsymbol{\lambda}} = \{\lambda^{(1)}, \dots \lambda^{(n)}\}$ and known as a basis of generalised Jack polynomials. The eigenvalues may be conveniently written in terms of contents of the Young diagrams

$$h(u)|\vec{\boldsymbol{\lambda}}\rangle = \prod_{\square\in\vec{\boldsymbol{\lambda}}} \frac{(u-c_\square)}{(u-c_\square-\epsilon_3)}|\vec{\boldsymbol{\lambda}}\rangle.$$

For a cell $\square = (i,j)$ the content $c_\square$ is defined as

$$c_\square = x_k - (i-1)\epsilon_1 - (j-1)\epsilon_2.$$

We proof that explicit formulas for the action of e, f generators in the eigenbasis of h are given by the formulas (1.60):

$$\begin{aligned} e(u)|\vec{\boldsymbol{\lambda}}\rangle &= \sum_{\square\in\text{addable}(\vec{\boldsymbol{\lambda}})} \frac{E(\vec{\boldsymbol{\lambda}}, \vec{\boldsymbol{\lambda}}+\square)}{u-c_\square}|\vec{\boldsymbol{\lambda}}+\square\rangle, \\ f(u)|\vec{\boldsymbol{\lambda}}\rangle &= \sum_{\square\in\text{removable}(\vec{\boldsymbol{\lambda}})} \frac{F(\vec{\boldsymbol{\lambda}}, \vec{\boldsymbol{\lambda}}-\square)}{u-c_\square}|\vec{\boldsymbol{\lambda}}-\square\rangle, \end{aligned} \tag{15}$$

where the amplitudes $E(\vec{\boldsymbol{\lambda}}, \vec{\boldsymbol{\lambda}}+\square)$ and $F(\vec{\boldsymbol{\lambda}}, \vec{\boldsymbol{\lambda}}-\square)$ are given by the formulas

$$\begin{aligned} E(\vec{\boldsymbol{\lambda}}, \vec{\boldsymbol{\lambda}}+\square) &= \frac{\epsilon_1\epsilon_2}{\epsilon_3} \prod_{\square'\in\vec{\boldsymbol{\lambda}}+\square} S^{-1}(c_{\square'}-c_\square) \prod_{k=1}^{n} \frac{(c_\square - x_k + \epsilon_3)}{(c_\square - x_k)}, \\ F(\vec{\boldsymbol{\lambda}}, \vec{\boldsymbol{\lambda}}-\square) &= \prod_{\square'\in\vec{\boldsymbol{\lambda}}-\square} S(c_\square - c_{\square'}), \end{aligned} \tag{16}$$

with

$$S(x) = \frac{(x+\epsilon_1)(x+\epsilon_2)}{x(x-\epsilon_3)}.$$

This formulas plays the crucial role in definition of Bethe vector, study of its matrix elements.

[5] Note that a singular vector may exist only if evaluation parameters u_i are not arbitrary, but they are restricted by some resonance conditions.

Transfer matrix and ILW Integrals of Motion. At the beginning of section 1.4 we recall that the transfer matrix defined by

$$\mathbf{T}_q(u) = \mathrm{Tr}\Big(q^{L_0^{(0)}}\mathcal{R}_{0,1}(u-u_1)\mathcal{R}_{0,2}(u-u_2)\ldots\mathcal{R}_{0,n-1}(u-u_{n-1})\mathcal{R}_{0,n}(u-u_n)\Big)\Big|_{\mathcal{F}_u},$$

admits the following large u expansion

$$\mathbf{T}_q(u) = \Lambda(u,q)\exp\left(\frac{1}{u}\mathbf{I}_1(q) + \frac{1}{u^2}\mathbf{I}_2(q) + \ldots\right),$$

where $\Lambda(u,q)$ is a normalization factor and $\mathbf{I}_1$ and $\mathbf{I}_2$ are the first ILW$_n$ Integrals of Motion.

$$\mathbf{I}_1(q) = \frac{iQ}{2\pi}\int\left[\frac{1}{2}\sum_{k=1}^{n}(\partial\varphi_k)^2\right]dx,$$

$$\mathbf{I}_2(q) = \frac{iQ}{2\pi}\int\left[\frac{1}{3}\sum_{k=1}^{n}(\partial\varphi_k)^3 + Q\left(\frac{i}{2}\sum_{i,j}\partial\varphi_i D\partial\varphi_j + \sum_{i<j}\partial\varphi_i\partial^2\varphi_j\right)\right]dx,$$

$$\mathbf{I}_3(q) = \frac{iQ}{2\pi}\int\left[\frac{1}{4}\sum_{k=1}^{n}(\partial\varphi_k)^4 + \ldots\right]dx,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where ($Q = -\frac{\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}$), and D is the non-local operator whose Fourier image is

$$D(k) = k\frac{1+q^k}{1-q^k}.$$

Now let us define KZ Integral of Motion as $T_q(u)$ operator at the special value of the parameter $u = u_1$. Using the fact that $\mathcal{R}_{0,1}(0) = \mathcal{P}_{0,1}$ is a permutation operator, one finds for the KZ IOM:

$$\mathcal{I}_1^{\mathrm{KZ}}(q) \overset{def}{=} T_q(u_1) = q^{L_0^{(1)}}\mathcal{R}_{1,2}(u_1-u_2)\mathcal{R}_{1,3}(u_1-u_3)\ldots\mathcal{R}_{1,n}(u_1-u_n).$$

The rest of this section is aimed to show that the simultaneous spectrum of KZ and first few local Integrals of Motion is governed by Bethe ansatz equations (8).

Special vector $|\chi\rangle$, definition of Bethe vector. In section 1.4.1 we define the Bethe vector $B(\boldsymbol{x})$ which turns to the eigenvector of corresponding integrable system after imposing the Bethe equations. In order to define Bethe vector we introduce the tensor product of $n+N$ Fock spaces, with n "quantum" and N "auxiliary" spaces

$$\underbrace{\mathcal{F}_{u_1}\otimes\cdots\otimes\mathcal{F}_{u_n}}_{\text{quantum space}}\otimes\underbrace{\mathcal{F}_{x_1}\otimes\cdots\otimes\mathcal{F}_{x_N}}_{\text{auxiliary space}}$$

generated from the vacuum state

$$|\varnothing\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}} = |x_1\rangle\otimes\cdots\otimes|x_N\rangle\otimes|u_1\rangle\otimes\cdots\otimes|u_n\rangle.$$

We then searched for the Bethe vector in the form[6]:

$$|B(\boldsymbol{x})\rangle_{\boldsymbol{u}} \overset{\mathrm{def}}{=} {}_{\boldsymbol{x}}\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}} \quad \text{where} \quad \mathcal{R}(\boldsymbol{x},\boldsymbol{u}) = \mathcal{R}_{x_1u_1}\ldots\mathcal{R}_{x_Nu_1}\ldots\mathcal{R}_{x_1u_n}\ldots\mathcal{R}_{x_Nu_n},$$

[6]This definition is similar to the very general approach investigated in [AO17] (in particular this construction is explained in section 1.3.3 of [AO17]).

here $|\chi\rangle_{\boldsymbol{x}}$ is some vector in auxiliary space. The convenient choice for the vector $|\chi\rangle$ is to choose it equal to an eigenvector of zero twist integral of motion $h(u)$ acting on auxiliary Fock space. Among the various eigenvectors the simplest one is (see (1.68) for details)

$$|\chi\rangle_{\boldsymbol{x}} \overset{\text{def}}{=} |\underbrace{\square, \dots, \square}_{N}\rangle \sim \oint_{x_N} dz_N \cdots \oint_{x_1} dz_1\, e(z_N)\dots e(z_1)|\varnothing\rangle_{\boldsymbol{x}}.$$

Alternatively vector $|\chi\rangle_{\boldsymbol{x}}$ is fixed (up to proportionality factor) as an eigenvector of zero twist integrable system with concrete eigenvalue

$$h(u)|\chi\rangle_{\boldsymbol{x}} = \prod_{k=1}^{N} \frac{u - x_k}{u - x_k - \epsilon_3} |\chi\rangle_{\boldsymbol{x}}.$$

Explicit computation of Bethe vector and its properties. Here we continue to describe the results of section 1.4.1. A direct consequence of (15),(16) implies a convenient formula :

$${}_{\boldsymbol{x}}\langle\varnothing|f(z_N)\dots f(z_1)|\chi\rangle_{\boldsymbol{x}} = \text{Sym}_{\boldsymbol{x}}\left(\prod_{a=1}^{N} \frac{1}{z_a - x_a} \prod_{a<b} S(x_a - x_b)\right),$$

where $\text{Sym}_{\boldsymbol{x}}$ means the symmetrization over the x_i variables. Together with the formula (13) for an $\mathcal{L}$-operators in terms of f and h currents it allows to explicitly compute the matrix elements of Bethe vector - the so called weight functions:

$$\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) \overset{\text{def}}{=} {}_{\boldsymbol{u}}\langle\varnothing|a^{(1)}_{\boldsymbol{\lambda}^{(1)}} \dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}}|B(\boldsymbol{x})\rangle_{\boldsymbol{u}} = {}_{\boldsymbol{x}}\langle\varnothing|\mathcal{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)|\chi\rangle_{\boldsymbol{x}}.$$

After the straightforward computation we get

$$\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) = \frac{1}{(2\pi i)^N} \oint \cdots \oint \Omega_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\, \text{Sym}_{\boldsymbol{x}}\left(\prod_{a=1}^{N} \frac{1}{z_a - x_a} \prod_{a<b} S(x_a - x_b)\right) d\vec{\boldsymbol{z}},$$

where function

$$\Omega_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u}) = F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u}) \left(\prod_{j=1}^{|\boldsymbol{\lambda}^{(1)}|} \frac{u_2 - z_j^{(1)}}{u_2 - z_j^{(1)} - \epsilon_3}\right) \left(\prod_{j=1}^{|\boldsymbol{\lambda}^{(2)}|} \frac{u_3 - z_j^{(2)}}{u_3 - z_j^{(2)} - \epsilon_3} \prod_{j=1}^{|\boldsymbol{\lambda}^{(1)}|} \frac{u_3 - z_j^{(1)}}{u_3 - z_j^{(1)} - \epsilon_3}\right) \cdots$$
$$\cdots \left(\prod_{j=1}^{|\boldsymbol{\lambda}^{(n-1)}|} \frac{u_n - z_j^{(n-1)}}{u_n - z_j^{(n-1)} - \epsilon_3} \prod_{j=1}^{|\boldsymbol{\lambda}^{(n-2)}|} \frac{u_n - z_j^{(n-2)}}{u_n - z_j^{(n-2)} - \epsilon_3} \cdots \prod_{j=1}^{|\boldsymbol{\lambda}^{(1)}|} \frac{u_n - z_j^{(1)}}{u_n - z_j^{(1)} - \epsilon_3}\right)$$

The integral shrinks to the points $\boldsymbol{x}$ and one obtains explicit formula (see (1.78) for details)

$$\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) = \text{Sym}_{\boldsymbol{x}}\left(\Omega_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{x}}|\boldsymbol{u}) \prod_{a<b} S(x_a - x_b)\right).$$

The simplicity of this formula explains our choice of vector $|\chi\rangle$.

Diagonalization of local and KZ Integrals of Motion. Using the computation methods described above, in sections 1.4.2,1.4.4 we were able to compute the action of local and KZ Integrals of Motion on a Bethe vector. Namely we were able to prove that upon the Bethe equations:

$$q \prod_{j \neq i} \prod_{\alpha=1}^{3} \frac{x_i - x_j - \epsilon_\alpha}{x_i - x_j + \epsilon_\alpha} \prod_{k=1}^{n} \frac{x_i - u_k + \epsilon_3}{x_i - u_k} = 1 \quad \text{for all} \quad i = 1, \dots, N,$$

the Bethe vector becomes an eigenvector of KZ integral of motion $\mathcal{I}_1^{\mathrm{KZ}} = q^{L_0^{(1)}}\mathcal{R}_{1,2}\mathcal{R}_{1,3}\dots\mathcal{R}_{1,n-1}\mathcal{R}_{1,n}$, with eigenvalue:

$$t_q^1(\boldsymbol{u}) = \prod_{k=1}^{N}\frac{x_k-u_1}{x_k-u_1+\epsilon_3}.$$

And also becomes an eigenvector of Local integral of motion $\mathbf{I}_2$:

$$-\epsilon_3\int\left[\frac{1}{3}\sqrt{\epsilon_1\epsilon_2}\sum_i(\partial\phi_i)^3-\epsilon_3\left(\frac{1}{2}\sum_{i,j}\partial\phi_i D(q)\partial\phi_j+\sum_{i<j}\partial\phi_i\partial^2\phi_j\right)\right]\frac{dx}{2\pi}-\frac{\epsilon_3\mathbf{I}_1(q)}{2}-\frac{\epsilon_3}{3}\sqrt{\epsilon_1\epsilon_2}\sum_i u_i^3,$$

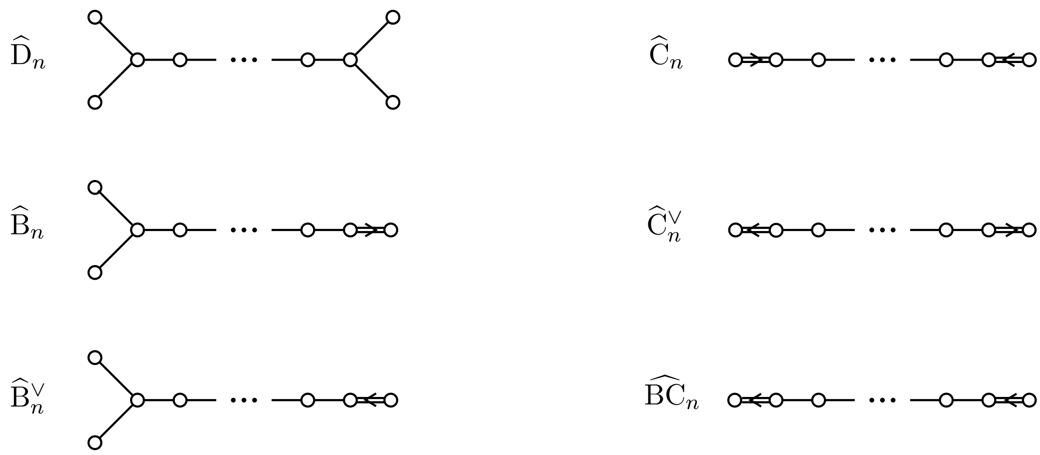
with eigenvalue $\left(\sum_1^N x_k\right)$. We were also able to write explicitly the solution (1.83) of a difference Knizhnik-Zamolodchikov (KZ) (1.84) and Okounkov-Pandharipande (OP) (1.96) equation in terms of Bethe vector. This finishes the review of the first chapter.

Integrable structure of B, C, D **conformal field theory.** The second chapter is devoted to the study of integrable structure of B, C, D conformal field theory and its relation to boundary Bethe ansatz of affine Yangian.
In section 2.2 we introduce the affine Toda QFT associated to an affine Lie algebra $\mathfrak{g}$ of BCD type. We recall that Integrals of Motion can be found as a commutant of the affine set of screenings:

$$\mathcal{S}_r = \oint e^{b(\boldsymbol{\alpha_r}\cdot\boldsymbol{\varphi(z)})}\frac{dz}{2\pi}, \tag{17}$$

$$[\mathbf{I}_s,\mathcal{S}_r]=0,$$

where vectors $\boldsymbol{\alpha_r}$ have the Gram matrix of BCD type affine Lie algebra and $b=\frac{\epsilon_2}{\sqrt{\epsilon_1\epsilon_2}}$ is the coupling constant.



Using the standard parametrization for the roots one can express the scalar products in the exponents in (17) as

$$(\boldsymbol{\alpha}_0\cdot\boldsymbol{\varphi})=\begin{cases}-\varphi_1\\-2\varphi_1\\-\varphi_1-\varphi_2\end{cases}\qquad(\boldsymbol{\alpha}_r\cdot\boldsymbol{\varphi})=\varphi_r-\varphi_{r+1}\quad\text{for}\quad 0<r<n,\quad(\boldsymbol{\alpha}_n\cdot\boldsymbol{\varphi})=\begin{cases}\varphi_n\\2\varphi_n\\\varphi_{n-1}+\varphi_n\end{cases}$$

That is each of the affine diagrams can be interpreted as non-affine A_{n-1} diagram with two boundary conditions which can be of three types B, C or D corresponding to the short root, the long root or the root of the length $\sqrt{2}$ correspondingly.

As in the first chapter, we will search local Integrals of Motion in terms of integrals of local densities $\mathbf{I}_s = \int\limits_0^{2\pi} G_{s+1}(z)\frac{dz}{2\pi}$. First few local Integrals of Motion can be computed explicitly by solving the equation

$$\frac{1}{2\pi i}\oint_z e^{b(\boldsymbol{\alpha}_r\cdot\boldsymbol{\varphi}(\xi))}G_{s+1}(z)d\xi = \partial X_s(z),$$

where $X_s(z)$ is some local field. The first non trivial density has the form

$$\begin{aligned}
G_4(z) = \big(\partial\boldsymbol{\varphi}\cdot\partial\boldsymbol{\varphi}\big)^2 - \frac{1}{3}\left(2n - \frac{\epsilon_\alpha+\epsilon_\beta}{\epsilon_3}\right)\sum_{k=1}^{n}\big(\partial\varphi_k\big)^4 + \\
+ \frac{4\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}\sum_{k=1}^{n}\partial\varphi_k^2\left(\sum_{j<k}\Big(j-1+\frac{\epsilon_3-\epsilon_\alpha}{2\epsilon_3}\Big)\partial^2\varphi_j - \sum_{j>k}\Big(n-j+\frac{\epsilon_3-\epsilon_\beta}{2\epsilon_3}\Big)\partial^2\varphi_j\right) + \\
+\left(2n + \frac{4(n-1)(\epsilon_1^2+\epsilon_2^2)}{3\epsilon_1\epsilon_2} + \frac{(\epsilon_1\epsilon_2-2\epsilon_3^2)(\epsilon_\alpha+\epsilon_\beta-2\epsilon_3)}{3\epsilon_1\epsilon_2\epsilon_3}\right)\big(\partial^2\boldsymbol{\varphi}\cdot\partial^2\boldsymbol{\varphi}\big) - \\
- \frac{4\epsilon_3^2}{\epsilon_1\epsilon_2}\sum_{i\le j}\left(i-1+\frac{\epsilon_3-\epsilon_\alpha}{2\epsilon_3}\right)\left(n-j+\frac{\epsilon_3-\epsilon_\beta}{2\epsilon_3}\right)(2-\delta_{ij})\partial^2\varphi_i\partial^2\varphi_j, \quad (18)
\end{aligned}$$

here $\alpha,\beta=\{1,2,3\}$ for the B,C or D type of endings correspondingly.

Sklyanin's $K-$matrix of affine Yangian. The crucial step in understanding the relation of this integrable structure to the boundary affine Yangian is to introduce reflection $K-$matrix. This is done in section 2.3. The idea is to consider reflection operator K as an intertwining operator of W algebra, analogically to how it was done for the $R-$matrix (10).

Let us introduce two currents of W_4 algebra acting in the space of two bosonic Fock modules $\mathcal{F}_{u_1}\otimes\mathcal{F}_{u_2}$:

$$W^{(2)} = (\partial\varphi_1)^2 + (\partial\varphi_2)^2 + \frac{2\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}\partial^2\varphi_1 + \frac{\epsilon_3-\epsilon_\alpha}{\sqrt{\epsilon_1\epsilon_2}}(\partial^2\varphi_2+\partial^2\varphi_1)$$

and

$$\begin{aligned}
W^{(4)} = (\partial\varphi_1)^2(\partial\varphi_2)^2 + \frac{2\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}\partial\varphi_1\partial\varphi_2\partial^2\varphi_2 + \frac{\epsilon_3-\epsilon_\alpha}{\sqrt{\epsilon_1\epsilon_2}}\left((\partial\varphi_1)^2\partial^2\varphi_2 + (\partial\varphi_2)^2\partial^2\varphi_1\right) - \\
- \frac{\epsilon_3\epsilon_\alpha}{\epsilon_1\epsilon_2}(\partial^2\varphi_1)^2 + \frac{(\epsilon_3-\epsilon_\alpha)^2}{\epsilon_1\epsilon_2}\partial^2\varphi_1\partial^2\varphi_2 - \frac{(\epsilon_1-\epsilon_\alpha)(\epsilon_2-\epsilon_\alpha)}{2\epsilon_1\epsilon_2}\left(\partial\varphi_1\partial^3\varphi_1 + \partial\varphi_2\partial^3\varphi_2\right) - \\
- \frac{\epsilon_3(\epsilon_3-\epsilon_\alpha)}{\epsilon_1\epsilon_2}\left(\partial\varphi_1\partial^3\varphi_1 - \partial\varphi_1\partial^3\varphi_2\right) + \frac{\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}\left(\frac{\epsilon_\alpha(\epsilon_3-\epsilon_\alpha)}{2\epsilon_1\epsilon_2} - \frac{\epsilon_3^2}{\epsilon_1\epsilon_2} - \frac{1}{3}\right)\partial^4\varphi_1
\end{aligned}$$

where $\alpha=1,2,3$ correspond to the W algebras of types B, C or D correspondingly.

By definition the R and K operators are defined by the following intertwining relations:

$$\mathcal{R}_{1,2}W^{(s)} = W^{(s)}\Big|_{\varphi_1\leftrightarrow\varphi_2}\mathcal{R}_{1,2}, \qquad \mathcal{K}_2W^{(s)} = W^{(s)}\Big|_{\varphi_2\to-\varphi_2}\mathcal{K}_2, \quad (19)$$

for $s=2,4$. The $R_{1,2}$ operator is identified with the Maulik-Okounkov $R-$matrix defined earlier (10) $\mathcal{R}_{1,2}=\mathcal{R}[\partial\varphi_1-\partial\varphi_2]$, while $\mathcal{K}_2$ is also equal to the MO $R-$matrix of the re-scaled argument

$$\begin{aligned}
\mathcal{K}_2^1 &= \mathcal{R}[\sqrt{2}\partial\varphi_2]\Big|_{\epsilon_1\to\sqrt{2}\epsilon_1,\epsilon_2\to\epsilon_2/\sqrt{2}} && \text{for B series}\\
\mathcal{K}_2^2 &= \mathcal{R}[\sqrt{2}\partial\varphi_2]\Big|_{\epsilon_1\to\epsilon_1/\sqrt{2},\epsilon_2\to\sqrt{2}\epsilon_2} && \text{for C series}\\
\mathcal{K}_2^3 &= \mathrm{Id} && \text{for D series}
\end{aligned}$$

Note that the simplest K operator is very explicit $\mathcal{K}_2^3 = \text{Id}$ and it does not depend on spectral parameter.

Now, similar to the argument of Maulik and Okounkov, the $K-$operator obeys Sklyanin's KRKR equation[7]

$$\mathcal{R}[\partial\varphi_1 - \partial\varphi_2]\mathcal{K}_1^\alpha\mathcal{R}[\partial\varphi_1 + \partial\varphi_2]\mathcal{K}_2^\alpha = \mathcal{K}_2^\alpha\mathcal{R}[\partial\varphi_1 + \partial\varphi_2]\mathcal{K}_1^\alpha\mathcal{R}[\partial\varphi_1 - \partial\varphi_2]. \tag{20}$$

KZ Integrals of Motion. In section 2.3.1 we have defined KZ Integrals of Motion:

$$\begin{aligned}\mathcal{T}_i^+ &= \mathcal{R}_{i,\overline{i+1}}\dots\mathcal{R}_{i,\overline{n}}\mathcal{K}_i^\alpha\mathcal{R}_{i,n}\dots\mathcal{R}_{i,i+1},\\ \mathcal{T}_i^- &= \mathcal{R}_{i,1}\dots\mathcal{R}_{i,i-1}\mathcal{K}_i^\beta\mathcal{R}_{1,\bar{i}}\dots\mathcal{R}_{i-1,\bar{i}},\\ \mathcal{I}_i^{\text{KZ}} &= \mathcal{T}_i^-\mathcal{T}_i^+\end{aligned} \tag{21}$$

where the barred index $\bar{i}$ means the conjugation by the operator of sign reflection D_i

$$\begin{aligned}D_i f(\boldsymbol{\varphi}) &= f(\boldsymbol{\varphi})\Big|_{\varphi_i\to-\varphi_i} D_i,\\ \mathcal{R}_{i,\bar{j}} &= D_j\mathcal{R}_{i,j}D_j = \mathcal{R}[\partial\varphi_i + \partial\varphi_j],\\ \mathcal{R}_{\bar{i},j} &= D_i\mathcal{R}_{i,j}D_i = \mathcal{R}[-\partial\varphi_i - \partial\varphi_j],\end{aligned}$$

Their mutual commutativity is provided by KRKR equation (20)

$$[\mathcal{I}_i^{\text{KZ}}, \mathcal{I}_j^{\text{KZ}}] = 0.$$

We also proved a commutativity between KZ and local Integrals of Motion $[\mathbf{I}_s, \mathcal{I}_i^{\text{KZ}}] = 0$ which follows from the intertwining relations (19) (see (2.15) for details)

$$\mathcal{T}_i^+\mathbf{I}_s = \mathbf{I}_s\Big|_{\varphi_i\to-\varphi_i}\mathcal{T}_i^+, \qquad \mathcal{T}_i^-\ \mathbf{I}_s\Big|_{\varphi_i\to-\varphi_i} = \mathbf{I}_s\ \mathcal{T}_i^-.$$

Of-shell Bethe vector. The section 2.4 goes in parallel to the section 1.4.1 where we considered the type A integrable structures. We introduce a product of $n+N$ Fock spaces where the first n products is a quantum $\mathcal{F}_{\boldsymbol{u}}$ Fock space and the second N products is an auxiliary Fock space $\mathcal{F}_{\boldsymbol{x}}$

$$\underbrace{\mathcal{F}_{u_n}\otimes\dots\otimes\mathcal{F}_{u_1}}_{\text{quantum space}}\otimes\underbrace{\mathcal{F}_{x_1}\otimes\dots\otimes\mathcal{F}_{x_N}}_{\text{auxiliary space}} = \mathcal{F}_{\boldsymbol{u}}\otimes F_{\boldsymbol{x}}.$$

We then define two types of $L-$ operators (2.21),(2.22), and $\mathcal{K}_{\boldsymbol{u}|\boldsymbol{x}}$ operator fixed by the recurrent relations (2.24).

Finally we define an of-shell Bethe vector by the formula (see (2.25) for details)

$$|B(\boldsymbol{x})\rangle = {}_x\langle\varnothing|\bar{\mathcal{L}}_{\boldsymbol{v}}\mathcal{K}_{\boldsymbol{x}}L_{\boldsymbol{v}}|\varnothing\rangle_v|\chi\rangle_x = {}_x\langle\varnothing|\mathcal{K}_{\boldsymbol{v}|\boldsymbol{x}}|\varnothing\rangle_v|\chi\rangle_x, \tag{22}$$

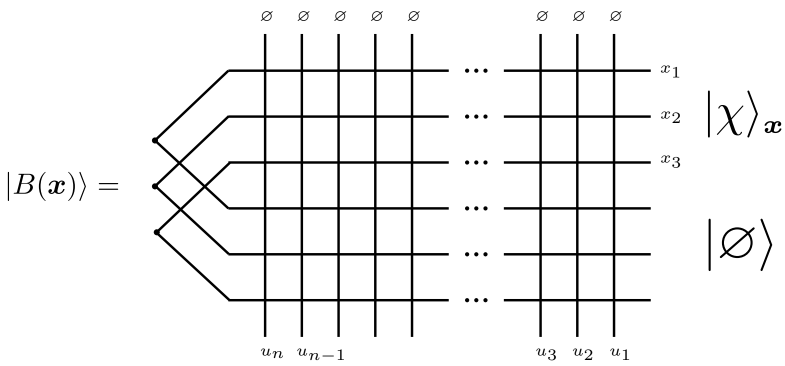
where $|\chi\rangle_{\boldsymbol{x}}$ is the same vector as in the first chapter (1.68). The definition of Bethe vector may be

[7] Let us note that originally [Skl88] the KRKR equation was written in a quite different form:

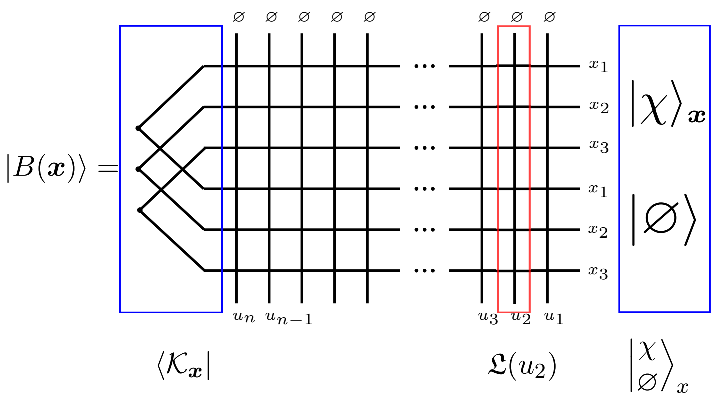
$$\mathcal{R}_{1,2}(u_1-u_2)\tilde{\mathcal{K}}_1(u_1)\mathcal{R}_{2,1}(u_2+u_1)\tilde{\mathcal{K}}_2(u_2) = \tilde{\mathcal{K}}_2(u_2)\mathcal{R}_{1,2}(u_1+u_2)\tilde{\mathcal{K}}_1(u_1)\mathcal{R}_{2,1}(u_1-u_2).$$

The difference is actually insufficient as the two equations are differ by the redefinition of $K-$operator and overall conjugation by the reflection of bosonic modes $a_n^{1,2}\to -a_n^{1,2}$, $n\neq 0$

illustrated by a picture:



In the beginning of section (2.5) we suggest to interpret the Bethe vector $|B(\boldsymbol{x})\rangle$ as a product of some $L-$operators $\mathfrak{L}(u_n)\ldots\mathfrak{L}(u_1)$ sandwiched between bra and ket states $\langle\mathcal{K}_{\boldsymbol{x}}|$ and $\left|\begin{matrix}\chi\\ \varnothing\end{matrix}\right\rangle_x$, see the picture below. This bra and ket vectors then should live in the tensor product of the Fock space and its dual $\mathcal{F}_{\boldsymbol{x}}\otimes\mathcal{F}^\star_{\boldsymbol{x}}$.



Strange module. In section 2.5 we observe that modified operators $\mathfrak{L}$ obeys the same RLL commutation relations:

$$\mathcal{R}_{ij}(u-v)\mathfrak{L}_i(u)\mathfrak{L}_j(v) = \mathfrak{L}_j(v)\mathfrak{L}_i(u)\mathcal{R}_{ij}(u-v).$$

And we still can define $\mathfrak{h} \overset{def}{=} \mathfrak{L}_{\varnothing,\varnothing}, \mathfrak{e} \overset{def}{=} \mathfrak{h}^{-1}\mathfrak{L}_{\varnothing,\square}, \mathfrak{f} \overset{def}{=} \mathfrak{L}_{\square,\varnothing}\mathfrak{h}^{-1}$ operators.

The difference is that $\mathfrak{L}-$operators act in the tensor product of Fock module and its dual $\mathcal{F}_{\boldsymbol{x}}\otimes\mathcal{F}^\star_{\boldsymbol{x}}$. This representation for the $\mathfrak{L}-$operator doesn't have a highest weight, however the action of $\mathfrak{h}(z)$ still can be diagonalized, the eigenvectors of $\mathfrak{h}(u),\psi(u)$ in $\mathcal{F}_{\boldsymbol{x}}\otimes\mathcal{F}^\star_{\boldsymbol{x}}$ are enumerated by the collection of $2N$ Young diagrams and denoted by $\left|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle$. The eigenvalues are given by the formulas:

$$\mathfrak{h}(u)\left|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle = \prod_{\square\in\vec{\boldsymbol{\lambda}}}\frac{(u-c_\square)}{(u-c_\square-\epsilon_3)}\prod_{\square\in\vec{\boldsymbol{\mu}}}\frac{(u-c_\square-\epsilon_3)}{(u-c_\square)}\left|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle,$$

where

$$\begin{aligned}
c_\square &= x_k-(i-1)\epsilon_1-(j-1)\epsilon_2, \quad \text{for } \square=(i,j)\in\vec{\boldsymbol{\lambda}},\\
c_\square &= -\epsilon_3-x_k+(i-1)\epsilon_1+(j-1)\epsilon_2, \quad \text{for } \square=(i,j)\in\vec{\boldsymbol{\mu}}.
\end{aligned}$$

One can also find the action of $\mathfrak{e},\mathfrak{f}$ currents:

$$\mathfrak{e}(u)\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle=\sum_{\square\in\text{addable}(\vec{\boldsymbol{\lambda}})}\frac{E\begin{pmatrix}\vec{\boldsymbol{\lambda}}&\to&\vec{\boldsymbol{\lambda}}+\square\\ \vec{\boldsymbol{\mu}}&\to&\vec{\boldsymbol{\mu}}\end{pmatrix}}{u-c_\square}\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}+\square\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle+\sum_{\square\in\text{removable}(\vec{\boldsymbol{\mu}})}\frac{E\begin{pmatrix}\vec{\boldsymbol{\lambda}}&\to&\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}&\to&\vec{\boldsymbol{\mu}}-\square\end{pmatrix}}{u-c_\square}\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}-\square\end{matrix}\Big\rangle,$$

$$\mathfrak{f}(u)\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle=\sum_{\square\in\text{removable}(\vec{\boldsymbol{\lambda}})}\frac{F\begin{pmatrix}\vec{\boldsymbol{\lambda}}&\to&\vec{\boldsymbol{\lambda}}-\square\\ \vec{\boldsymbol{\mu}}&\to&\vec{\boldsymbol{\mu}}\end{pmatrix}}{u-c_\square}\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}-\square\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle+\sum_{\square\in\text{addable}(\vec{\boldsymbol{\mu}})}\frac{F\begin{pmatrix}\vec{\boldsymbol{\lambda}}&\to&\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}&\to&\vec{\boldsymbol{\mu}}+\square\end{pmatrix}}{u-c_\square}\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}+\square\end{matrix}\Big\rangle.$$

The E,F coefficients are given in (2.31),(2.32). Note that now operators $\mathfrak{e},\mathfrak{f}$ not only add or remove boxes, but do both.

Reflection property of the $\langle\mathcal{K}|$ state. The final ingredient which allows to calculate the matrix elements of Bethe vector the so called of-shell Bethe functions is the formula which describe the action of the operator $\mathfrak{f}$ on the state $\langle\mathcal{K}|$. In section 2.5.2 we derive the following reflection properties (2.39):

$$\langle\mathcal{K}|\mathfrak{h}(u)=\langle\mathcal{K}|\mathfrak{h}(-u)$$
$$\langle\mathcal{K}|\mathfrak{f}(u)=r(u)\langle\mathcal{K}|\mathfrak{f}(-\epsilon_3-u),$$

with

$$r(u-\epsilon_3/2)=-\frac{u+\epsilon_3/2}{u-\epsilon_3/2}\quad\text{for the D case,}$$
$$r(u-\epsilon_3/2)=-\frac{u+\epsilon_i/2}{u-\epsilon_i/2}\quad\text{for the BC case,}$$

where in the last line $i=1$ corresponds to the B case and $i=2$ corresponds to the C case.
This formula allows to compute the coupling between $\langle\mathcal{K}|$ and $\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle$ state (2.5.2),(2.5.2).

Diagonalization of KZ and local IOMs. In section 2.5.3 we derive the Bethe ansatz equation for the diagonalization of KZ Integrals of Motion:

$$\begin{gathered}\mathrm{BAE}(\boldsymbol{x})\overset{\text{def}}{=}r^\alpha(x_i)r^\beta(x_i)A(x_i)A^{-1}(-x_i)\prod_{j\neq i}G(x_i-x_j)G^{-1}(-x_i-x_j)=1,\\ G(x)=\frac{(x-\epsilon_1)(x-\epsilon_2)(x-\epsilon_3)}{(x+\epsilon_1)(x+\epsilon_2)(x+\epsilon_3)},\qquad A(x)=\prod_{k=1}^{n}\frac{x-u_k+\frac{\epsilon_3}{2}}{x-u_k-\frac{\epsilon_3}{2}},\quad r^\alpha(x)=-\frac{x+\epsilon_\alpha/2}{x-\epsilon_\alpha/2}.\end{gathered}\tag{23}$$

We also prove that the on-shell Bethe vector with shifted x parameters $|B(\boldsymbol{x}-\frac{\epsilon_3}{2})\rangle$ are the eigenvectors of KZ IOMs $\mathcal{I}_i^{\mathrm{KZ}}$ (21):

$$\mathcal{I}_i^{\mathrm{KZ}}|B(\boldsymbol{x}-\frac{\epsilon_3}{2})\rangle\overset{\mathrm{BAE}(\boldsymbol{x})=1}{=}\prod_a\frac{(u_i+\frac{\epsilon_3}{2})^2-x_a^2}{(u_i-\frac{\epsilon_3}{2})^2-x_a^2}|B(\boldsymbol{x}-\frac{\epsilon_3}{2})\rangle.\tag{24}$$

Equations (23) and (24) together with the explicit form of off-shell Bethe vector (22) are the main results of the second chapter.

In contrast to the A case we will not provide a proof for the diagonalization of local Integrals of Motion, however we conjectured and checked numerically the formula for eigenvalues of
$\mathbf{I}_3=\frac{1}{2\pi}\int G_4(x)dx$, the local density G_4 is given by (18). Namely, on level N one has an eigenvalue:

$$\mathbf{I}_3^{\mathrm{vac}}+\left(4N-4\sum_{k=1}^{n}\frac{u_k^2}{\epsilon_1\epsilon_2}+\frac{\epsilon_1^2+\epsilon_2^2}{3\epsilon_1\epsilon_2}\Big(2n-\frac{\epsilon_\alpha+\epsilon_\beta}{\epsilon_3}\Big)\right)N+\frac{4}{\epsilon_1\epsilon_2}\left(2n-\frac{\epsilon_\alpha+\epsilon_\beta}{\epsilon_3}\right)\sum_{k=1}^{N}x_k^2,$$

where $\mathbf{I}_3^{\mathrm{vac}}={}_{\boldsymbol{u}}\langle\varnothing|\mathbf{I}_3|\varnothing\rangle_{\boldsymbol{u}}$ - is the vacuum expectation value.

More general integrable systems. One may note that affine Yangian commutation relations (1.40) are symmetric with respect to permutations of all ϵ_α. Nevertheless Bethe Ansatz equations (23) are not symmetric in all ϵ_α, because of the source term $A(x) = \prod_{k=1}^{n} \frac{x-u_k+\frac{\epsilon_3}{2}}{x-u_k-\frac{\epsilon_3}{2}}$. In fact this symmetry is broken by a choice of a particular Fock representation, in order to restore the symmetry back one should introduce three types of Fock modules $\mathcal{F}^\alpha$ (see [FJMM13, BFM18, LS16]). The whole machinery then may be applied to associate an integrable system to the chain of colored Fock spaces with two colored boundaries $\beta_L \Big| \mathcal{F}_1^{\alpha_1} \otimes \mathcal{F}_2^{\alpha_2} \cdots \otimes \mathcal{F}_n^{\alpha_n} \Big| \beta_R \quad , \quad \alpha_i, \beta_{L,R} = 1,2,3$. The corresponding systems of screenings are summarised in picture (B.1). We present the details in Appendix B.1, here we just mention a particular interesting model given by: $1 \Big| \mathcal{F}_1^1 \otimes \mathcal{F}_2^3 \cdots \otimes \mathcal{F}_{2n-1}^1 \otimes \mathcal{F}_{2n}^3 \Big| 3$. This model provides a UV limit for the (dual of) $O(2n+1)$ sigma model considered in [LS18]. Similarly the model $3 \Big| \mathcal{F}_1^3 \otimes \mathcal{F}_2^1 \cdots \otimes \mathcal{F}_{2n+1}^3 \Big| 3$ provides the UV limit of $O(2n)$ sigma model.

q-deformation of local and KZ IOMs. In the last chapter we provide the q-deformation of objects considered in first two chapter. In section 3.2 we review the definition of the q-deformed W algebra as a commutant of the screenings. In section 3.3 we provide a construction of a commutant of affine set of screenings, it turns out that in a q-deformed case the commutant can be found explicitly. We provide explicit formulas for q-deformed Integrals of Motion of arbitrary high spin (3.59) for W algebras of BCD cases, and considered in details an example of affine Lie algebra of type D in section 3.4. We found that all W algebras of BCD case fits into the same scheme, which allows to introduce a new algebra $\mathcal{K}$ which unifies the W algebras of type BCD. The detailed study of algebra $\mathcal{K}$ is reported in paper [FJMV21], while in this thesis we restrict ourselves to a more elementary approach. Finally in section 3.5 we provide a construction for a q-deformed versions of R and K reflection operators, as well as q-deformed KZ IOMs.

Chapter 1

Affine Yangian and Bethe ansatz

In this chapter we study integrable structure of conformal field theory by means of Liouville reflection operator/Maulik Okounkov $R-$matrix. We discuss relation between RLL and current realization of the affine Yangian of $\mathfrak{gl}(1)$. We construct the family of commuting transfer matrices related to the Intermediate Long Wave hierarchy and derive Bethe ansatz equations for their spectra discovered by N.Nekrasov and A.Okounkov and independently by A.Litvinov.

1.1 Introduction

There is a large class of $2D$ QFT's defined by Toda action

$$S_0 = \int \left(\frac{1}{4\pi} (\partial_\mu \boldsymbol{\varphi} \cdot \partial_\mu \boldsymbol{\varphi}) + \Lambda \sum_{r=1}^{n} e^{\left(\boldsymbol{\alpha}_r \cdot \boldsymbol{\varphi}\right)} \right) d^2 x, \tag{1.1}$$

where $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$ is the $n-$component bosonic field and $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n)$ is a set of linearly independent vectors. The theory (1.1), properly coupled to a background metric, defines a conformal field theory. However, it is well known, that under some conditions on the set $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n)$ it also enjoys enlarged conformal symmetry usually referred as W algebra [Zam85]. There is a class of such distinguishable sets $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n)$ with semi-classical behavior

$$\boldsymbol{\alpha}_r = b \boldsymbol{e}_r \quad \text{for all} \quad r = 1, \dots, n,$$

where $\boldsymbol{e}_r$ are finite in the limit $b \to 0$. The vectors $\boldsymbol{e}_r$ have to be simple roots of a semi-simple Lie algebra $\mathfrak{g}$ of rank n.

An interesting question arises if one perturbs the theory (1.1) by an additional exponential field

$$S_0 \to S_0 + \lambda \int e^{\left(\boldsymbol{\alpha}_{n+1} \cdot \boldsymbol{\varphi}\right)} d^2 x. \tag{1.2}$$

Typically this perturbation breaks down all the W algebra symmetry down to Poincaré symmetry. However, there is a special class of perturbations, called the integrable ones, which survive an infinite symmetry of the original theory in a very non-trivial way [Zam89]. Namely, one can argue that there are infinitely many mutually commuting local Integrals of Motion $\mathbf{I}_m^\lambda$ and $\bar{\mathbf{I}}_m^\lambda$ which are perturbative in λ

$$\mathbf{I}_m^\lambda = \mathbf{I}_m + O(\lambda), \qquad \bar{\mathbf{I}}_m^\lambda = \bar{\mathbf{I}}_m + O(\lambda),$$

where $(\mathbf{I}_m, \bar{\mathbf{I}}_m)$ are defined in CFT.

Thus any integrable perturbation (1.2) induces a distinguished set of local IM's $\mathbf{I}_m$ in conformal field theory. The seminal program devoted to calculation of simultaneous spectra of $\mathbf{I}_m$ has been initiated by Bazhanov, Lukyanov and Zamolodchikov in [BLZ96, BLZ97, BLZ99] for $\mathfrak{sl}(2)$/KdV case.

The culmination was the discovery [BLZ04] of Gaudin-like equations for the spectrum. In current notes we use an alternative approach, based on affine Yangian symmetry. We consider the case of $\mathfrak{sl}(n)$ symmetry. Actually, it will be convenient for us to extend the theory by adding an auxiliary non-interacting bosonic field, leading to the action

$$S = \int \left(\frac{1}{4\pi}(\partial_\mu \boldsymbol{\varphi} \cdot \partial_\mu \boldsymbol{\varphi}) + \Lambda \sum_{k=1}^{n-1} e^{b(\varphi_{k+1}-\varphi_k)} + \Lambda e^{b(\varphi_1-\varphi_n)} \right) d^2x, \tag{1.3}$$

where the last term, corresponding to the affine root of $\mathfrak{sl}(n)$, is known to lead to an integrable perturbation. With the last term dropped, the theory (1.3) defines the conformal field theory, whose symmetry algebra can be described by quantum Miura-Gelfand-Dikii transformation [FL88, Luk88]

$$\big(Q\partial - \partial\varphi_n\big)\big(Q\partial - \partial\varphi_{n-1}\big)\dots\big(Q\partial - \partial\varphi_2\big)\big(Q\partial - \partial\varphi_1\big) = (Q\partial)^n + \sum_{k=1}^{n} W^{(k)}(z)(Q\partial)^{n-k}, \tag{1.4}$$

where $Q = b + b^{-1}$. In fact, one can drop any other exponent in (1.3), leading to different, but isomorphic W algebra. For example, dropping the term $e^{b(\varphi_2-\varphi_1)}$, one has different formula

$$\big(Q\partial - \partial\varphi_1\big)\big(Q\partial - \partial\varphi_n\big)\dots\big(Q\partial - \partial\varphi_3\big)\big(Q\partial - \partial\varphi_2\big) = (Q\partial)^n + \sum_{k=1}^{n} \tilde{W}^{(k)}(z)(Q\partial)^{n-k}. \tag{1.5}$$

By symmetry arguments, it is clear that local Integrals of Motion $\mathbf{I}_m$ should belong to the intersection of these two W algebras. In particular, one can check that (for n large enough)

$$\begin{gathered} \mathbf{I}_1 = -\frac{1}{2\pi}\int \left[\sum_{i<j}^{n} \partial\phi_i\partial\phi_j\right] dx, \qquad \mathbf{I}_2 = \frac{1}{2\pi}\int \left[\sum_{i<j<k}^{n} \partial\phi_i\partial\phi_j\partial\phi_k + Q\sum_{i<j} \partial\phi_i\partial^2\phi_j\right] dx, \\ \mathbf{I}_3 = \frac{1}{2\pi}\int \left[\sum_{i<j<k<l}^{n} \partial\phi_i\partial\phi_j\partial\phi_k\partial\phi_l + \dots\right] dx, \qquad \dots\dots \end{gathered} \tag{1.6}$$

where

$$\phi_k \overset{\text{def}}{=} \varphi_k - \frac{1}{n}\sum_{j=1}^{n} \varphi_j.$$

indeed satisfy this requirement. We note that in (1.6) we excluded trivial IM's build out of $U(1)$ field

$$U = \frac{1}{n}\sum_{k=1}^{n} \partial\varphi_k. \tag{1.7}$$

In general, there are local Integrals of Motion for all $m \neq 0 (\mathrm{mod}\, n)$.

This point of view that IM's should belong to intersection of two W algebras given by (1.4) and (1.5) automatically implies that the intertwining operator $\mathcal{I}_1^{\mathrm{KZ}}$

$$\mathcal{I}_1^{\mathrm{KZ}} \tilde{W}^{(k)}(z) = W^{(k)}(z)\mathcal{I}_1^{\mathrm{KZ}}, \tag{1.8}$$

will be itself an Integral of Motion. The operator $\mathcal{I}_1^{\mathrm{KZ}}$ will be primarily important for us. We call it Knizhnik-Zamolodchikov operator (see section 1.4). Actually it is natural to define more operators, which will map between different W algebras corresponding to different permutations of factors in (1.4). The Maulik-Okounkov $R-$matrix [MO19] corresponds to elementary transposition

$$\mathcal{R}_{i,j}\big(Q\partial - \partial\varphi_i\big)\big(Q\partial - \partial\varphi_j\big) = \big(Q\partial - \partial\varphi_j\big)\big(Q\partial - \partial\varphi_i\big)\mathcal{R}_{i,j}, \tag{1.9}$$

while the operator $\mathcal{I}_1^{\mathrm{KZ}}$ introduced in (1.8) corresponds to the long cycle permutation

$$\mathcal{I}_1^{\mathrm{KZ}} = \mathcal{R}_{1,2}\mathcal{R}_{1,3}\dots\mathcal{R}_{1,n-1}\mathcal{R}_{1,n}.$$

Apart from its simplicity, equation (1.9) cannot be solved in a closed form. Actually the $\mathcal{R}$ matrix is closely related to the well known Liouville reflection operator [ZZ96], and has been studied a lot in the past, see discussion in section 1.2. The operator $\mathcal{R}_{i,j}$ acts in the tensor product of two Fock representations of Heisenberg algebra with the highest weight parameters u_i and u_j

$$\mathcal{F}_{u_i}\otimes\mathcal{F}_{u_j}\xrightarrow{\mathcal{R}_{i,j}}\mathcal{F}_{u_i}\otimes\mathcal{F}_{u_j}$$

and its matrix depends on the difference u_i-u_j. Then it follows immediately from the definition (1.9) that $\mathcal{R}_{i,j}(u_i-u_j)$ satisfies the Yang-Baxter equation

$$\mathcal{R}_{1,2}(u_1-u_2)\mathcal{R}_{1,3}(u_1-u_3)\mathcal{R}_{2,3}(u_2-u_3)=\mathcal{R}_{2,3}(u_2-u_3)\mathcal{R}_{1,3}(u_1-u_3)\mathcal{R}_{1,2}(u_1-u_2),$$

and hence the whole machinery of quantum inverse scattering method can be applied. In particular, one can construct a family of commuting transfer-matrices on $n-$sites

$$\mathbf{T}(u)=\mathrm{Tr}'\big(\mathcal{R}_{0,1}(u-u_1)\mathcal{R}_{0,2}(u-u_2)\dots\mathcal{R}_{0,n-1}(u-u_{n-1})\mathcal{R}_{0,n}(u-u_n)\big)\Big|_{\mathcal{F}_u}. \tag{1.10}$$

At $u=u_1$ one has $\mathcal{R}_{0,1}=\mathcal{P}_{0,1}$ a permutation operator and hence

$$\mathbf{T}(u_1)=\mathcal{R}_{1,2}\mathcal{R}_{1,3}\dots\mathcal{R}_{1,n-1}\mathcal{R}_{1,n}=\mathcal{I}_1^{\mathrm{KZ}}, \tag{1.11}$$

which implies that $\mathbf{T}(u)$ commutes with local Integrals of Motion $\mathbf{I}_m$ and can be taken as a generating function.

In (1.10) the notation Tr' corresponds to certain regularization of the trace, which goes through the introduction of the twist parameter q

$$\mathrm{Tr}'(\dots)\overset{\mathrm{def}}{=}\lim_{q\to1}\frac{1}{\chi(q)}\,\mathrm{Tr}\left(q^{L_0^{(0)}}\dots\right),\quad\text{where}\quad \chi(q)=\prod_{k=1}^{\infty}\frac{1}{1-q^k}$$

and $L_0^{(0)}=\sum_{k>0}a_{-k}^{(0)}a_k^{(0)}$ is the level operator in auxiliary space $\mathcal{F}_u$. Remarkably, the introduction of the twist parameter does not spoil the integrability, indeed commutativity of the twist deformed transfer-matrices

$$\mathbf{T}_q(u)=\mathrm{Tr}\big(q^{L_0^{(0)}}\mathcal{R}_{0,1}(u-u_1)\mathcal{R}_{0,2}(u-u_2)\dots\mathcal{R}_{0,n-1}(u-u_{n-1})\mathcal{R}_{0,n}(u-u_n)\big)\Big|_{\mathcal{F}_u}, \tag{1.12}$$

follows from the Yang-Baxter equation (1.17) and the fact that the $\mathcal{R}$ matrix commutes with the tensor square of the twist: $[\mathcal{R}_{1,2},q^{L_0^{(1)}}\otimes q^{L_0^{(2)}}]=0$.

On the level of local Integrals of Motion (1.6) the twist deformation corresponds to the non-local deformation $\mathbf{I}_m\to\mathbf{I}_m(q)$ called quantum ILW$_n$ (Intermediate Long Wave) integrable system [Lit13]. In particular

$$\begin{aligned}
&\mathbf{I}_1(q)=\frac{1}{2\pi}\int\left[\frac{1}{2}\sum_{k=1}^{n}(\partial\varphi_k)^2\right]dx,\\
&\mathbf{I}_2(q)=\frac{1}{2\pi}\int\left[\frac{1}{3}\sum_{k=1}^{n}(\partial\varphi_k)^3+Q\left(\frac{i}{2}\sum_{i,j}\partial\varphi_iD\partial\varphi_j+\sum_{i<j}\partial\varphi_i\partial^2\varphi_j\right)\right]dx,\\
&\mathbf{I}_3(q)=\frac{1}{2\pi}\int\left[\frac{1}{4}\sum_{k=1}^{n}(\partial\varphi_k)^4+\dots\right]dx,\\
&\dots\dots\dots\dots\dots\dots
\end{aligned} \tag{1.13}$$

where D is the non-local operator whose Fourier image is

$$D(k) = k\frac{1+q^k}{1-q^k}.$$

We note that the limit $q \to 1$ is a little subtle since the operator D is singular at $q \to 1$ and hence some eigenvalues of $\mathbf{I}_m(q)$ become infinite. However, one can show that on a subspace spanned by eigenfunctions with *finite* eigenvalues the modes of the $U(x)$ field (1.7) are not excited. It implies in particular that

$$\mathbf{I}_2 = \mathbf{I}_2(q)\Big|_{U\to 0}.$$

The spectrum of ILW_n integrable system is governed by finite type Bethe ansatz equations which have been conjectured by Nekrasov and Okounkov[1] and later independently by one of the authors in [Lit13]

$$q\prod_{j\neq i}\frac{(x_i-x_j-b)(x_i-x_j-b^{-1})(x_i-x_j+Q)}{(x_i-x_j+b)(x_i-x_j+b^{-1})(x_i-x_j-Q)}\prod_{k=1}^{n}\frac{x_i-u_k-\frac{Q}{2}}{x_i-u_k+\frac{Q}{2}} = 1 \quad \text{for all} \quad i=1,\dots,N, \tag{1.14}$$

such that the eigenvalues of $\mathbf{I}_m(q)$ are symmetric polynomials in Bethe roots

$$\mathbf{I}_1(q) \sim -\frac{1}{2}\sum_{k=1}^{n}u_k^2 + N, \qquad \mathbf{I}_2(q) \sim \frac{1}{3}\sum_{k=1}^{n}u_k^3 - 2i\sum_{j=1}^{N}x_j, \qquad \dots \tag{1.15}$$

Equations (1.14)-(1.15) have been checked in [Lit13] by explicit calculations on lower levels. Later these equations have been proven in the trigonometric ($\mathfrak{q}$-deformed) case independently by two groups: using the algebraic methods of shuffle algebras by Feigin, Jimbo, Miwa and Mukhin [FJMM15] and using the geometric method of quasi-maps by Aganagic and Okounkov [AO17]. Rational Bethe anzatz equations (1.14) follow from the trigonometric ones in the limit $\mathfrak{q},\mathfrak{t}\to 1$, $\frac{\log(\mathfrak{q})}{\log(\mathfrak{t})} = b^2$ and thus (1.14) are currently proven by two independent methods. In this notes we give a more direct proof which is qualitatively different from [FJMM15] and [AO17].

We note that Bethe ansatz equations (1.15) are simplified drastically for $q^{\pm 1} \to 0$, which is equivalent to $D(k) \to \pm|k|$. The limit of ILW_n system at $q^{\pm 1} \to 0$ is known as BO_n integrable system (Benjamin-Ono). The basis of its eigenfunctions stands behind AGT correspondence [AGT10]. Namely, it has been shown in [AFLT11,FL12] that the matrix elements of semi-degenerate W_n-primary fields, dressed by suitably chosen $U(1)$ vertex operators, sandwiched between the BO_n eigenfunctions coincide with bi-fundamental contribution to the Nekrasov partition function [Nek04] for corresponding quiver gauge theory.

In the opposite limit $q \to 1$ the Bethe equations (1.14) describe the spectrum of local Integrals of Motion. In particular, the spectrum of $\mathbf{I}_2$ is given by (1.15). We note that the same system can be studied by Bazhanov-Lukyanov-Zamolodchikov approach which leads to different Gaudin-like algebraic equations for the spectrum. We have a remarkable phenomenon: there are two different systems of algebraic equations, namely BAE (1.14) with $q = 1$ and Gaudin-like equations found in [BLZ04] which describe the same integrable system[2]. See [Lit13] and discussions in section 1.5.

The Maulik-Okounkov $R-$matrix defines in a standard way the Yang-Baxter algebra (RLL algebra). We note that $R_{i,j}$ intertwines two representations of Heisenberg algebra (1.9) ($\widehat{\mathfrak{gl}}(1)$ current algebra). Since the matrix elements of $R_{i,j}$ are rational functions of the highest weight/spectral parameter (see below), it is natural to call the corresponding Yang-Baxter algebra the Yangian of

[1]See Okounkov's talk at Facets of Integrability conference, SCGP January 2013.

[2]Technically, these Gaudin-like equations has been found in [BLZ04] only for qKdV case, i.e. for $n = 2$. However, a generalization for $n > 2$ should not be a problem.

$\widehat{\mathfrak{gl}}(1)$, or affine Yangian of $\mathfrak{gl}(1)$. The algebra under the same name has been introduced by Tsymbaliuk in [Tsy17, Tsy14]. It has been given by explicit commutation relations (the so called current realization). However both algebras do not coincide, but rather we conjecture that Tsymbaliuk's algebra $\mathrm{Y}(\widehat{\mathfrak{gl}}(1))$ is obtained from the Yang-Baxter algebra $\mathrm{YB}(\widehat{\mathfrak{gl}}(1))$ by factorization over infinite-dimensional center. In section 1.3 we will show that the central elements of $\mathrm{YB}(\widehat{\mathfrak{gl}}(1))$ correspond to singular vectors of W_n algebra in the tensor product of n Fock spaces.

As we already mentioned, the Yangian $\mathrm{Y}(\widehat{\mathfrak{gl}}(1))$ is the rational counterpart of the trigonometric algebra called Ding-Iohara-Miki algebra or quantum toroidal $\mathfrak{gl}(1)$ algebra [Tsy17]. This algebra has been extensively studied by Feigin and collaborators in [AFH+11,FJMM,FJMM15,FKSW07]. Another but equivalent approach through the methods of geometric representation theory was developed by Okounkov and collaborators [OS16, AO17]. We borrow many ideas developed in [AFH+11, FJMM, FJMM15, FKSW07] and [OS16, AO17] for our study. In particular, Bethe ansatz equations as well as Bethe vectors can be found in the $\mathfrak{q}$-deformed case in [FJMM15] and in [AO17].

This chapter is organized as follows. In section 1.2 we define the main actor of our study – Liouville reflection operator/Maulik-Okounkov $R-$matrix and discuss its general properties and various representations. In section 1.3 we study corresponding RLL algebra and discuss its relation to affine Yangian of $\mathfrak{gl}(1)$. In section 1.4 we introduce quantum Integrals of Motion corresponding to ILW system and prove Bethe ansatz equations for the spectrum. In section 1.5 we give some conclusions an emphasize future possible directions of study. In appendices we present some explicit formulae and calculations, used in the main text.

1.2 Maulik-Okounkov R-matrix as Liouville reflection operator

Let us recall the definition of Maulik-Okounkov $\mathcal{R}$-matrix [MO19]

$$\mathcal{R}_{i,j}\big(Q\partial-\partial\varphi_i\big)\big(Q\partial-\partial\varphi_j\big)=\big(Q\partial-\partial\varphi_j\big)\big(Q\partial-\partial\varphi_i\big)\mathcal{R}_{i,j}. \tag{1.16}$$

Here φ_i are free bosonic fields

$$\partial\varphi_i(x)=u_i+\sum_{n\neq 0}a_n^i e^{inx},\quad [a_n^i,a_m^j]=m\delta_{i,j}\delta_{m,-n}.$$

Let us permute the product of three Miura terms, it can be done in a two ways:

$$\begin{aligned}\mathcal{R}_{1,2}(u_1-u_2)\mathcal{R}_{1,3}(u_1-u_3)\mathcal{R}_{2,3}(u_2-u_3)\big(Q\partial-\partial\varphi_1\big)\big(Q\partial-\partial\varphi_2\big)\big(Q\partial-\partial\varphi_3\big)=\\ =\big(Q\partial-\partial\varphi_3\big)\big(Q\partial-\partial\varphi_2\big)\big(Q\partial-\partial\varphi_1\big)\mathcal{R}_{1,2}(u_1-u_2)\mathcal{R}_{1,3}(u_1-u_3)\mathcal{R}_{2,3}(u_2-u_3),\end{aligned}$$

$$\begin{aligned}\mathcal{R}_{2,3}(u_2-u_3)\mathcal{R}_{1,3}(u_1-u_3)\mathcal{R}_{1,2}(u_1-u_2)\big(Q\partial-\partial\varphi_1\big)\big(Q\partial-\partial\varphi_2\big)\big(Q\partial-\partial\varphi_3\big)=\\ =\big(Q\partial-\partial\varphi_3\big)\big(Q\partial-\partial\varphi_2\big)\big(Q\partial-\partial\varphi_1\big)\mathcal{R}_{2,3}(u_2-u_3)\mathcal{R}_{1,3}(u_1-u_3)\mathcal{R}_{1,2}(u_1-u_2).\end{aligned}$$

If we take into account that Miura map provides an irreducible representation of W-3 algebra in the tensor product of three Fock spaces, then it follows immediately that $\mathcal{R}_{i,j}(u_i-u_j)$ satisfies the Yang-Baxter equation

$$\mathcal{R}_{1,2}(u_1-u_2)\mathcal{R}_{1,3}(u_1-u_3)\mathcal{R}_{2,3}(u_2-u_3)=\mathcal{R}_{2,3}(u_2-u_3)\mathcal{R}_{1,3}(u_1-u_3)\mathcal{R}_{1,2}(u_1-u_2), \tag{1.17}$$

From the definition of $\mathcal{R}$-matrix (1.16), it follows that $\mathcal{R}_{i,j}$ trivially commutes with the center of mass field $\varphi_i+\varphi_j$. So, it is natural to introduce the operator acting on the space of a single bosonic field J

$$\mathcal{R}=\mathcal{R}_{i,j}|_{\partial\varphi_i+\partial\varphi_j=0,\ \partial\varphi_i-\partial\varphi_j=2J},\qquad \mathcal{R}_{i,j}=\mathcal{R}\Big|_{J\to\frac{\partial\varphi_i-\partial\varphi_j}{2}},$$

The operator $\mathcal{R}$ is known as the Liouville reflection operator and it is closely related to the Liouville S-matrix introduced in [ZZ96] (see also paragraph 1.2 at the end of this section)

$$J(x) = u + \sum_{n\neq 0} a_n e^{-inx}, \quad 2u = u_i - u_j \tag{1.18}$$

$$[a_m, a_n] = \frac{m}{2}\delta_{m,-n}.$$

which is defined as

$$\mathcal{R}(-J^2 + Q\partial J) = (-J^2 - Q\partial J)\mathcal{R}. \tag{1.19}$$

The relation (1.19) can be used for calculation of $\mathcal{R}$. Consider highest weight representation of the Heisenberg algebra (1.18). It is generated by the negative mode operators a_{-k} from the vacuum state $|u\rangle$

$$a_n|u\rangle = 0 \quad \text{for} \quad n > 0.$$

Then (1.19) is equivalent to the infinite set of relations

$$\mathcal{R}L^{(+)}_{-\lambda_1}\dots L^{(+)}_{-\lambda_n}|u\rangle = \mathcal{R}^{\text{vac}}(u)L^{(-)}_{-\lambda_1}\dots L^{(-)}_{-\lambda_n}|u\rangle, \tag{1.20}$$

where $L^{(\pm)}_n$ are the components of $T^{(\pm)} = -J^2 \pm Q\partial J$

$$L^{(\pm)}_n = \sum_{k\neq 0,n} a_k a_{n-k} + (2a_0 \pm inQ)a_n, \quad L^{(+)}_0 = L^{(-)}_0 = \frac{Q^2}{4} + a_0^2 + 2\sum_{k>0} a_{-k}a_k.$$

and $\mathcal{R}^{\text{vac}}(u)$ is an eigenvalue for the vacuum state. In the following we will usually take

$$\mathcal{R}^{\text{vac}}(u) = 1. \tag{1.21}$$

Using (1.20) as a set of equations one can compute the matrix of $\mathcal{R}$. For example at the level 1 one has

$$\mathcal{R}L^{(+)}_{-1}|u\rangle = L^{(-)}_{-1}|u\rangle \implies \mathcal{R}a_{-1}|u\rangle = \frac{2u+iQ}{2u-iQ}a_{-1}|u\rangle.$$

Similarly, at the level 2 one obtains

$$\mathcal{R}a_{-2}|u\rangle = \frac{\big(\big(8u^3 + 2u(3Q^2-1) - iQ(2Q^2+1)\big)a_{-2} - 8iQua_{-1}^2\big)\,|u\rangle}{(2u-iQ)(2u-iQ-ib)(2u-iQ-ib^{-1})},$$

$$\mathcal{R}a_{-1}^2|u\rangle = \frac{\big(-4iQua_{-2} + \big(8u^3 + 2u(3Q^2-1) + iQ(2Q^2+1)\big)a_{-1}^2\big)\,|u\rangle}{(2u-iQ)(2u-iQ-ib)(2u-iQ-ib^{-1})}.$$

Apart from explicit expressions on lower levels the reflection operator is not known in a closed form. However it shares several properties allowing to judge about its structure:

Poles. It is clear that apart from the normalization factor the operator $\mathcal{R}$ is a meromorphic functions of the momentum u. In fact, it can be argued that it has only simple poles located at the Kac points

$$u = u_{m,n} = i\left(\frac{mb}{2} + \frac{n}{2b}\right), \quad m,n > 0, \tag{1.22}$$

i.e. $\mathcal{R}(u)$ can be written in the form

$$\mathcal{R}(u) = 1 + \sum_{m,n>0} \frac{R_{m,n}}{u - u_{m,n}}. \tag{1.23}$$

Indeed it is well known that for the values (1.22) the map from the Fock module $\mathcal{F}_u$ to Verma module $\mathcal{V}_\Delta$ given by the L^+ generators has a kernel. More precisely all the states of the form

$$L^{(+)}_{-\lambda_1}\dots L^{(+)}_{-\lambda_n}\Big((L^+_{-1})^{mn}+\dots\Big)|u\rangle,$$

where $|\chi_{m,n}\rangle \overset{\text{def}}{=} \Big((L^+_{-1})^{mn}+\dots\Big)|u\rangle$ is a special state called co-singular vector, vanish at $u=u_{m,n}$. Explicitly, one has

$$|\chi_{1,1}\rangle = L^+_{-1}|u\rangle,\quad |\chi_{2,1}\rangle = \Big((L^+_{-1})^2-b^2L^+_{-2}\Big)|u\rangle,\quad |\chi_{1,2}\rangle = \Big((L^+_{-1})^2-b^{-2}L^+_{-2}\Big)|u\rangle \quad \text{etc}$$

At the same time the reflected states

$$L^{(-)}_{-\lambda_1}\dots L^{(-)}_{-\lambda_n}\Big((L^-_{-1})^{mn}+\dots\Big)|u\rangle, \tag{1.24}$$

do not vanish for $u=u_{m,n}$ which implies that $\mathcal{R}$ should exhibit a singularity at (1.24), namely a simple pole, which implies (1.23).

We note that the formula (1.23) is a reminiscent of the Alyosha Zamolodchikov's recurrence formula for conformal block [Zam84]. In particular, one can use (1.23) as a tool for calculation of the matrix of $\mathcal{R}(u)$.

Relation to Liouville $S-$matrix The Liouville reflection operator $\mathcal{R}$ is closely related to the Liouville $S-$matrix introduced in Zamolodchikov's paper [ZZ96]. Namely, they differ by the sign change operator $\boldsymbol{\pi}J(x)=-J(x)\boldsymbol{\pi}$ as

$$\mathcal{R}(u)=\boldsymbol{\pi}\hat{S}(u).$$

According to (1.20) the S-matrix $\hat{S}(u)$ acts between different Fock modules $\mathcal{F}_u \xrightarrow{\hat{S}(u)} \mathcal{F}_{-u}$ as follows

$$\hat{S}(u)L^{(+)}_{-\lambda_1}\dots L^{(+)}_{-\lambda_n}|u\rangle = L^{(+)}_{-\lambda_1}\dots L^{(+)}_{-\lambda_n}|-u\rangle. \tag{1.25}$$

Expression through Screening operators. Given the stress energy tensor $T^+=-J^2+Q\partial J$ with $J=\partial\varphi$, one finds that the exponential fields $e^{2b^{\pm1}\varphi(z)}$ satisfy

$$\oint_{\mathcal{C}_\xi} e^{2b^{\pm1}\varphi(z)}T^+(\xi)dz=0. \tag{1.26}$$

Then suppose that $u=-u_{m,n}$ for $m,n\geq 0$. In this case one can define a *closed* contour $\mathcal{C}$ such that the normalized operator $\mathcal{F}_{-u_{m.n}} \xrightarrow{\mathcal{Q}_{m,n}} \mathcal{F}_{u_{m.n}}$

$$\mathcal{Q}_{m,n}=\Omega_{m,n}\oint_{\mathcal{C}} e^{2b\varphi(z_1)}\dots e^{2b\varphi(z_m)}e^{2b^{-1}\varphi(z_{m+1})}\dots e^{2b^{-1}\varphi(z_{m+n})}dz_1\dots dz_{m+n}: \qquad \mathcal{Q}_{m,n}|{-u_{m,n}}\rangle = |u_{m,n}\rangle$$

called the screening operator, is well defined. Then the formula (1.26) implies that

$$\mathcal{Q}_{m,n}L^{(+)}_{-\lambda_1}\dots L^{(+)}_{-\lambda_n}|-u_{m,n}\rangle = L^{(+)}_{-\lambda_1}\dots L^{(+)}_{-\lambda_n}|u_{m,n}\rangle. \tag{1.27}$$

Comparing (1.25) and (1.27) one finds

$$\hat{S}(-u_{m,n})=\mathcal{Q}_{m,n} \implies \mathcal{R}(-u_{m,n})=\boldsymbol{\pi}\mathcal{Q}_{m,n}.$$

Large momenta expansion. The details of large u expansion are collected in Appendix A.1.

We note that $\mathcal{R}_{1,2}$ coincides with the KZ operator (1.11) for $n=2$ and hence $\mathcal{R}$ commutes with the system of local Integrals of Motion of quantum KdV (mKdV) system

$$\mathbf{I}_{2n-1}=\frac{1}{2\pi}\int\Big(J^{2n}+\text{higher derivatives}\Big)dx. \tag{1.28}$$

It can be shown that $\mathcal{R}$ is an exponent of semi-local (non-polynomial) Integral of Motion, see (A.2)-(A.5) for details.[3]

$$\mathcal{R}=\exp\left[\frac{iQ}{2\pi}\int\Big(2J\log J+\frac{1-2Q^2}{24}\frac{J_x^2}{J^3}+\text{higher derivatives}\Big)dx\right]. \tag{1.29}$$

The formula (1.29) is rather symbolic and requires a regularization prescription to make sense. Such a regularization can be defined as a large u expansion. Namely, if one splits J into constant and zero-mean parts $J=u+\tilde{J}$, then the expansion coefficients

$$J\log J=u\log u+\tilde{J}(\log u+1)+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)u^k}\,\tilde{J}^{k+1},\qquad \frac{J_x^2}{J^3}=\frac{\tilde{J}_x^2}{2}\sum_{k>1}\frac{(-1)^k(k-2)(k-3)}{2u^{k-1}}\,\tilde{J}^{k-4},\qquad\ldots$$

are zeta-valued regularized (similar regularization is used in the definition of local IM's (1.28)). So that (1.29) leads to large u expansion

$$\mathcal{R}(u)=\exp\left[\frac{iQ}{2\pi}\int\left(2u\log u+\frac{\tilde{J}^2}{u}-\frac{\tilde{J}^3}{3u^2}+O\Big(u^{-3}\Big)\right)dx\right]. \tag{1.30}$$

We note that in (1.29) and (1.30) the normalization is different from the one used before, i.e. $\mathcal{R}_{\text{vac}}(u)\neq 1$.

Free-fermion point. One can show that $\mathcal{R}$ admits simple representation at the free-fermion point $c=-2$. Namely, if one uses boson-fermion correspondence to represent

$$J(x)=u+\frac{1}{\sqrt{2}}:\psi^+(x)\psi(x):,$$

where $(\psi(x),\psi^+(x))$ is the chiral part of Dirac fermion, then up to normalization factor one has an explicit formula (see appendix A.4)

$$\mathcal{R}(u)\Big|_{c=-2}\sim\exp\left(\frac{1}{2\pi}\int_0^{2\pi}:\psi^+(x)\log\left(1+\frac{i}{u\sqrt{2}}\partial\right)\psi(x):\,dx\right). \tag{1.31}$$

For $c\neq-2$ formula is more complicated and (1.31) will include multiple fermion terms.

Smirnov's fermion formula. There is also Smirnov's formula for Maulik-Okounkov $R-$matrix involving an infinite product of fermionic operators [Smi16]. Unfortunately, we do not known any practical use of it for our purposes.

[3]The existence of such semi-local Integrals of Motion for KdV (mKdV) equation has been noticed by Boris Dubrovin [Dub06].

1.3 Yang-Baxter algebra

The Maulik-Okounkov $R-$matrix defines the Yang-Baxter algebra in a standard way

$$\mathcal{R}_{ij}(u-v)\mathcal{L}_i(u)\mathcal{L}_j(v) = \mathcal{L}_j(v)\mathcal{L}_i(u)\mathcal{R}_{ij}(u-v). \tag{1.32}$$

Here $\mathcal{L}_i(u)$ is treated as an operator in some quantum space, a tensor product of n Fock spaces in our case, and as a matrix in auxiliary Fock space $\mathcal{F}_u$. The algebra (1.32) becomes an infinite set of quadratic relations between the matrix elements labeled by two partitions

$$\mathcal{L}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(u) \overset{\text{def}}{=} \langle u|a_{\boldsymbol{\lambda}}\mathcal{L}(u)a_{-\boldsymbol{\mu}}|u\rangle \quad \text{where} \quad a_{-\boldsymbol{\mu}}|u\rangle = a_{-\mu_1}a_{-\mu_2}\dots|u\rangle.$$

It is well known that commutation relations of RLL algebras could be rewritten in an equivalent current form, see [DF93] where such an analysis was performed for $U_q(gl(n))$. Here we provide similar analysis for the case of Maulik-Okounkov $R-$matrix. A clear candidate for the current realization is the Affine Yangian algebra introduced in [Tsy17, Tsy14] from quite different perspectives. Our goal is to derive current realisation out of RLL algebra. As we will see, the two algebras are not literally coincide. We rather conjecture that (1.32) is related to the Yangian of $\widehat{\mathfrak{gl}}(1)$ by factorization over its center. This is similar to the well known fact that the Yangians of $\mathfrak{gl}(n)$ and of $\mathfrak{sl}(n)$ are differ by central elements [KS82]. We note that, compared to the non-affine case, the center of (1.32) is infinite dimensional. We will denote the Yang-Baxter algebra as $\mathrm{YB}\big(\widehat{\mathfrak{gl}}(1)\big)$, reserving the notation $\mathrm{Y}\big(\widehat{\mathfrak{gl}}(1)\big)$ for Tsymbaliuk's algebra.

In discussions below we will mainly follow the analysis performed in [DF93]. We introduce three basic currents of degree 0, 1 and -1 (see appendix A.2 for more details)

$$h(u) \overset{\text{def}}{=} \mathcal{L}_{\varnothing,\varnothing}(u), \qquad e(u) \overset{\text{def}}{=} h^{-1}(u)\cdot\mathcal{L}_{\varnothing,\square}(u) \quad \text{and} \quad f(u) \overset{\text{def}}{=} \mathcal{L}_{\square,\varnothing}(u)\cdot h^{-1}(u), \tag{1.33}$$

as well as an auxiliary current (as we will see (1.40a) it also belongs to the Cartan subalgebra of $\mathrm{YB}\big(\widehat{\mathfrak{gl}}(1)\big)$)

$$\psi(u) \overset{\text{def}}{=} \Big(\mathcal{L}_{\square,\square}(u-Q) - \mathcal{L}_{\varnothing,\square}(u-Q)h^{-1}(u-Q)\mathcal{L}_{\square,\varnothing}(u-Q)\Big)h^{-1}(u-Q) \tag{1.34}$$

As follows from definition of the $R-$matrix these currents admit large u expansion

$$h(u) = 1+\frac{h_0}{u}+\frac{h_1}{u^2}+\dots, \quad e(u) = \frac{e_0}{u}+\frac{e_1}{u^2}+\dots, \quad f(u) = \frac{f_0}{u}+\frac{f_1}{u^2}+\dots, \quad \psi(u) = 1+\frac{\psi_0}{u}+\frac{\psi_1}{u^2}+\dots \tag{1.35}$$

As we will see below, it proves convenient to introduce higher currents labeled by $3D$ partitions. In particular, on level 2 one has three $e_{\boldsymbol{\lambda}}(u)$ currents

$$e_{\substack{\square\\\square}}(u) = \frac{ibQ}{(b^2-1)(b^2+2)}h^{-1}(u)\left(\mathcal{L}_{\varnothing,\square\!\square}(u) - ib\mathcal{L}_{\varnothing,\square\,\square}(u)\right),$$

$$e_{\square\square}(u) = \frac{ib^{-1}Q}{(b^{-2}-1)(b^{-2}+2)}h^{-1}(u)\left(\mathcal{L}_{\varnothing,\square\!\square}(u) - ib^{-1}\mathcal{L}_{\varnothing,\square\,\square}(u)\right), \quad e_{\square\!\!\square}(u) = Q\left[be_{\substack{\square\\\square}}(u)+b^{-1}e_{\square\square}(u)-e^2(u)\right]. \tag{1.36}$$

and similarly

$$f_{\substack{\square\\\square}}(u) = \frac{ibQ}{(b^2-1)(b^2+2)}h^{-1}(u)\left(\mathcal{L}_{\square\!\square,\varnothing}(u) - ib\mathcal{L}_{\square\,\square,\varnothing}(u)\right),$$

$$f_{\square\square}(u) = \frac{ib^{-1}Q}{(b^{-2}-1)(b^{-2}+2)}h^{-1}(u)\left(\mathcal{L}_{\square\!\square,\varnothing}(u) - ib^{-1}\mathcal{L}_{\square\,\square,\varnothing}(u)\right), \quad f_{\square\!\!\square}(u) = Q\left[bf_{\substack{\square\\\square}}(u)+b^{-1}f_{\square\square}(u)-f^2(u)\right]. \tag{1.37}$$

As we will see below these currents are algebraically depending on the basic currents (1.33).

It will be more convenient to use Nekrasov epsilon notations rather than Liouville notations. Formally, they are obtained by replacing central charge parameter

$$b \to \frac{\epsilon_2}{\sqrt{\epsilon_1\epsilon_2}}, \quad b^{-1} \to \frac{\epsilon_1}{\sqrt{\epsilon_1\epsilon_2}}, \quad Q \to -\frac{\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}} \implies \epsilon_1+\epsilon_2+\epsilon_3=0, \tag{1.38}$$

together with the normalization of the bosonic fields:

$$\varphi_j \to \phi(x) = -i\frac{\varphi_j}{\sqrt{\epsilon_1\epsilon_2}}. \tag{1.39}$$

Altogether, this leads to the following Miura transformation

$$\mathcal{W}^{(2)}(z) = (i\epsilon_3\partial - \partial\phi_1)(i\epsilon_3\partial-\partial\phi_2)$$

We also have to scale our basic current $e(u)$ and $f(u)$ as

$$e(u)\to\sqrt{\epsilon_3}e(u), \qquad f(u)\to\sqrt{\epsilon_3}f(u).$$

1.3.1 Current realisation of the Yang-Baxter algebra $\mathrm{YB}\big(\widehat{\mathfrak{gl}}(1)\big)$

Using the definition (1.33) and (1.34) and explicit expression for the $R-$matrix on first three levels one finds (see appendix A.2 for details)

$$[h(u),\psi(v)]=0, \quad [\psi(u),\psi(v)]=0, \quad [h(u),h(v)]=0, \tag{1.40a}$$

$$(u-v-\epsilon_3)h(u)e(v) = (u-v)e(v)h(u) - \epsilon_3 h(u)e(u), \tag{1.40b}$$

$$(u-v-\epsilon_3)f(v)h(u) = (u-v)h(u)f(v) - \epsilon_3 f(u)h(u), \tag{1.40c}$$

$$[e(u),f(v)] = \frac{\psi(u)-\psi(v)}{u-v}, \tag{1.40d}$$

as well as ee, ff relations

$$g(u-v)\Big[e(u)e(v) - \frac{e_{\square\square}(v)}{u-v+\epsilon_1} - \frac{e_{\substack{\square\\\square}}(v)}{u-v+\epsilon_2} - \frac{e_{\square\!\!\square}(v)}{u-v+\epsilon_3}\Big] =$$

$$= \bar g(u-v)\Big[e(v)e(u) - \frac{e_{\square\square}(u)}{u-v-\epsilon_1} - \frac{e_{\substack{\square\\\square}}(u)}{u-v-\epsilon_2} - \frac{e_{\square\!\!\square}(u)}{u-v-\epsilon_3}\Big], \tag{1.40e}$$

$$\bar g(u-v)\Big[f(u)f(v) - \frac{f_{\square\square}(v)}{u-v-\epsilon_1} - \frac{f_{\substack{\square\\\square}}(v)}{u-v-\epsilon_2} - \frac{f_{\square\!\!\square}(v)}{u-v-\epsilon_3}\Big] =$$

$$= g(u-v)\Big[f(v)f(u) - \frac{f_{\square\square}(u)}{u-v+\epsilon_1} - \frac{f_{\substack{\square\\\square}}(u)}{u-v+\epsilon_2} - \frac{f_{\square\!\!\square}(u)}{u-v+\epsilon_3}\Big], \tag{1.40f}$$

ψe, ψf relations

$$\begin{aligned} g(u-v)\psi(u)e(v) &= \bar g(u-v)e(v)\psi(u) + \text{locals}, \\ g(u-v)f(v)\psi(u) &= \bar g(u-v)\psi(u)f(v) + \text{locals}, \end{aligned} \tag{1.40g}$$

and Serre relations

$$\begin{aligned} &\sum_{\sigma\in\mathbb{S}_3}(u_{\sigma_1}-2u_{\sigma_2}+u_{\sigma_3})e(u_{\sigma_1})e(u_{\sigma_2})e(u_{\sigma_3}) + \sum_{\sigma\in\mathbb{S}_3}\big[e(u_{\sigma_1}), e_{\square\!\!\square}(u_{\sigma_2}) + e_{\square\square}(u_{\sigma_2}) + e_{\substack{\square\\\square}}(u_{\sigma_2})\big] = 0, \\ &\sum_{\sigma\in\mathbb{S}_3}(u_{\sigma_1}-2u_{\sigma_2}+u_{\sigma_3})f(u_{\sigma_1})f(u_{\sigma_2})f(u_{\sigma_3}) + \sum_{\sigma\in\mathbb{S}_3}\big[f(u_{\sigma_1}), f_{\square\!\!\square}(u_{\sigma_2}) + f_{\square\square}(u_{\sigma_2}) + f_{\substack{\square\\\square}}(u_{\sigma_2})\big] = 0. \end{aligned} \tag{1.40h}$$

In the relations above we have used the following notations

$$g(x) \overset{\text{def}}{=} (x+\epsilon_1)(x+\epsilon_2)(x+\epsilon_3), \quad \bar{g}(x) \overset{\text{def}}{=} (x-\epsilon_1)(x-\epsilon_2)(x-\epsilon_3).$$

The higher currents $e_{\boldsymbol{\lambda}}$ and $f_{\boldsymbol{\lambda}}$ in (1.40e)-(1.40h) are related to (1.36) and (1.37) by change of notations (1.38)-(1.39) and by certain scaling factors.

We note that the terms shown by blue in (1.40b)-(1.40g) depend only on one parameter either u or v (in (1.40g) these terms are so complicated, that we do no write them explicitly) and in (1.40h) they depend only on two parameters instead of three. We call such terms *local* and use shorthand notation *locals* in formulas instead of writing them explicitly. If one writes the commutation relations for the modes of the currents (1.35), *local* terms affect only few of them. Indeed if we apply

$$\frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_\infty}\oint_{\mathcal{C}_\infty} u^i v^j \, du \, dv \tag{1.41}$$

to (1.40b) for $i, j \geq 0$ the *local* term $h(u)e(u)$ does not contribute and we obtain

$$[h_{i+1}, e_j] - [h_i, e_{j+1}] = \epsilon_3 h_i e_j.$$

The local term appears if we apply (1.41) with $j = -1$

$$[h(u), e_0] = \epsilon_3 h(u)e(u) \overset{(1.33)}{\Longrightarrow} \mathcal{L}_{\varnothing,\square}(u) = [e_0, h(u)]. \tag{1.42}$$

Similarly, applying

$$\frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_\infty}\oint_{\mathcal{C}_\infty} u^i v^j \, du \, dv\cdot$$

for $i, j \geq 0$ to (1.40e), local terms represented by $e_{\boldsymbol{\lambda}}(u)$ do not contribute and we obtain

$$[e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] + \sigma_2\left([e_{i+1}, e_j] - [e_i, e_{j+1}]\right) = \sigma_3\{e_i, e_j\},$$

where σ_k are elementary symmetric polynomials in ϵ_j. However, taking either i or j negative allow to express the higher currents $e_{\boldsymbol{\lambda}}$ in terms of commutators

$$\begin{aligned}
e_{\square\square}(u) &= \frac{1}{(\epsilon_1-\epsilon_2)(\epsilon_1-\epsilon_3)}\Big((u-\epsilon_2)(u-\epsilon_3)[e(u), e_0] - (2u+\epsilon_1)[e(u), e_1] + [e(u), e_2] - 3[e_1, e_0]\Big),\\
e_{\substack{\square\\\square}}(u) &= \frac{1}{(\epsilon_2-\epsilon_1)(\epsilon_2-\epsilon_3)}\Big((u-\epsilon_1)(u-\epsilon_3)[e(u), e_0] - (2u+\epsilon_2)[e(u), e_1] + [e(u), e_2] - 3[e_1, e_0]\Big),\\
e_{\square\!\!\diagup\!\!\square}(u) &= \frac{1}{(\epsilon_3-\epsilon_1)(\epsilon_3-\epsilon_2)}\Big((u-\epsilon_1)(u-\epsilon_2)[e(u), e_0] - (2u+\epsilon_3)[e(u), e_1] + [e(u), e_2] - 3[e_1, e_0]\Big),
\end{aligned} \tag{1.43}$$

and similar expressions for $f_{\boldsymbol{\lambda}}(u)$.

Using the relations (1.43) one can express generators of the Yangian $\mathcal{L}_{\varnothing,\boldsymbol{\lambda}}(u)$ with $|\boldsymbol{\lambda}| = 2$ as

$$\mathcal{L}_{\varnothing,\square\square}(u) = [e_0, [e_0, h(u)]], \qquad \mathcal{L}_{\varnothing,\substack{\square\\\square}}(u) = [[e_1, e_0], h(u)]$$

and similarly for $\mathcal{L}_{\boldsymbol{\lambda},\varnothing}(u)$ with e_k being replaced by f_k. These equations as well as (1.42) suggest that generic generator $\mathcal{L}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(u)$ can be obtained as an adjoint action of e_k and f_k generators on $h(u)$. Using the RLL relations (1.32) at level 3 one can find

$$\mathcal{L}_{\varnothing,\square\square\square}(u) = [e_0, [e_0, [e_0, h(u)]]], \quad \mathcal{L}_{\varnothing,\substack{\square\\\square\square}}(u) = [e_0, [[e_1, e_0], h(u)], \quad \mathcal{L}_{\varnothing,\substack{\square\\\square\\\square}}(u) = [[e_1, [e_1, e_0]], h(u)]$$

In general, we have found nice representation for the generating function

$$\langle u| e^{\sum_n t_{-n}a_n} \mathcal{L}(u) e^{\sum_n t_n a_{-n}} |u\rangle = e^{\sum_n t_{-n}t_n} e^{\sum_n U_{-n}t_n} e^{-\sum_n U_n t_{-n}} h(u) e^{\sum_n U_n t_{-n}} e^{-\sum_n U_{-n}t_n}.$$

Here U_n are the modes of the $U(1)$ current $W^{(1)}$ (A.20), which can be expressed in terms of e and f currents (A.22)

$$U_{-1} = e_0, \qquad U_{-2} = [e_1, e_0], \qquad U_{-3} = [e_1, [e_1, e_0]], \qquad \dots$$
$$U_1 = f_0, \qquad U_2 = [f_1, f_0], \qquad U_3 = [f_1, [f_1, f_0]], \qquad \dots$$

One can compare current relations (1.40) with commutation Y0-Y6 in Tsymbaliuk's paper [Tsy17]. In fact, it has been noticed already in [Tsy14] that these relations can be compactly written in terms of the generating functions (see also section 2.1 of [Pro16] or section 2.1 of [GGLP17], where these relations were written in a convenient form). The conceptual difference between our relations (1.40) and Tsymbaliuk ones is that we have an auxiliary current $h(u)$. However, one can immediately see that Tsymbaliuk's relations, spanned by the currents $e(u)$, $f(u)$ and $\psi(u)$, are contained in ours, that is $\mathrm{Y}(\widehat{\mathfrak{gl}}(1))$ is a subalgebra in $\mathrm{YB}(\widehat{\mathfrak{gl}}(1))$. Moreover, as we will show in the next subsection, the algebra $\mathrm{YB}(\widehat{\mathfrak{gl}}(1))$ contains infinitely many central elements, with the simplest one given by (1.44), which allows in principle to exclude additional generator $h(u)$.

1.3.2 Center of $\mathrm{YB}(\widehat{\mathfrak{gl}}(1))$

In this section we will show that the algebra $\mathrm{YB}(\widehat{\mathfrak{gl}}(1))$ contains a huge center. Namely for any singular vector $|s\rangle$ of W_n algebra in the space of n bosons we assign a central element D_s (1.52). First element of this series is related to the operator $\psi(u)$ in (1.34) as

$$D_{1,1}(u) = \psi(u)\frac{h(u)h(u+\epsilon_3)}{h(u-\epsilon_1)h(u-\epsilon_2)}. \tag{1.44}$$

For the representation of $\mathrm{YB}(\widehat{\mathfrak{gl}}(1))$ in the space of n bosons $\mathcal{F}_{u_1} \otimes \dots \mathcal{F}_{u_n}$ the element $D_{1,1}(u)$ acts by the function:

$$D_{1,1}(u)|\varnothing\rangle = V(u)|\varnothing\rangle \quad \text{where} \quad V(u) = \prod_{k=1}^{n} \frac{u-u_k+\epsilon_3}{u-u_k},$$

which we call the weight of the representation.

In order to see that operator (1.44) is indeed central and act in any representation by a number, we note that the algebra (1.40) contains additional Hamiltonian $\psi(u)$ which commutes with $h(v)$. One can derive that

$$\psi(u)e(v) = \prod_{\alpha=1}^{3} \frac{(u-v-\epsilon_\alpha)}{(u-v+\epsilon_\alpha)} e(v)\psi(u) + \text{locals}. \tag{1.45}$$

Here, locals denote the terms which contain operators depending only on one parameter either u or v instead of two. Such terms should cancel the poles of the RHS, and may be computed explicitly, see also some discussion right after the formulas (1.40).

Using the relation

$$e(v+\epsilon_3)h^{-1}(v+\epsilon_3) = h^{-1}(v+\epsilon_3)e(v),$$

which immediately follows from (1.40b) at $u = v + \epsilon_3$, we may transform the operator $\psi(u)$ to the more convenient form

$$\psi(u) = -h^{-1}(u)\, {}_u\langle\varnothing| \otimes {}_{u+\epsilon_3}\langle\varnothing| a_1^{(2)} |\mathcal{L}^1(u)\mathcal{L}^2(u+\epsilon_3)\big(a_{-1}^{(1)} - a_{-1}^{(2)}\big)|\varnothing\rangle_u \otimes |\varnothing\rangle_{u+\epsilon_3} h^{-1}(u+\epsilon_3).$$

Using another identity

$${}_{u+\epsilon_3}\langle\varnothing| \otimes {}_u\langle\varnothing|(a_1^{(1)} + a_1^{(2)})\mathcal{L}^1(u)\mathcal{L}^2(u+\epsilon_3)(a_{-1}^{(1)} - a_{-1}^{(2)})|\varnothing\rangle_u \otimes |\varnothing\rangle_{u+\epsilon_3} = 0,$$

we find

$$\psi(u) = \frac{\langle s_{1,1}|\mathcal{L}^1(u)\mathcal{L}^2(u+\epsilon_3)|s_{1,1}\rangle}{h(u)h(u+\epsilon_3)},$$

where

$$|s_{1,1}\rangle_u \stackrel{\text{def}}{=} \big(a^{(1)}_{-1} - a^{(2)}_{-1}\big)|\varnothing\rangle_u \otimes |\varnothing\rangle_{u+\epsilon_3}$$

is a singular vector of a W algebra which appears in the tensor product of two Fock spaces $\mathcal{F}_{u_1} \otimes \mathcal{F}_{u_2}$ at $u_2 = u_1 + \epsilon_3$. Indeed it can be checked that under the resonance condition $u_2 = u_1 + \epsilon_3$ the vector $|s_{1,1}\rangle$ is annihilated by positive modes of the W currents defined by Miura formula:

$$-\epsilon_3^2\partial^2 - i\epsilon_3\mathcal{W}^{(1)}(z)\partial + \mathcal{W}^{(2)}(z) = (i\epsilon_3\partial - \partial\phi_1)(i\epsilon_3\partial - \partial\phi_2)$$

Due to the property that the singular vector is annihilated by all positive modes of W currents it follows that the $R-$matrix acts trivially on the tensor product of the vacuum and the singular vector. In our particular case we have

$$R_{0,1}(u-v)R_{0,2}(u-v+\epsilon_3)|\varnothing\rangle_u \otimes |s_{1,1}\rangle_v = \frac{u-v+\epsilon_3}{u-v}|\varnothing\rangle_u \otimes |s_{1,1}\rangle_v \tag{1.46}$$

Relation (1.46) implies the commutativity of $h(v)$ and $\psi(u)$ and ensures that the Hamiltonian $\psi(u)$ acts on the vacuum $|\varnothing\rangle_v$ by the highest weight

$$\psi(u)|\varnothing\rangle_v = \frac{u-v+\epsilon_3}{u-v}|\varnothing\rangle_v$$

We also found by explicit calculation that $\psi(u)\frac{h(u)h(u+\epsilon_3)}{h(u-\epsilon_1)h(u-\epsilon_2)}$ commutes with $e(v)$ and $f(v)$ and so belongs to the center of RLL algebra [4]. In order to understand this phenomenon, let us note that R-matrix between two vector spaces which are representations of W_∞ algebra is completely (up to a normalization constant) fixed by the eigenvalues of zero modes W_0 of W currents on vacuum and intertwining identity

$$\mathcal{R}\Big(\sum_{k\geq 0}\big(\mathcal{W}^{(k)}(z)\otimes 1\big)\big(i\epsilon_3\partial\big)^{n_1-k}\Big)\Big(\sum_{k\geq 0}\big(\mathcal{W}^{(k)}(z)\otimes 1\big)\big(i\epsilon_3\partial\big)^{n_1-k}\Big)\Big(\sum_{k\geq 0}\big(1\otimes \mathcal{W}^{(k)}(z)\big)\big(i\epsilon_3\partial\big)^{n_2-k}\Big) =$$
$$= \Big(\sum_{k\geq 0}\big(1\otimes \mathcal{W}^{(k)}(z)\big)\big(i\epsilon_3\partial\big)^{n_2-k}\Big)\Big(\sum_{k\geq 0}\big(\mathcal{W}^{(k)}(z)\otimes 1\big)\big(i\epsilon_3\partial\big)^{n_1-k}\Big)\mathcal{R} \tag{1.47}$$

We will consider two representations of W algebra, one in the space of one boson, and other in the space of finite number of bosons n. We take two different representations of W_n algebra - one is the standard Fock representation and the other is the highest weight representation arising from the singular vector $|s\rangle_u$. Let us compute exchanging relation of higher Hamiltonian

$$h_s = \langle s|\mathcal{L}^1(u-u_1)\ldots\mathcal{L}^n(u-u_n)|s.$$

and the current $e(v)$. On general grounds, it has the form

$$h_s(u)e(v) = F_s(u-v)e(v)h_s(u) + \text{locals}, \tag{1.48}$$

where $F_s(u-v)$ is some rational function. Let us concentrate on the first term of (1.48), because local terms are fixed by a demand that l.h.s of (1.48) doesn't have poles[5]. According to the RLL relation, the function $F_s(u-v)$ is equal to the matrix element

$$F_s(u-v) = V^{-1}(u-v)_u\langle s| \otimes {}_v\langle\varnothing|a_1\mathcal{R}(u-v)a_{-1}|\varnothing\rangle_v|s\rangle_u, \tag{1.49}$$

[4] This fact is an analog of similar relation in $\mathfrak{gl}(2)$ Yangian: operator $\psi(u)$ is a direct analog of q-determinant [KS82]

$$qDet\big[L^{gl(2)}(u)\big] = \langle\uparrow\otimes\downarrow - \downarrow\otimes\uparrow|R^{(1)}(u)R^{(2)}(u+\epsilon_3)|\uparrow\otimes\downarrow - \downarrow\otimes\uparrow|\rangle$$

and $qDet\big[L^{gl(2)}(u)\big]$ belongs to the center of $Y\big(\mathfrak{gl}(2)\big)$.

[5] First term of (1.48) obviously has poles because of the rational function $F_s(u-v)$. Its residues should be canceled by local terms which fixes them unambiguously.

where $V(u-v)$ is the weight of representation arising from the singular vector $|s\rangle$

$$\mathcal{R}(u-v)|\varnothing\rangle_v|s\rangle_u = V(u-v)|\varnothing\rangle_v|s\rangle_u$$

In order to calculate the matrix element (1.49) let us act by the minus first mode of intertwining identity (1.47), specified to the case n_1=1, $n_2=n$, $\Big(\sum\limits_{k\geq 0}\big(\mathcal{W}^{(k)}(z)\otimes 1\big)\big(i\epsilon_3\partial\big)^{n_1-k}\Big)\to i\epsilon_3\partial-\partial\phi(z)$, on vacuum

$$\mathcal{R}\Big[a_{-1}\sum_k \mathcal{W}_0^{(k)}\big(i\epsilon_3\partial\big)^{n-k}+\sum_k W_{-1}^{(k)}(-\epsilon_3+i\epsilon_3\partial+u)\big(i\epsilon_3\partial\big)^{n-k}\Big]|\varnothing\rangle_u\otimes|s\rangle_v =$$
$$= V(u-v)\Big[a_{-1}\sum_k \mathcal{W}_0^{(k)}\big(-\epsilon_3+i\epsilon_3\partial\big)^{n-k}+\sum_k \mathcal{W}_{-1}^{(k)}(i\epsilon_3\partial+u)\big(i\epsilon_3\partial\big)^{n-k}\Big]|\varnothing\rangle_u|\otimes|s\rangle_v \quad (1.50)$$

The desired matrix element can be found by solving a linear system and excluding all $W_{-1}^{(k)}$ modes in the l.h.s of (1.50). However one can avoid this complicated calculation simply by substitution $i\epsilon_3\partial\to -u+\epsilon_3$:

$$_v\langle s|\otimes\,_u\langle\square|R(u-v)|\square\rangle_u\times|s\rangle_v = V(u-v)\frac{\sum_k W_0^{(k)}\big(-u+\epsilon_3\big)^{n-k}}{\sum_k W_0^{(k)}\big(-u\big)^{n-k}}.$$

Thus we find that the exchanging function in (1.48) depends only on the polynomial $P_s(u)$:

$$P_s(u)=\sum_k W_0^{(k)}\big(-u\big)^{n-k}=\prod_{k=1}^{n}(u-v_k) \quad (1.51)$$

as

$$F_s(u-v)=\frac{P_s(u+\epsilon_3)}{P_s(u)}.$$

For example, explicit calculation for singular vector on the first level $|s_{1,1}\rangle = (a_{-1}^{(1)}-a_{-1}^{(2)})|\varnothing\rangle_{v,v+\epsilon_3}$ provides

$$P_{s_{1,1}}(u)=(u-v-\epsilon_1)(u-v-\epsilon_2)$$

More generally for a singular vector in W_2 algebra $s_{m,n}$ at level mn

$$P_{s_{m,n}}=(u-v-m\epsilon_1)(u-v-n\epsilon_2)$$

Let us note that the same polynomial corresponds to a vacuum vector in two Fock spaces $F_{v-m\epsilon_1}\otimes F_{v-n\epsilon_2}$. This calculation immediately implies that current $D_{m,n}(u)=\frac{h_{s_{m,n}}(u)}{h(u-m\epsilon_1)h(u-n\epsilon_2)}$ commute with $e(v), f(v), h(v)$ and so belongs to the center of $YB(\widehat{\mathfrak{gl}}(1))$. Indeed:

$$D_{m,n}(u)e(v)=e(v)D_{m,n}(u)+\text{locals}$$

However, as we have seen, all local terms came up with poles which should be canceled with residues of non local term. Since non-local terms do not have poles no local terms allowed. Thus, we proved

$$D_{m,n}(u)e(v)=e(v)D_{m,n}(u)$$

Exchanging relation with $f(u)$ is similar, and hence we prove that $D_{m,n}(u)$ is indeed belongs to the center of $YB(\widehat{\mathfrak{gl}}(1))$.

In general, by the same argument, any singular vector of W_n algebra in the space of n Fock modules gives rise to central element of $YB(\widehat{\mathfrak{gl}}(1))$. As we explained exchanging relations of higher Hamiltonian h_s with $e(v), f(v)$ currents are encoded in a single polynomial (1.51). And the operator:

$$D_s=\frac{h_s(u)}{\prod\limits_{i=1}^{n}h(u-v_i)} \quad (1.52)$$

is the central element of the algebra YB$(\widehat{\mathfrak{gl}}(1))$.

1.3.3 Zero twist integrable system

The Yang-Baxter algebra $\mathrm{YB}(\widehat{\mathfrak{gl}}(1))$ contains commutative subalgebra spanned by modes of the current $h(u)$. If one consider a representation of the $\mathrm{YB}(\hat{\mathfrak{gl}}(1))$ algebra on n sites, this integrable system is known to coincide with matrix generalization of quantum Benjamin-Ono integrable hierarchy. It attracted some attention because it is directly related to AGT representation for conformal blocks [AGT10]. Much is known about this integrable system. In particular, its spectra and eigenfunctions can be written rather explicitly.

Suppose, one has an eigenvector of $h(u)$

$$h(u)|\Lambda\rangle = h_\Lambda(u)|\Lambda\rangle.$$

Then one can try to create new states by repetitive application of $e(v)$. Using (1.40b), one finds that

$$h(u)e(v)|\Lambda\rangle = \frac{u-v}{u-v-\epsilon_3} h_\Lambda(u)e(v)|\Lambda\rangle - \frac{\epsilon_3}{u-v-\epsilon_3} L_{\varnothing,\square}(u)|\Lambda\rangle, \tag{1.53}$$

and hence in general $e(v)|\Lambda\rangle$ is not an eigenvector of $h(u)$. However if $e(v)|\Lambda\rangle$ develops a singularity at some value $v = x$, typically a pole, then the second term in the r.h.s. of (1.53) is negligible and we have a new eigenvector

$$|\tilde{\Lambda}\rangle = \frac{1}{2\pi i}\oint_{\mathcal{C}_x} e(v)|\Lambda\rangle dv, \qquad h(u)|\tilde{\Lambda}\rangle = \frac{(u-x)}{(u-x-\epsilon_3)} h_\Lambda(u)|\tilde{\Lambda}\rangle \tag{1.54}$$

Similar argument applies to the operator $\psi(u)$

$$\psi(u)|\tilde{\Lambda}\rangle = \prod_{\alpha=1}^{3} \frac{(u-x-\epsilon_\alpha)}{(u-x+\epsilon_\alpha)} \psi_\lambda(u)|\tilde{\Lambda}\rangle \tag{1.55}$$

and to any higher Hamiltonian $h_s(u)$ from the previous section.

Using (1.54)-(1.55), one can generate any eigenvector from the vacuum state by successive application of $e(u)$. We note that the operators $e(u)$ do not commute. However the structure of commutation relations (1.40e) implies the following property

$$\oint_{\mathcal{C}_y} dv \oint_{\mathcal{C}_x} du\, e(u)e(v)|\Lambda\rangle = \prod_{\alpha=1}^{3} \frac{(x-y-\epsilon_\alpha)}{(x-y+\epsilon_\alpha)} \oint_{\mathcal{C}_y} dv \oint_{\mathcal{C}_x} du\, e(v)e(u)|\Lambda\rangle \tag{1.56}$$

provided that x and y are *simple* poles and that $y \neq x + \epsilon_\alpha$.

The properties (1.54)-(1.55) and (1.56) are used to show that the eigenstates are in correspondence with tuples of Young diagrams or more generally with 3D partitions. In order to demonstrate how it works, we take our quantum space to be the tensor product of n Fock modules generated from the vacuum state $|\varnothing\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$

$$\mathcal{F}_{x_1} \otimes \cdots \otimes \mathcal{F}_{x_n} = \operatorname{span}\{a^{(1)}_{-\boldsymbol{\lambda}^{(1)}} \dots a^{(n)}_{-\boldsymbol{\lambda}^{(n)}}|\varnothing\rangle : \boldsymbol{\lambda}^{(k)} = \lambda^{(k)}_1 \geq \lambda^{(k)}_2 \geq \dots\}.$$

Our normalization of $h(u)$, which is inherited from our normalization of the $R-$matrix (1.21), implies that $h(u)|\varnothing\rangle = |\varnothing\rangle$. Then it follows from the definition of $\psi(u)$ (1.34) that

$$\psi(u)|\varnothing\rangle = \prod_{k=1}^{n} \frac{u-x_k+\epsilon_3}{u-x_k}|\varnothing\rangle. \tag{1.57}$$

Moreover the vacuum state is annihilated by $f(u)$

$$f(u)|\varnothing\rangle = 0,$$

while the new states are generated by the modes of $e(u)$. In principle, one can rewrite a generic state in $\mathcal{F}_{x_1} \otimes \cdots \otimes \mathcal{F}_{x_n}$ as an integral

$$a^{(1)}_{-\boldsymbol{\lambda}^{(1)}} \dots a^{(n)}_{-\boldsymbol{\lambda}^{(n)}} |\varnothing\rangle = \int \cdots \int \rho_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{u}) e(u_N) \dots e(u_1) |\varnothing\rangle \, du_1 \dots du_N \quad \text{where} \quad N = \sum_{k=1}^{n} |\boldsymbol{\lambda}^{(k)}|,$$

for some function $\rho_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{u})$ (see [Pro16, GGLP17] for explicit formulae on lowest levels). The eigenfunctions of $h(u)$ provide another basis $|\vec{\boldsymbol{\lambda}}\rangle$ in $\mathcal{F}_{x_1} \otimes \cdots \otimes \mathcal{F}_{x_n}$ which has very simple form in terms of $e(u)$ generators

$$|\vec{\boldsymbol{\lambda}}\rangle \sim \oint_{\mathcal{C}_N} du_N \cdots \oint_{\mathcal{C}_1} du_1 \, e(u_N) \dots e(u_1) |\varnothing\rangle, \qquad N = |\vec{\boldsymbol{\lambda}}| = \sum_{k=1}^{n} |\boldsymbol{\lambda}^{(k)}|, \tag{1.58}$$

We will specify the proportionality coefficient in (1.58) later. In fact it depends on the order in which we perform integrations. The contours in (1.58) go counterclockwise around simple poles located at the contents of Young diagrams in $\vec{\boldsymbol{\lambda}}$. By definition a content of a cell with coordinates (i, j) in Young diagram $\boldsymbol{\lambda}^{(k)}$ is

$$c_{\square} = x_k - (i-1)\epsilon_1 - (j-1)\epsilon_2.$$

The order of the contours $\mathcal{C}_i$ in (1.58) should follow the order of any standard Young tableaux associated to $\boldsymbol{\lambda}^{(k)}$. Different choices of the ordering would lead to the same state which might differ by a factor, later we will provide a formula for eigenvector $|\vec{\boldsymbol{\lambda}}\rangle$ which is independent of the ordering (see (1.64)).

The state defined by (1.58) is an eigenstate of $h(u)$ and $\psi(u)$ with eigenvalues

$$h(u)|\vec{\boldsymbol{\lambda}}\rangle = \prod_{\square \in \vec{\boldsymbol{\lambda}}} \frac{(u - c_{\square})}{(u - c_{\square} - \epsilon_3)} |\vec{\boldsymbol{\lambda}}\rangle, \qquad \psi(u)|\vec{\boldsymbol{\lambda}}\rangle = \prod_{\alpha=1}^{3} \prod_{\square \in \vec{\boldsymbol{\lambda}}} \frac{(u - c_{\square} - \epsilon_\alpha)}{(u - c_{\square} + \epsilon_\alpha)} \prod_{k=1}^{n} \frac{(u - x_k + \epsilon_3)}{(u - x_k)} |\vec{\boldsymbol{\lambda}}\rangle \tag{1.59}$$

We note that (1.59) follows immediately from (1.40b), (1.45) and (1.57) provided that the surrounded singularities of the integrand in (1.58) are all simple poles. This statement can be proven by induction in level N:

- The base of induction. Let us consider generic states at level one: $e(u)|\varnothing\rangle$. In order to find its poles we use (1.40d)

  $$f(v)e(u)|\varnothing\rangle = -\frac{\psi(u) - \psi(v)}{u - v} |\varnothing\rangle,$$

  which implies that poles of $e(u)|\varnothing\rangle$ are located exactly at $u = x_k$ and hence

  $$|\square_k\rangle \sim \oint_{\mathcal{C}_k} du \, e(u) |\varnothing\rangle$$

  are the eigenstates of $h(u)$.

- Let us assume that up to level N the operators $e(u)$ and $f(u)$ act as follows

$$\begin{aligned} e(u)|\vec{\boldsymbol{\lambda}}\rangle &= \sum_{\square \in \text{addable}(\vec{\boldsymbol{\lambda}})} \frac{E(\vec{\boldsymbol{\lambda}}, \vec{\boldsymbol{\lambda}} + \square)}{u - c_{\square}} |\vec{\boldsymbol{\lambda}} + \square\rangle \qquad \text{for} \quad |\vec{\boldsymbol{\lambda}}| < N, \\ f(u)|\vec{\boldsymbol{\lambda}}\rangle &= \sum_{\square \in \text{removable}(\vec{\boldsymbol{\lambda}})} \frac{F(\vec{\boldsymbol{\lambda}}, \vec{\boldsymbol{\lambda}} - \square)}{u - c_{\square}} |\vec{\boldsymbol{\lambda}} - \square\rangle \qquad \text{for} \quad |\vec{\boldsymbol{\lambda}}| \leq N, \end{aligned} \tag{1.60}$$

where the amplitudes $E(\vec{\boldsymbol{\lambda}},\vec{\boldsymbol{\lambda}}+\square)$ and $F(\vec{\boldsymbol{\lambda}},\vec{\boldsymbol{\lambda}}-\square)$ are given by

$$\begin{aligned} E(\vec{\boldsymbol{\lambda}},\vec{\boldsymbol{\lambda}}+\square) &= \frac{\epsilon_1\epsilon_2}{\epsilon_3}\prod_{\square'\in\vec{\boldsymbol{\lambda}}+\square} S^{-1}(c_{\square'}-c_{\square})\prod_{k=1}^{n}\frac{(c_{\square}-x_k+\epsilon_3)}{(c_{\square}-x_k)},\\ F(\vec{\boldsymbol{\lambda}},\vec{\boldsymbol{\lambda}}-\square) &= \prod_{\square'\in\vec{\boldsymbol{\lambda}}-\square} S(c_{\square}-c_{\square'}), \end{aligned} \tag{1.61}$$

with

$$S(x)=\frac{(x+\epsilon_1)(x+\epsilon_2)}{x(x-\epsilon_3)}. \tag{1.62}$$

In (1.60) the sets addable$(\vec{\boldsymbol{\lambda}})$ and removable$(\vec{\boldsymbol{\lambda}})$ corresponds to the sets of all boxes which can be either added or removed from $\vec{\boldsymbol{\lambda}}$.

- We have to show that $e(u)|\vec{\boldsymbol{\lambda}}\rangle$ with $|\vec{\boldsymbol{\lambda}}|=N$ has poles at addable points. Consider u poles of the following vector
$$f(v)e(u)|\vec{\boldsymbol{\lambda}}\rangle = -\frac{\psi(u)-\psi(v)}{u-v}|\vec{\boldsymbol{\lambda}}\rangle + e(u)f(v)|\vec{\boldsymbol{\lambda}}\rangle. \tag{1.63}$$
There are two sources of poles in the r.h.s of (1.63): the eigenvalue of $\psi(u)$ and the $e(u)f(v)|\vec{\boldsymbol{\lambda}}\rangle$ term. It is easy to show that both terms have poles only at addable and removable points. Formula (1.61) provides exact cancellation of poles at removable points, which implies the statement.

Finally, we provide the normalized formula (1.58) for the eigenvector $|\vec{\boldsymbol{\lambda}}\rangle$ which agrees with formulas (1.60)

$$|\vec{\boldsymbol{\lambda}}\rangle = \lim_{u_i\to c_i}\prod_{i,k}\frac{u_i-x_k}{u_i-x_k-\epsilon_3}\prod_{i<j}S(u_i-u_j)e(u_N)...e(u_1)|\varnothing\rangle \tag{1.64}$$

The state $e(u_N)...e(u_1)|\varnothing\rangle$ contains a lot of poles at $u_i\to c_i$ which are cancelled by a numerical prefactor, so the formula (1.64) should be understood by L'Hôpital's rule - careful analysis of this formula leads to the same residues as in (1.58). c_i variables are equal to the contents of Young diagrams $\vec{\boldsymbol{\lambda}}$ and ordered to follow the order of any standard Young tableaux associated to $\boldsymbol{\lambda}^{(k)}$. Note that this prescription doesn't completely define an ordering, however each ordering leads to the same formula. Indeed two admissible orderings are different by a number of transposition of $e(u_i)$ currents in (1.64) note that for such transpositions corresponding arguments u_i are not in a resonance, using (1.40e) we have:

$$S(u_i-u_j)e(u_i)e(u_j)=S(u_j-u_i)e(u_j)e(u_i)+\text{locals}$$

The *local* terms are regular at $u_i\to c_i$ and so doesn't contribute to the (1.64).

1.4 ILW Integrals of Motion and Bethe ansatz

Consider the transfer matrix on n sites $\mathbf{T}_q(u)$ defined by (1.12). Using the formulas (A.2),(A.3) from the Appendix A.1 one can easily see that $\mathbf{T}_q(u)$ admits the following large u expansion

$$\mathbf{T}_q(u)=\Lambda(u,q)\exp\left(\frac{1}{u}\mathbf{I}_1(q)+\frac{1}{u^2}\mathbf{I}_2(q)+\dots\right),$$

where $\Lambda(u,q)$ is a normalization factor and $\mathbf{I}_1$ and $\mathbf{I}_2$ are the first ILW$_n$ Integrals of Motion (1.13). As explained in Introduction among other Integrals of Motion there is a particular one called KZ integral

$$\mathcal{I}_1^{\text{KZ}}(q)\overset{\text{def}}{=}\mathbf{T}_q(u_1). \tag{1.65}$$

Using the fact that $\mathcal{R}_{0,1}(0) = \mathcal{P}_{0,1}$ a permutation operator, one finds

$$\mathcal{I}_1^{\mathrm{KZ}}(q) = q^{L_0^{(1)}} \mathcal{R}_{1,2}(u_1 - u_2)\mathcal{R}_{1,3}(u_1 - u_3)\dots\mathcal{R}_{1,n}(u_1 - u_n). \tag{1.66}$$

As announced in Introduction the simultaneous spectrum of $\mathbf{T}_q(u)$ is governed by Bethe ansatz equations (1.15). In this section we will derive these equations.

1.4.1 Off-shell Bethe vector

The basic ingredient of algebraic Bethe ansatz is the construction of the so-called off-shell Bethe vector. We take the tensor product of $n+N$ Fock spaces, with n "quantum" and N "auxiliary" spaces

$$\underbrace{\mathcal{F}_{u_1}\otimes\dots\otimes\mathcal{F}_{u_n}}_{\text{quantum space}}\otimes\underbrace{\mathcal{F}_{x_1}\otimes\dots\otimes\mathcal{F}_{x_N}}_{\text{auxiliary space}} \tag{1.67}$$

generated from the vacuum state

$$|\varnothing\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}} = |x_1\rangle\otimes\dots\otimes|x_N\rangle\otimes|u_1\rangle\otimes\dots\otimes|u_n\rangle.$$

Consider the special eigenstate in the auxiliary space (1.58),(1.64)

$$|\chi\rangle_{\boldsymbol{x}} \overset{\text{def}}{=} |\underbrace{\square,\dots,\square}_{N}\rangle \sim \oint_{\mathcal{C}_N} dz_N\cdots\oint_{\mathcal{C}_1} dz_1\, e(z_N)\dots e(z_1)|\varnothing\rangle_{\boldsymbol{x}}, \tag{1.68}$$

where the contour $\mathcal{C}_k$ encircles the point x_k in counter-clockwise direction. The vector $|\chi\rangle_{\boldsymbol{x}}$ is an eigenvector of zero twist integrable system with eigenvalue

$$h(u)|\chi\rangle_{\boldsymbol{x}} = \prod_{k=1}^{N}\frac{u-x_k}{u-x_k-\epsilon_3}|\chi\rangle_{\boldsymbol{x}}. \tag{1.69}$$

A direct consequence of (1.60) implies a convenient formula :

$${}_{\boldsymbol{x}}\langle\varnothing|f(z_N)\dots f(z_1)|\chi\rangle_{\boldsymbol{x}} = \mathrm{Sym}_{\boldsymbol{x}}\left(\prod_{a=1}^{N}\frac{1}{z_a-x_a}\prod_{a<b}S(x_a-x_b)\right), \tag{1.70}$$

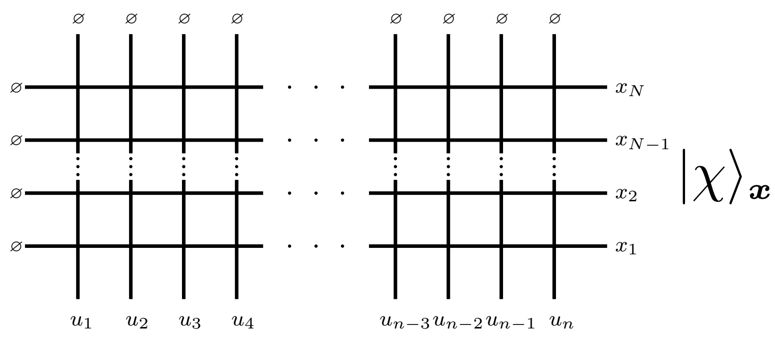
which is an immediate consequence of (1.60) and (1.61).

Now we define the off-shell Bethe vector, our definition is similar to the very general approach investigated in [AO17] (in particular this construction is explained in section 1.3.3 of [AO17])

$$|B(\boldsymbol{x})\rangle_{\boldsymbol{u}} \overset{\text{def}}{=} {}_{\boldsymbol{x}}\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}} \quad \text{where} \quad \mathcal{R}(\boldsymbol{x},\boldsymbol{u}) = \mathcal{R}_{x_1u_1}\dots\mathcal{R}_{x_Nu_1}\dots\mathcal{R}_{x_1u_n}\dots\mathcal{R}_{x_Nu_n}. \tag{1.71}$$

The off-shell Bethe vector $|B(\boldsymbol{x})\rangle$ can be represented by the following picture (similar to the figure 2 in [AO17])

$|B(\boldsymbol{x})\rangle =$



Consider the matrix element between $|B(\boldsymbol{x})\rangle_{\boldsymbol{u}}$ and generic state

$$\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) \stackrel{\text{def}}{=} {}_{\boldsymbol{u}}\langle\varnothing|a^{(1)}_{\boldsymbol{\lambda}^{(1)}}\dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}}|B(\boldsymbol{x})\rangle_{\boldsymbol{u}} = {}_{\boldsymbol{x}}\langle\varnothing|\mathcal{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)|\chi\rangle_{\boldsymbol{x}}, \tag{1.72}$$

which is non-zero only if

$$|\vec{\boldsymbol{\lambda}}| = \sum_{k=1}^{n}|\boldsymbol{\lambda}^{(k)}| = N.$$

Following [TV95], we call $\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u})$ the weight function. It can be simplified by noting that the matrix element of Lax operator $\mathcal{L}_{\boldsymbol{\lambda},\varnothing}(u)$ can be expressed through $h(u)$ and $f(z)$ via contour integral[6]

$$\mathcal{L}_{\boldsymbol{\lambda},\varnothing}(u) = \frac{1}{(2\pi i)^{|\boldsymbol{\lambda}|}}\oint_{\mathcal{C}_1}\cdots\oint_{\mathcal{C}_{|\boldsymbol{\lambda}|}} F_{\boldsymbol{\lambda}}(\boldsymbol{z}|u)\,h(u)f(z_{|\boldsymbol{\lambda}|})\dots f(z_1)dz_1\dots dz_{|\boldsymbol{\lambda}|}, \tag{1.73}$$

where each contour $\mathcal{C}_k$ goes clockwise around ∞ and $u-\epsilon_3$. Using (1.73) the weight function (1.72) can be rewritten as

$$\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) = \frac{1}{(2\pi i)^N}\times$$
$$\times\oint F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\;{}_{\boldsymbol{x}}\langle\varnothing|h(u_1)\underbrace{f(z_1^{(1)})f(z_2^{(1)})\dots}_{|\boldsymbol{\lambda}^{(1)}|}\;h(u_2)\underbrace{f(z_1^{(2)})f(z_2^{(2)})\dots}_{|\boldsymbol{\lambda}^{(2)}|}\;\dots\;h(u_n)\underbrace{f(z_1^{(n)})f(z_2^{(n)})\dots}_{|\boldsymbol{\lambda}^{(n)}|}|\chi\rangle_{\boldsymbol{x}}\,d\vec{\boldsymbol{z}}, \tag{1.74}$$

where

$$F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u}) = \prod_{k=1}^{n}F_{\boldsymbol{\lambda}^{(k)}}\left(z_1^{(k)},\dots,z^{(k)}_{|\boldsymbol{\lambda}^{(k)}|}|u_k\right). \tag{1.75}$$

Then the matrix element in (1.74) can be explicitly computed using (1.40c) and (1.70). One obtains

$$\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) = \frac{1}{(2\pi i)^N}\oint\cdots\oint\Omega_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\;\mathrm{Sym}_{\boldsymbol{x}}\left(\prod_{a=1}^{N}\frac{1}{z_a-x_a}\prod_{a<b}S(x_a-x_b)\right)d\vec{\boldsymbol{z}}, \tag{1.76}$$

where $(z_1,\dots,z_N) = (z_1^{(1)},\dots z^{(1)}_{\boldsymbol{\lambda}^{(1)}}, z_1^{(2)},\dots z^{(2)}_{\boldsymbol{\lambda}^{(2)}},\dots,z_1^{(n)},\dots z^{(n)}_{\boldsymbol{\lambda}^{(n)}})$ and the function

$$\Omega_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u}) = F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\left(\prod_{j=1}^{|\boldsymbol{\lambda}^{(1)}|}\frac{u_2-z_j^{(1)}}{u_2-z_j^{(1)}-\epsilon_3}\right)\left(\prod_{j=1}^{|\boldsymbol{\lambda}^{(2)}|}\frac{u_3-z_j^{(2)}}{u_3-z_j^{(2)}-\epsilon_3}\prod_{j=1}^{|\boldsymbol{\lambda}^{(1)}|}\frac{u_3-z_j^{(1)}}{u_3-z_j^{(1)}-\epsilon_3}\right)\cdots$$
$$\cdots\left(\prod_{j=1}^{|\boldsymbol{\lambda}^{(n-1)}|}\frac{u_n-z_j^{(n-1)}}{u_n-z_j^{(n-1)}-\epsilon_3}\prod_{j=1}^{|\boldsymbol{\lambda}^{(n-2)}|}\frac{u_n-z_j^{(n-2)}}{u_n-z_j^{(n-2)}-\epsilon_3}\cdots\prod_{j=1}^{|\boldsymbol{\lambda}^{(1)}|}\frac{u_n-z_j^{(1)}}{u_n-z_j^{(1)}-\epsilon_3}\right) \tag{1.77}$$

has been obtained from $F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})$ as a result of application of (1.40c). We note that as explained in appendix A.3 the local terms do not appear in (1.76) if one extends the integration contour to include all singularities of (1.77). It implies that the integral shrinks to the points $\boldsymbol{x}$ and one obtains

$$\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) = \mathrm{Sym}_{\boldsymbol{x}}\left(\Omega_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{x}}|\boldsymbol{u})\prod_{a<b}S(x_a-x_b)\right). \tag{1.78}$$

Let us note finally that this last equation implies the well known co-product property of weight function:

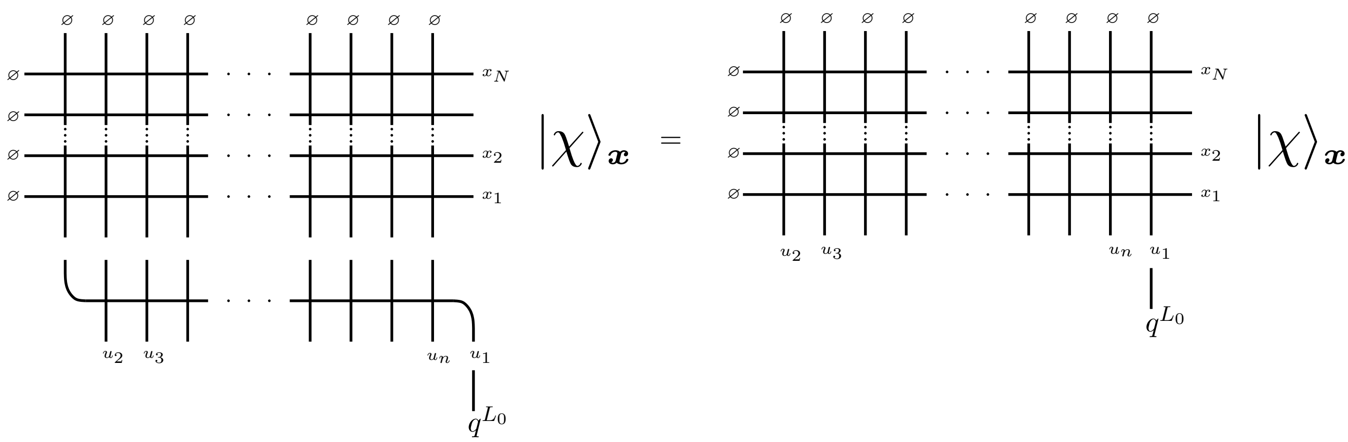
$$\omega^{V_1,V_2}_{\vec{\lambda}^{(1)},\vec{\lambda}^{(2)}} = \sum_{I=I_1+I_2}\omega^{V_1}_{\vec{\lambda}_1}(x^{(1)})\omega^{V_2}_{\vec{\lambda}_2}(x^{(2)})\prod_{x_i\in I_2}V_1(x_i)\prod_{i\in I_1,j\in I_2}S(x_i-x_j),$$

Where the sum is over partitions of set of indices $I = 1,2,\dots N$ into two sets (I_1,I_2) of lengths (N_1,N_2), we also denote all x_i variables from set I_a $x^{(a)}$.

[6] See Appendix A.3 for details.

1.4.2 Diagonalization of KZ Integral

The action of the KZ Integral of Motion (1.65) on off-shell Bethe vector $|B(\boldsymbol{x})\rangle_{\boldsymbol{u}}$ is very simple and can be explained by the following picture



Projecting this equation on arbitrary state, one obtains

$$_{\boldsymbol{u}}\langle\varnothing|a^{(1)}_{\boldsymbol{\lambda}^{(1)}}\dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}}|\mathcal{I}_1^{\mathrm{KZ}}(q)|B(\boldsymbol{x})\rangle_{\boldsymbol{u}} = q^{|\boldsymbol{\lambda}^{(1)}|}\,_{\boldsymbol{x}}\langle\varnothing|\mathcal{L}_{\boldsymbol{\lambda}^{(2)},\varnothing}(u_2)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)\mathcal{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)|\chi\rangle_{\boldsymbol{x}} \tag{1.79}$$

If we require that $|B(\boldsymbol{x})\rangle_{\boldsymbol{u}}$ is an eigenstate for $\mathcal{I}_1^{\mathrm{KZ}}(q)$ we have to demand

$$q^{|\boldsymbol{\lambda}^{(1)}|}\,_{\boldsymbol{x}}\langle\varnothing|\mathcal{L}_{\boldsymbol{\lambda}^{(2)},\varnothing}(u_2)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)\mathcal{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)|\chi\rangle_{\boldsymbol{x}} = t_q^1(\boldsymbol{u})\,_{\boldsymbol{x}}\langle\varnothing|\mathcal{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)|\chi\rangle_{\boldsymbol{x}}, \tag{1.80}$$

which should hold for any set of partitions $\vec{\boldsymbol{\lambda}}$. The eigenvalue $t_q^1(\boldsymbol{u})$ can be found from (1.80) by taking $\boldsymbol{\lambda}^{(1)} = \varnothing$

$$t_q^1(\boldsymbol{u}) = \prod_{k=1}^{N}\frac{x_k - u_1}{x_k - u_1 + \epsilon_3}.$$

For generic $\vec{\boldsymbol{\lambda}}$ the eigenstate equation (1.80) implies the integral identity

$$q^{|\boldsymbol{\lambda}^{(1)}|}\oint F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\,_{\boldsymbol{x}}\langle\varnothing|h(u_2)\underbrace{f(z_1^{(2)})\dots}_{|\boldsymbol{\lambda}^{(2)}|}\,\dots\,h(u_n)\underbrace{f(z_1^{(n)})\dots}_{|\boldsymbol{\lambda}^{(n)}|}h(u_1)\underbrace{f(z_1^{(1)})\dots}_{|\boldsymbol{\lambda}^{(1)}|}|\chi\rangle_{\boldsymbol{x}}\,d\vec{\boldsymbol{z}} =$$

$$= T_1(\boldsymbol{u})\oint F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\,_{\boldsymbol{x}}\langle\varnothing|\,h(u_1)\underbrace{f(z_1^{(1)})\dots}_{|\boldsymbol{\lambda}^{(1)}|}\,h(u_2)\underbrace{f(z_1^{(2)})\dots}_{|\boldsymbol{\lambda}^{(2)}|}\,\dots\,h(u_n)\underbrace{f(z_1^{(n)})\dots}_{|\boldsymbol{\lambda}^{(n)}|}|\chi\rangle_{\boldsymbol{x}}\,d\vec{\boldsymbol{z}},$$

which holds provided that $\boldsymbol{x}$ obeys Bethe ansatz equations

$$q\prod_{j\neq i}\prod_{\alpha=1}^{3}\frac{x_i - x_j - \epsilon_\alpha}{x_i - x_j + \epsilon_\alpha}\prod_{k=1}^{n}\frac{x_i - u_k + \epsilon_3}{x_i - u_k} = 1 \quad \text{for all} \quad i = 1,\dots,N. \tag{1.81}$$

Indeed using (1.40c) and (1.40f) one can drag all $f(z_j^{(1)})$'s to the left. As the integrals over $z_i^{(j)}$ go around simple poles located at the points $x_1,\dots,x_N$ (see (1.76)), the local terms in (1.40c) and (1.40f) do not contribute and we will have

$$q^{|\boldsymbol{\lambda}^{(1)}|}\oint F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\ {}_{\boldsymbol{x}}\langle\varnothing|h(u_2)\underbrace{f(z_1^{(2)})\dots}_{|\boldsymbol{\lambda}^{(2)}|}\ \dots\ h(u_n)\underbrace{f(z_1^{(n)})\dots}_{|\boldsymbol{\lambda}^{(n)}|}h(u_1)\underbrace{f(z_1^{(1)})\dots}_{|\boldsymbol{\lambda}^{(1)}|}|\chi\rangle_{\boldsymbol{x}}\,d\vec{\boldsymbol{z}}=$$

$$=\oint F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\prod_{k=1}^{|\boldsymbol{\lambda}^{(1)}|}\mathcal{D}(z_k^{(1)}|\boldsymbol{z})\ {}_{\boldsymbol{x}}\langle\varnothing|\underbrace{f(z_1^{(1)})\dots}_{|\boldsymbol{\lambda}^{(1)}|}h(u_2)\underbrace{f(z_1^{(2)})\dots}_{|\boldsymbol{\lambda}^{(2)}|}\ \dots\ h(u_n)\underbrace{f(z_1^{(n)})\dots}_{|\boldsymbol{\lambda}^{(n)}|}h(u_1)|\chi\rangle_{\boldsymbol{x}}\,d\vec{\boldsymbol{z}}=$$

$$=T_1(\boldsymbol{u})\oint F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\prod_{k=1}^{|\boldsymbol{\lambda}^{(1)}|}\mathcal{D}(z_k^{(1)}|\boldsymbol{z})\ {}_{\boldsymbol{x}}\langle\varnothing|\underbrace{f(z_1^{(1)})\dots}_{|\boldsymbol{\lambda}^{(1)}|}h(u_2)\underbrace{f(z_1^{(2)})\dots}_{|\boldsymbol{\lambda}^{(2)}|}\ \dots\ h(u_n)\underbrace{f(z_1^{(n)})\dots}_{|\boldsymbol{\lambda}^{(n)}|}|\chi\rangle_{\boldsymbol{x}}\,d\vec{\boldsymbol{z}},\tag{1.82}$$

where $\boldsymbol{z}$ denotes the set of all $z_i^{(j)}$ and

$$\mathcal{D}(z|\boldsymbol{z})=q\prod_{z_j\neq z}\prod_{\alpha=1}^{3}\frac{z-z_j-\epsilon_\alpha}{z-z_j+\epsilon_\alpha}\prod_{k=1}^{n}\frac{z-u_k+\epsilon_3}{z-u_k}.$$

One can easily show that under Bethe ansatz equations (1.81) each additional factor $\mathcal{D}(z_k^{(1)}|\boldsymbol{z})$ in (1.82) equals to 1, which implies the statement.

1.4.3 Quantum KZ equation

Off-shell Bethe vectors (1.71), are closely related to solutions of difference KZ equation. Namely, let us introduce auxiliary functions

$$V^{(\hbar)}(x)=\frac{\Gamma(\frac{x}{\hbar})}{\Gamma(\frac{x-\epsilon_3}{\hbar})},\qquad \Phi^{(\hbar)}(x)=\frac{\Gamma(\frac{x-\epsilon_1}{\hbar})\Gamma(\frac{x-\epsilon_2}{\hbar})}{\Gamma(\frac{x}{\hbar})\Gamma(\frac{x+\epsilon_3}{\hbar})},$$

where $S(x)$ is given by (1.62). The main property of these functions which will be used is the shift relation

$$V^{(\hbar)}(x+\hbar)=V^{(\hbar)}(x)\frac{x}{x-\epsilon_3},\quad \Phi^{(\hbar)}(x+\hbar)=S(-x)\Phi^{(\hbar)}(x)$$

Then the wave function

$$|\psi(\boldsymbol{u})\rangle\stackrel{\text{def}}{=}\oint q^{\sum\frac{x_a}{\hbar}}\prod_{a=1}^{N}\prod_{j=1}^{n}V^{(\hbar)}(u_j-x_a)\prod_{a\neq b}\Phi^{(\hbar)}(x_a-x_b)|B(\boldsymbol{x})\rangle_{\boldsymbol{u}}\frac{d^N\boldsymbol{x}}{(2\pi i)^N},\tag{1.83}$$

is a solution of difference KZ equation:

$$|\psi(u_1+\hbar,u_2,...,u_n)\rangle=\mathcal{I}_1^{\mathrm{KZ}}|\psi(u_1,...,u_n)\rangle,\tag{1.84}$$

where $\mathcal{I}_1^{\mathrm{KZ}}$ is the first KZ operator (1.65).

The proof of (1.84) is simple. Let us pick a tuple of Young diagrams $\vec{\boldsymbol{\lambda}}=\{\boldsymbol{\lambda}^{(1)},...,\boldsymbol{\lambda}^{(n)}\}$ and consider the projection of the wave function $|\psi(\boldsymbol{u})\rangle$

$${}_{\boldsymbol{u}}\langle\varnothing|a^{(1)}_{\boldsymbol{\lambda}^{(1)}}\dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}}|\psi(\boldsymbol{u})\rangle=\oint q^{\sum_a\frac{x_a}{\hbar}}\prod_{a=1}^{N}\prod_{j=1}^{n}V^{(\hbar)}(u_j-x_a)\prod_{a\neq b}\Phi^{(\hbar)}(x_a-x_b)\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u})\frac{d^N\boldsymbol{x}}{(2\pi i)^N},\tag{1.85}$$

where the weight function $\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u})$ is given by (1.78). At the same time using (1.79) one finds

$$
{}_{\boldsymbol{u}}\langle\varnothing|a^{(1)}_{\boldsymbol{\lambda}^{(1)}}\dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}}|\mathcal{I}_1^{\mathrm{KZ}}|\psi(\boldsymbol{u})\rangle = \oint q^{\sum_a \frac{x_a}{\hbar}}\prod_{a=1}^{N}\prod_{j=1}^{n}V^{(\hbar)}(u_j-x_a)\prod_{a\neq b}\Phi^{(\hbar)}(x_a-x_b)q^{|\boldsymbol{\lambda}^{|1|}|}\omega_{\vec{\boldsymbol{\lambda}}'}(\boldsymbol{x}|\boldsymbol{u}')\frac{d^N\boldsymbol{x}}{(2\pi i)^N}, \tag{1.86}
$$

where $\boldsymbol{u'}=(u_2,\dots,u_n,u_1)$ and $\vec{\boldsymbol{\lambda}}'=\{\boldsymbol{\lambda}^{(2)},\dots,\boldsymbol{\lambda}^{(n)},\boldsymbol{\lambda}^{(1)}\}$. As the integration kernel in (1.85) and (1.86) is $\boldsymbol{x}$ symmetric, one can replace (see (1.77))

$$
\omega_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u})\to F^{\boldsymbol{u}}_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x})\prod_{a<b}S(x_a-x_b),\qquad \omega_{\vec{\boldsymbol{\lambda}}'}(\boldsymbol{x}|\boldsymbol{u'})\to F^{\boldsymbol{u}'}_{\vec{\boldsymbol{\lambda}}'}(\boldsymbol{x})\prod_{a<b}S(x_a-x_b).
$$

Then it is immediately to see that the shift

$$
u_1\to u_1+\hbar,\qquad \left(x_1,\dots,x_{|\boldsymbol{\lambda}^{(1)}|}\right)\to\left(x_1+\hbar,\dots,x_{|\boldsymbol{\lambda}^{(1)}|}+\hbar\right)
$$

reduces (1.85) to (1.86) after relabelling of integration variables. The statement (1.84) follows.

Of course these considerations are correct modulo choice of integration contour. Integrals of the form (1.83) have been discussed in details in the literature [AHKS13, Zen19]. Following these approaches, we treat the integral (1.83) as a sum over residues, the poles contributing to the integral are in one to one correspondence with a collection of n 3D partitions, with fixed floor shape $\boldsymbol{\lambda}$:

$$
x_{\boldsymbol{I}} = v_k-(i_k-1)\epsilon_1-(j_k-1)\epsilon_2+\hbar n^{(k)}_{i,j}
$$

Here we treat the 3D partition as 2D Young diagram filled with integer numbers $n^{(k)}_{i,j}$ such that:

$$
n^{(k)}_{i,j}\geq n^{(k)}_{i+1,j}\quad n^{(k)}_{i,j}\geq n^{(k)}_{i,j+1}
$$

1.4.4 Diagonalization of $\mathbf{I}_2$ Integral

The diagonalization problem of KZ integral given above does not work for $n=1$, because in that case there is no KZ operator. Direct formula (1.65) provides an identity operator. Specially for this case and also for academic purposes we consider diagonalization problem for $\mathbf{I}_2(q)$ IM (1.13). We have to remember that we have changed normalization in (1.38)-(1.39). It is also convenient to subtract the vacuum eigenvalue and $\frac{\epsilon_3}{2}\mathbf{I}_1(q)$ from $\mathbf{I}_2(q)$. Altogether, one has

$$
\tilde{\mathbf{I}}_2(q)=-\epsilon_3\int\left[\frac{1}{3}\sqrt{\epsilon_1\epsilon_2}\sum_i(\partial\phi_i)^3-\epsilon_3\left(\frac{1}{2}\sum_{i,j}\partial\phi_iD(q)\partial\phi_j+\sum_{i<j}\partial\phi_i\partial^2\phi_j\right)\right]\frac{dx}{2\pi}-\frac{\epsilon_3\mathbf{I}_1(q)}{2}-\frac{\epsilon_3}{3}\sqrt{\epsilon_1\epsilon_2}\sum_i u_i^3
$$

where the last two terms are added for a convenience, in order to have more simple formula for the eigenvalues (1.93).

Since our total Fock space splits into quantum and auxiliary parts (1.67), it will be convenient to add an upper index to $\tilde{\mathbf{I}}_2(q)$, either $\boldsymbol{x}$, $\boldsymbol{u}$ or $(\boldsymbol{x},\boldsymbol{u})$ referring to auxiliary, quantum or total spaces respectively. The key observation is that the Integral of Motion $\tilde{\mathbf{I}}_2^{(\boldsymbol{x},\boldsymbol{u})}(q)$ is almost the sum of terms acting separately on spaces $\mathcal{F}_{\boldsymbol{u}}$ and $\mathcal{F}_{\boldsymbol{x}}$ plus a cross term

$$
\tilde{\mathbf{I}}_2^{(\boldsymbol{x},\boldsymbol{u})}(q)=\tilde{\mathbf{I}}_2^{\boldsymbol{x}}(q)+\tilde{\mathbf{I}}_2^{(\boldsymbol{u}}(q)-\frac{1}{2\pi}\int U^{\boldsymbol{u}}(\xi)(D(q)+\partial)J^{\boldsymbol{x}}(\xi)d\xi, \tag{1.87}
$$

where $U^{\boldsymbol{x}/\boldsymbol{v}}(\xi)$ is the $U(1)$ mode (A.20), in our particular representation $U^{\boldsymbol{x}/\boldsymbol{v}}(\xi)=\sqrt{\epsilon_3}\sum_{i\in\mathcal{F}_{\boldsymbol{x}/\boldsymbol{u}}}\partial\phi_i(\xi)$.

We will show that on-shell Bethe vector, is an eigenvector of $\tilde{\mathbf{I}}_2^u(q)$

$$\tilde{\mathbf{I}}_2^{\boldsymbol{u}}(q)\,{}_{\boldsymbol{x}}\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}}=\left(\sum_{k=1}^{N}x_k\right)\,{}_{\boldsymbol{x}}\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}}$$

where x_k obeys Bethe anzatz equations (1.81). We start with off-shell Bethe vector (1.71) and insert Integral of Motion for a system with zero twist $q=0$ acting on $\boldsymbol{x}$ space

$$|B(\boldsymbol{x})\rangle_{\boldsymbol{u}}={}_{\boldsymbol{x}}\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}}\longrightarrow{}_{\boldsymbol{x}}\,\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})\tilde{\boldsymbol{I}}_2^{\boldsymbol{x}}(0)|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}}$$

We have the following chain of arguments

1. Since $|\chi\rangle_{\boldsymbol{x}}$ is an eigenvector of zero twist integrable system (1.69), it is also an eigenvector for $\tilde{\boldsymbol{I}}_2^{\boldsymbol{x}}(0)$ with eigenvalue $\sum_k x_k$. It implies

$$\,{}_{\boldsymbol{x}}\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})\tilde{\boldsymbol{I}}_2^{\boldsymbol{x}}(0)|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}}=\sum_{k=1}^{N}x_k|B(\boldsymbol{x})\rangle_{\boldsymbol{u}} \tag{1.88}$$

2. On the other hand, the Integral $\tilde{\boldsymbol{I}}_2^{\boldsymbol{x}}(0)$ can be completed by (1.87) to $\tilde{\boldsymbol{I}}_2^{(\boldsymbol{x},\boldsymbol{u})}(0)$ which acts on the whole $(\boldsymbol{x},\boldsymbol{u})$ space

$$\tilde{\boldsymbol{I}}_2^{(\boldsymbol{x},\boldsymbol{u})}(0)=\tilde{\boldsymbol{I}}_2^{\boldsymbol{x}}(0)+\tilde{\boldsymbol{I}}_2^{\boldsymbol{u}}(0)-i\sum_{k\in\mathbb{Z}}(|k|+k)U_k^{\boldsymbol{u}}U_{-k}^{\boldsymbol{x}},$$

because the last two terms vanish on $|\varnothing\rangle_{\boldsymbol{u}}$. It implies

$$\,{}_{\boldsymbol{x}}\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})\tilde{\boldsymbol{I}}_2^{\boldsymbol{x}}(0)|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}}={}_{\boldsymbol{x}}\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})\tilde{\boldsymbol{I}}_2^{\boldsymbol{x},\boldsymbol{u}}(0)|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}} \tag{1.89}$$

3. From the definition of $\mathcal{R}(\boldsymbol{x},\boldsymbol{u})$ (see (1.71)) we have

$$\mathcal{R}(\boldsymbol{x},\boldsymbol{u})\tilde{\boldsymbol{I}}_2^{(\boldsymbol{x},\boldsymbol{u})}(0)=\tilde{\boldsymbol{I}}_2^{(\boldsymbol{u},\boldsymbol{x})}(0)\mathcal{R}(\boldsymbol{x},\boldsymbol{u})\quad\text{where}\quad\boldsymbol{I}_2^{(\boldsymbol{u},\boldsymbol{x})}(0)=\tilde{\boldsymbol{I}}_2^{\boldsymbol{x}}(0)+\tilde{\boldsymbol{I}}_2^{\boldsymbol{u}}(0)-i\sum_{k\in\mathbb{Z}}(|k|+k)U_k^{\boldsymbol{x}}U_{-k}^{\boldsymbol{u}},$$

and hence

$$\,{}_{\boldsymbol{x}}\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})\tilde{\boldsymbol{I}}_2^{\boldsymbol{x},\boldsymbol{u}}(0)|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}}={}_{\boldsymbol{x}}\langle\varnothing|\left(\tilde{\boldsymbol{I}}_2^{\boldsymbol{u}}(0)-2i\sum_{k>0}kU_{-k}^{\boldsymbol{u}}U_k^{\boldsymbol{x}}\right)\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}} \tag{1.90}$$

4. One has a remarkable property

$$\,{}_{\boldsymbol{x}}\langle\varnothing|U_k^{\boldsymbol{x}}\,\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}}=q^k\,{}_{\boldsymbol{x}}\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})U_k^{\boldsymbol{x}}|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}}, \tag{1.91}$$

which holds provided that $\boldsymbol{x}$ satisfy (1.81).

5. One can replace $U_k^{\boldsymbol{x}}\to U_k^{\boldsymbol{x}}+U_k^{\boldsymbol{u}}$ in the r.h.s. of (1.91) and use the property

$$[\mathcal{R}(\boldsymbol{x},\boldsymbol{u}),U_k^{\boldsymbol{x}}+U_k^{\boldsymbol{u}}]=0,$$

to obtain

$$\,{}_{\boldsymbol{x}}\langle\varnothing|J_k^{\boldsymbol{x}}\,\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}}=\frac{q^k}{1-q^k}\,{}_{\boldsymbol{x}}\langle\varnothing|\,J_k^{\boldsymbol{u}}\,\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}}\otimes|\varnothing\rangle_{\boldsymbol{u}}. \tag{1.92}$$

6. Equations (1.88), (1.89), (1.90), (1.91) and (1.92) imply that

$$\tilde{I}_2^{\boldsymbol{u}}(q)|B(\boldsymbol{x})\rangle_{\boldsymbol{u}} = \sum_{k=1}^{N} x_k |B(\boldsymbol{x})\rangle_{\boldsymbol{u}}. \tag{1.93}$$

on Bethe ansatz equations (1.81).

In the above reasoning (1.91) requires explanation. In Appendix A.3 we have shown that

$$U_k^{\boldsymbol{x}} = \mathrm{Ad}_{f_1}^{k-1} f_0$$

Using this formula, one finds explicitly

$$U_k^{\boldsymbol{x}} = \oint g_k(\boldsymbol{\xi}) f(\xi_1)...f(\xi_k) d\boldsymbol{\xi} \quad \text{with} \quad g_n(\vec{\xi}) = \prod_i \xi_i \left(\sum (-1)^i C_n^i \xi_i^{-1}\right) \tag{1.94}$$

where C_n^i are the binomial coefficients.

Consider a matrix element of the l.h.s. of (1.91) with generic state

$${}_{\boldsymbol{u}}\langle\varnothing| a^{(1)}_{\boldsymbol{\lambda}^{(1)}} \dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}} \otimes {}_{\boldsymbol{x}}\langle\varnothing|\, J_k^{\boldsymbol{x}}\, \mathcal{R}(\boldsymbol{x},\boldsymbol{u}) |\chi\rangle_{\boldsymbol{x}} \otimes |\varnothing\rangle_{\boldsymbol{u}},$$

with

$$\sum_{j=1}^{n} |\boldsymbol{\lambda}^{(j)}| + k = N.$$

It can be rewritten as

$$\begin{aligned}
&{}_{\boldsymbol{u}}\langle\varnothing| a^{(1)}_{\boldsymbol{\lambda}^{(1)}} \dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}} \otimes {}_{\boldsymbol{x}}\langle\varnothing|\, J_k^{\boldsymbol{x}}\, \mathcal{R}(\boldsymbol{x},\boldsymbol{u}) |\chi\rangle_{\boldsymbol{x}} \otimes |\varnothing\rangle_{\boldsymbol{u}} = {}_{\boldsymbol{x}}\langle\varnothing| J_k^{\boldsymbol{x}} \mathcal{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)|\chi\rangle_{\boldsymbol{x}} = \\
&= \oint g_k(\boldsymbol{\xi}) F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\, {}_{\boldsymbol{x}}\langle\varnothing| f(\xi_1)...f(\xi_k)\, h(u_1) \underbrace{f(z_1^{(1)})\dots}_{|\boldsymbol{\lambda}^{(1)}|} h(u_2) \underbrace{f(z_1^{(2)})\dots}_{|\boldsymbol{\lambda}^{(2)}|} \dots\ h(u_n) \underbrace{f(z_1^{(n)})\dots}_{|\boldsymbol{\lambda}^{(n)}|} |\chi\rangle_{\boldsymbol{x}}\, d\boldsymbol{\xi}\, d\vec{\boldsymbol{z}} = \\
&= q^k \oint g_k(\boldsymbol{\xi}) F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u})\, {}_{\boldsymbol{x}}\langle\varnothing| h(u_1) \underbrace{f(z_1^{(1)})\dots}_{|\boldsymbol{\lambda}^{(1)}|} h(u_2) \underbrace{f(z_1^{(2)})\dots}_{|\boldsymbol{\lambda}^{(2)}|} \dots\ h(u_n) \underbrace{f(z_1^{(n)})\dots}_{|\boldsymbol{\lambda}^{(n)}|} f(\xi_1)...f(\xi_k)|\chi\rangle_{\boldsymbol{x}}\, d\boldsymbol{\xi}\, d\vec{\boldsymbol{z}},
\end{aligned} \tag{1.95}$$

which is equivalent to (1.91). In the first line in (1.95) we have used definition of $\mathcal{L}_{\boldsymbol{\lambda},\varnothing}(u)$, in the second line (1.94), (1.73) and definition (1.75), while in the third line we have used argument similar to the one in (1.82) that is dragging all $f(\xi_j)$'s to the right, abandoning local terms in commutation relations (1.40c) and (1.40f) and using the fact that all the factors (here $\boldsymbol{z}$ denotes the set of all ξ_j and $z_i^{(k)}$)

$$\prod_{z_j \neq z} \prod_{\alpha=1}^{3} \frac{z - z_j + \epsilon_\alpha}{z - z_j - \epsilon_\alpha} \prod_{k=1}^{n} \frac{z - u_k}{z - u_k + \epsilon_3}$$

are equal to q on Bethe ansatz equations (1.81).

1.4.5 Okounkov-Pandharipande equation

We saw that there are two related problems: diagonalization of KZ integral and solution of KZ difference equation. Both problems can be solved in terms of Bethe vector. Similarly to KZ case, both counterparts exists for local Integrals of Motion. Let us consider the following equation [OP10]:

$$\tilde{\boldsymbol{I}}_2(q)|\psi\rangle = \hbar q \frac{d}{dq}|\psi\rangle \tag{1.96}$$

We will show now that this equation is solved by the same wave function (1.83). In order to do that let us notice that while acting (1.83) differential operator $\hbar q \frac{d}{dq}$ is equal to multiplication on $\sum x_k$. The later can be expressed as an action of Integral of Motion in auxiliary space:

$$\sum_{k=1}^{N} x_k |B(\boldsymbol{x})\rangle_{\boldsymbol{u}} = \sum_{k=1}^{N} x_k \,{}_{\boldsymbol{x}}\langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}} \otimes |\varnothing\rangle_{\boldsymbol{u}} ={}_{\boldsymbol{x}} \langle\varnothing|\mathcal{R}(\boldsymbol{x},\boldsymbol{u})\tilde{\boldsymbol{I}}_2^{\boldsymbol{x}}(0)|\chi\rangle_{\boldsymbol{x}} \otimes |\varnothing\rangle_{\boldsymbol{u}}$$

Then we may repeat all the steps from previous section and find that under the integral over Bethe roots, action of $I_2^{\boldsymbol{x}}(0)$ on auxiliary space is equal to the action of $I_2^{\boldsymbol{v}}(q)$ on the quantum space. The only problematic point is number 4, let us explain it more details.

Let us consider the matrix element of the wave function:

$$\begin{aligned}{}_{\boldsymbol{u}}\langle\varnothing|a^{(1)}_{\boldsymbol{\lambda}^{(1)}}\dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}} \otimes {}_{\boldsymbol{x}}\langle\varnothing|\; \mathcal{R}(\boldsymbol{x},\boldsymbol{u})U_k^{\boldsymbol{x}}|\chi\rangle_{\boldsymbol{x}} \otimes |\varnothing\rangle_{\boldsymbol{u}} &={}_{\boldsymbol{x}}\langle\varnothing|\mathcal{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)U_k^{\boldsymbol{x}}|\chi\rangle_{\boldsymbol{x}} = \\ &= \mathrm{Sym}_{\boldsymbol{x}}\Big(\Omega_{\vec{\boldsymbol{\lambda}}}(x_1,...,x_{N-k}|\boldsymbol{u})g_k(x_{N-k+1},...,x_N)\prod_{i<j}S(x_i-x_j)\Big)\end{aligned}$$

On the other hand, if we insert U_k^x from the left we will have:

$$\begin{aligned}{}_{\boldsymbol{u}}\langle\varnothing|a^{(1)}_{\boldsymbol{\lambda}^{(1)}}\dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}} \otimes {}_{\boldsymbol{x}}\langle\varnothing|\; U_k^{\boldsymbol{x}}\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}} \otimes |\varnothing\rangle_{\boldsymbol{u}} &={}_{\boldsymbol{x}}\langle\varnothing|U_k^{\boldsymbol{x}}\mathcal{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)\dots\mathcal{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)|\chi\rangle_{\boldsymbol{x}} = \\ = \mathrm{Sym}_{\boldsymbol{x}}\Big(\mathcal{D}(x_1,...,x_k|x_{k+1},...x_N)\Omega_{\vec{\boldsymbol{\lambda}}}(x_1,...,x_{N-k}|\boldsymbol{u})&g_k(x_{N-k+1},...,x_N)\prod_{i<j}S(x_i-x_j)\Big),\end{aligned}$$

where

$$\mathcal{D}(\boldsymbol{x_i}|\boldsymbol{y_j}) = \prod_{i,j}\prod_{\alpha=1}^{3}\frac{x_i-y_j-\epsilon_\alpha}{x_i-y_j+\epsilon_\alpha}\prod_{i}\prod_{k=1}^{n}\frac{x_i-u_k+\epsilon_3}{x_i-u_k}.$$

As we explain in Appendix A.3 (see (A.24)), the function $g_k(\boldsymbol{x_i})$ is transnational invariant under the matrix element. And under the integral (1.83) we can freely perform simultaneous shift of all x_i for $i > N-k$. After this shift factor $\mathcal{D}(\boldsymbol{x_i}|\boldsymbol{y_j})$ will be canceled, and we arrive to the desired identity

$${}_{\boldsymbol{u}}\langle\varnothing|a^{(1)}_{\boldsymbol{\lambda}^{(1)}}\dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}} \otimes {}_{\boldsymbol{x}}\langle\varnothing|\; J_k^{\boldsymbol{x}}\mathcal{R}(\boldsymbol{x},\boldsymbol{u})|\chi\rangle_{\boldsymbol{x}} \otimes |\varnothing\rangle_{\boldsymbol{u}} \sim q^k \,{}_{\boldsymbol{u}}\langle\varnothing|a^{(1)}_{\boldsymbol{\lambda}^{(1)}}\dots a^{(n)}_{\boldsymbol{\lambda}^{(n)}} \otimes {}_{\boldsymbol{x}}\langle\varnothing|\,\mathcal{R}(\boldsymbol{x},\boldsymbol{u})J_k^{\boldsymbol{x}}|\chi\rangle_{\boldsymbol{x}} \otimes |\varnothing\rangle_{\boldsymbol{u}},$$

where the " $\sim$ " means equivalence under the integral (1.83). Other steps are completely similar to the ones in section 1.4.4. And we recover the equation (1.96).

1.4.6 Difference equations and norms of Bethe eigenvectors

Both KZ (1.84) and OP (1.96) equations has an extra parameter $\hbar$. It is clear intuitively that in the limit $\hbar \to 0$ the corresponding solution of KZ,OP equations turns into a Bethe vector. In this section we will develop this intuition, and show that it helps to derive a determinant formula for the norms of Bethe vectors [Sla89], [RV94]. In this section we will treat $\hbar$ to be purely imaginary. As we already discussed in section (1.4.3), solutions of KZ-OP (1.84), (1.96) equations on level N are labeled by collections of Young diagrams with N boxes. Let us compute the scalar product of two different solutions

$$\langle\psi_\lambda(\boldsymbol{u})|\psi_\mu(\boldsymbol{u})\rangle$$

As a consequence of KZ and OP equations this scalar product obeys

$$D_\hbar^{v_i}\langle\psi_\lambda(\boldsymbol{u})|\psi_\mu(\boldsymbol{u})\rangle = q\frac{d}{dq}\langle\psi_\lambda(\boldsymbol{u})|\psi_\mu(\boldsymbol{u})\rangle = 0,$$

and can be computed at the point $q = 0$:

$$\langle \psi_\lambda(\boldsymbol{u})|\psi_\mu(\boldsymbol{u})\rangle\Big|_{q=0} = \delta_{\lambda,\mu} C(N,\hbar). \tag{1.97}$$

We note that the constant $C(N,\hbar)$ is independent on the spectral parameters u_i and the twist q. Let us compute the integral (1.83) for general q , but in the limit $\hbar \to 0$. In this limit (1.83) can be taken by the saddle point method. Using

$$\Gamma\left(\frac{x}{\hbar}\right) = \sqrt{\frac{2\pi\hbar}{x}} e^{\frac{x}{\hbar}(\log(x)-1)-\frac{x}{\hbar}log(\hbar)} + o(\hbar),$$

we find that the integration kernel in (1.83) turns to the exponent of the Yang-Yang function:

$$|\psi_\lambda(\boldsymbol{u})\rangle = \oint_{C_\lambda} \frac{1}{\mathcal{F}(\boldsymbol{x})} e^{\frac{\mathcal{Y}(\boldsymbol{x})}{\hbar}} |B(\boldsymbol{x})\rangle_{\boldsymbol{u}} \frac{d^N \boldsymbol{x}}{(2\pi i)^N}, \tag{1.98}$$

where

$$\mathcal{Y}(\boldsymbol{x}) = \sum_{i\neq j} \Big[\omega(x_i - x_j - \epsilon_1) + \omega(x_i - x_j - \epsilon_2) - \omega(x_i - x_j + \epsilon_3) - \omega(x_i - x_j)\Big] + \\ + \sum_{i,k} (\omega(v_k - x_i) - \omega(v_k - x_i - \epsilon_3)) + \sum_i x_i \log(q), \quad \text{with} \quad \omega(x) = x(\log(x) - 1).$$

and

$$\mathcal{F}(\boldsymbol{x}) = \prod_{i\neq j} S(x_i - x_j) \prod_{i,k} \frac{u_k - x_i}{u_k - x_i - \epsilon_3}$$

Computing the integral (1.98) by saddle point, we found:

$$|\psi_\lambda(\boldsymbol{u})\rangle = e^{\frac{\mathcal{Y}_{crit}(\boldsymbol{x})}{\hbar}} \frac{\hbar^{\frac{N}{2}}}{\sqrt{\mathcal{F}(\boldsymbol{x}) H(\mathcal{Y})}} |B_\lambda(\boldsymbol{x})\rangle_{\boldsymbol{u}},$$

where $H(\mathcal{Y})$ is a Hessian:

$$H(\mathcal{Y}) = \det\left(\frac{\partial^2 \mathcal{Y}}{\partial x_i \partial x_j}\right)$$

Comparing to the (1.97), we immediately recover the Slavnov's determinant formula for the norms of on-shell Bethe vectors [Sla89]

$$\langle B_\lambda | B_\mu \rangle = C(N) \delta_{\lambda,\mu} H(\mathcal{Y}) \mathcal{F}(\boldsymbol{x}) \tag{1.99}$$

Where $C(N)$ is the limit $C(N) = \lim_{\hbar\to 0} C(N,\hbar)\hbar^{-N}$. The formula (1.99) can be rewritten in a different way:

$$\frac{\langle B_\lambda | B_\lambda \rangle_q}{H(\mathcal{Y})\mathcal{F}(\boldsymbol{x})} = \frac{\langle B_\lambda | B_\lambda \rangle_q}{H(\mathcal{Y})\mathcal{F}(\boldsymbol{x})}\Big|_{q=0}$$

1.5 Concluding remarks

This notes represents our efforts to understand the affine Yangian of $\mathfrak{gl}(1)$ and its role in integrability of conformal field theory. Many aspects have not been touched. Below we present some open problems and preliminary results that will be left for future work.

Other representations of the Yangian. As we have seen, the commutation relations of $\mathrm{YB}(\widehat{\mathfrak{gl}}(1))$ (1.40) are symmetric with respect to permutations of ϵ_k. It implies that the algebra $\mathrm{YB}(\widehat{\mathfrak{gl}}(1))$ admits three types of Fock modules $\mathcal{F}_u^{(k)}$ with $k=1,2,3$. Taking a representation of generic type

$$\mathcal{F}_{u_1}^{(k_1)} \otimes \mathcal{F}_{u_2}^{(k_2)} \otimes \cdots \otimes \mathcal{F}_{u_n}^{(k_n)},$$

will lead to ILW type integrable system corresponding to more general W algebras introduced in [BFM18, LS16]. The corresponding Miura transformation is explicitly known [PR19, Pro19]. All the results obtained in current notes can be generalized with a mild modification to this case. We collect some details in appendix A.4.

Massive deformation of ILW$_n$ integrable system. The twist deformation of CFT integrable system (1.13) leads to certain $\tau-$deformation of Toda action (1.3). Namely, for our choice of twist deformation (1.12), one exponent in (1.3) gets replaced by its non-local counterpart

$$e^{b(\varphi_2(x,t)-\varphi_1(x,t))} \xrightarrow{\text{twist deformation}} e^{b(\varphi_2(x,t)-\varphi_1(x+\pi\tau,t))}, \quad \text{where} \quad q=e^{i\pi\tau}$$

The corresponding classical field theory called *non-local* $\mathfrak{gl}(n)$ Toda field theory is known to be integrable in a Lax sense [DLO$^+$91, LOPZ91]. Its quantization has not been studied in the literature so far.

The simplest model of this kind is a free boson perturbed by a single exponent

$$S=\int\left(\frac{1}{8\pi}\partial_\mu\varphi\partial_\mu\varphi+\Lambda e^{b(\varphi(x,t)-\varphi(x+\pi\tau,t))}\right)d^2x, \tag{1.100}$$

This model has an interesting feature, in a finite volume of circumference $L=\pi n\tau$: $(x\sim x+L)$, relabeling the fields: $\phi(x+\pi k\tau)\overset{def}{=}\phi_k(x)$ we found that non local theory (1.100) in a volume $L=\pi n\tau$ is mapped to a local affine A_n Toda in volume τ. So it will be interesting to study the S matrix and the spectrum of theory (1.100) in finite volume.

Relation to quantum KP equation. Bethe anzatz equations similar to the ones studied in this notes were recently obtained by Kozlowski, Sklyanin, Torrielli in a slightly different context of quantization of the first Hamiltonian structure of KP equation [KST17]. We believe that our results may be relevant in this context.

ODE/IM correspondence. The spectrum of untwisted integrable systems (i.e. at $q=1$) can be studied by means of ODE/IM correspondence (see eg [DDT07] for review). In our approach, the $q\to 1$ limit could be taken, both approaches works well and could be checked numerically, however We do not know any transparent relation between them. In particular the transfer matrices which naturally arise in both approaches are quite different, while they could be related order by order in $1/u$ expansion, we were unable to find any closed formula. We note also that algebraic equations for the spectrum are rather different in two approaches. In Yangian approach one has BA equations (1.81), while on ODE/IM side the spectrum is given by Gaudin-like equations (see for example [BLZ04, Luk13, KL20]). It looks similar to the known duality between trigonometric Gaudin and rational XXX models [MTV08], however it has not been clarified yet (see discussions in [FJM17]). Finally let us mention that conjecturing an ODE/IM relations is always a piece of art, while in our approach Bethe equations (1.81) are derived, and could be easily generalised to more general models, for example despite the long history of ODE/IM correspondence the spectrum of Reflection operator (KZ operator T_1 in our terminology) was studied very recently [KL20].

Relation to "Tsymbaliuks" Affine Yangian As we mentioned in introduction, current algebra relations (1.40) are not the same as Affine Yangian relations (Y0-Y6) revisted in [Tsy17]. In particular, relations (1.40) include one extra Cartan current $h(u)$, this leads to the existence of an infinite dimensional center 1.3.2 which is not presents in Affine Yangian of [Tsy17]. However two algebras are very similar, we conjecture that the factor of $YB(\widehat{\mathfrak{gl}}(1))$ algebra over it's center will be isomorphic to the Affine Yangian of [Tsy17].

Yangian Double. The algebra called Yangian Double has been introduced in [KT96] following Drinfeld's quantum double construction [Dri88]. The Yangian Double seems to be more appropriate for construction of Bethe vectors by the so called "method of projections" developed in [EKP07, KPT07] (see [HLP$^+$17, LPRS19] for latest results). Unfortunately, we were unable to repeat the procedure executed in [HLP$^+$17, LPRS19] and define the off-shell Bethe vector as a projection of a state build of "total" currents. This is an interesting open problem.

Chapter 2

BCD **integrable structures and boundary Bethe ansatz.**

In this chapter we study integrable structures of conformal field theory with BCD symmetry. We realise these integrable structures as $\mathfrak{gl}(1)$ affine Yangian "spin chains" with boundaries. We provide three solutions of Sklyanin's KRKR equation compatible with the affine Yangian R-matrix and derive Bethe ansatz equations for the spectrum. Our analysis provides a unified approach to the integrable structures with BCD symmetry including superalgebras.

2.1 Introduction

In current notes we generalize the results and the methods of previous chapter (see also [LV20]) to the W-algebras of BCD type. The key new ingredient, which appears in this case is the analog of Sklyanin's $K-$matrix [Skl88], introduced by him for studying of spin chains with boundary. The "boundary" in the current context corresponds to the endpoints of the affine Dynkin diagram for a given integrable system. This fact has been already noticed and studied in trigonometric case in [FJMV21]. Here we restrict ourselves to the conformal case, but consider the problem of diagonalization of Integrals of Motion. Similar to the A case [LV20], it is convenient to diagonalize KZ Integrals of Motion (called reflection operators in [KL20]) rather that local ones. We explicitly construct the off-shell Bethe vector, which depends on auxiliary parameters $x_1, \dots, x_N$, where N is the level, and show that the KZ operator acts diagonally on this vector provided that x_k's satisfy Bethe ansatz equations. These equations (formula (2.27)) together with the explicit form of off-shell Bethe vector (formula (2.25)) constitute the main results of the current chapter.

2.2 Integrable systems of BCD type in CFT

The integrable systems studied in this chapter can be realized by the $n-$component bosonic free field $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$. Local Integrals of Motion have the following general form

$$\mathbf{I}_s = \frac{1}{2\pi}\int_0^{2\pi} G_{s+1}(z)dz, \qquad \bar{\mathbf{I}}_s = \frac{1}{2\pi}\int_0^{2\pi} \bar{G}_{s+1}(\bar{z})d\bar{z}, \tag{2.1}$$

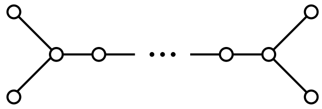
where $G_{s+1}(z)$ and $\bar{G}_{s+1}(\bar{z})$ are the local densities with the spins s belonging to some set, which is a characteristic property of a particular integrable system. The important property of local IM's is that they form the commutative set

$$[\mathbf{I}_r, \mathbf{I}_s] = 0.$$

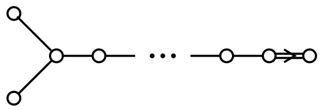
The best way to describe our integrable systems goes through affine Toda QFT

$$S = \int \Big(\frac{1}{4\pi}(\partial_a \boldsymbol{\varphi} \cdot \partial_a \boldsymbol{\varphi}) + \Lambda \sum_{r=0}^{n} e^{b(\boldsymbol{\alpha}_r \cdot \boldsymbol{\varphi})}\Big) d^2 z. \tag{2.2}$$

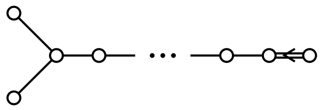
where the vectors $(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n)$ have the Gram matrix corresponding to one of the affine Dynkin diagrams of BCD type:

$\widehat{\mathrm{D}}_n$ 

$\widehat{\mathrm{C}}_n$ 

$\widehat{\mathrm{B}}_n$ 

$\widehat{\mathrm{C}}_n^\vee$ 

$\widehat{\mathrm{B}}_n^\vee$ 

$\widehat{\mathrm{BC}}_n$ 

and b is the coupling constant. Using the standard parametrization for the roots one can express the scalar products in the exponents in (2.2) as

$$(\boldsymbol{\alpha}_0 \cdot \boldsymbol{\varphi}) = \begin{cases} -\varphi_1 \\ -2\varphi_1 \\ -\varphi_1 - \varphi_2 \end{cases} \qquad (\boldsymbol{\alpha}_r \cdot \boldsymbol{\varphi}) = \varphi_r - \varphi_{r+1} \quad \text{for} \quad 0 < r < n, \quad (\boldsymbol{\alpha}_n \cdot \boldsymbol{\varphi}) = \begin{cases} \varphi_n \\ 2\varphi_n \\ \varphi_{n-1} + \varphi_n \end{cases} \tag{2.3}$$

That is each of the affine diagrams can be interpreted as non-affine A_{n-1} diagram with two boundary conditions which can be of three types B, C or D corresponding to the short root, the long root or the root of the length $\sqrt{2}$ correspondingly.

The theories (2.2) are known to be integrable both classically and quantum mechanically. They share an interesting property of the duality (see e.g. [Cor94]). Namely, both $\widehat{\mathrm{D}}_n$ and $\widehat{\mathrm{BC}}_n$ theories are self-dual with respect to the substitution $b \to b^{-1}$, while $\widehat{\mathrm{B}}_n$ and $\widehat{\mathrm{B}}_n^\vee$ as well as $\widehat{\mathrm{C}}_n$ and $\widehat{\mathrm{C}}_n^\vee$ are mapped to each other. The quantum integrability implies that the theory admits the set of local Integrals of Motion whose short distance limit coincides with $\mathbf{I}_s$ and $\bar{\mathbf{I}}_s$ from (2.1).

The integrals $\mathbf{I}_s$ and $\bar{\mathbf{I}}_s$ by themselves can be defined up to a total factor from the equation (and similar antiholomorphic equation)

$$\frac{1}{2\pi i} \oint_{\mathcal{C}_z} e^{b(\boldsymbol{\alpha}_r \cdot \boldsymbol{\varphi}(\xi))} G_{s+1}(z) d\xi = \partial V_s(z), \tag{2.4}$$

where $V_s(z)$ is some local field (and similar formula for $\bar{G}_{s+1}$). Using (2.4) one can construct first few local IM's explicitly. It is convenient to write them in Nekrasov epsilon notations[1]

$$b = \sqrt{\frac{\epsilon_2}{\epsilon_1}}, \qquad b^{-1} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \quad \text{and} \quad \epsilon_3 \overset{\text{def}}{=} -\epsilon_1 - \epsilon_2.$$

[1]The answer will depends only on the ratio $\frac{\epsilon_1}{\epsilon_2}$, so without loss of generality we may assume $\epsilon_1\epsilon_2 = 1$ and thus

$$b = \epsilon_2, \qquad b^{-1} = \epsilon_1 \quad \text{and} \quad Q = b + \frac{1}{b} = -\epsilon_3.$$

The first non-trivial local Integral of Motion is $\mathbf{I}_3$ and the corresponding Wick ordered density has the form

$$
\begin{aligned}
G_4(z) = & \big(\partial\boldsymbol{\varphi}\cdot\partial\boldsymbol{\varphi}\big)^2 - \frac{1}{3}\left(2n - \frac{\epsilon_\alpha+\epsilon_\beta}{\epsilon_3}\right)\sum_{k=1}^{n}\big(\partial\varphi_k\big)^4 + \\
& + \frac{4\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}\sum_{k=1}^{n}\partial\varphi_k^2\left(\sum_{j<k}\Big(j-1+\frac{\epsilon_3-\epsilon_\alpha}{2\epsilon_3}\Big)\partial^2\varphi_j - \sum_{j>k}\Big(n-j+\frac{\epsilon_3-\epsilon_\beta}{2\epsilon_3}\Big)\partial^2\varphi_j\right) + \\
& + \left(2n + \frac{4(n-1)(\epsilon_1^2+\epsilon_2^2)}{3\epsilon_1\epsilon_2} + \frac{(\epsilon_1\epsilon_2-2\epsilon_3^2)(\epsilon_\alpha+\epsilon_\beta-2\epsilon_3)}{3\epsilon_1\epsilon_2\epsilon_3}\right)\big(\partial^2\boldsymbol{\varphi}\cdot\partial^2\boldsymbol{\varphi}\big) - \\
& - \frac{4\epsilon_3^2}{\epsilon_1\epsilon_2}\sum_{i\leq j}\left(i-1+\frac{\epsilon_3-\epsilon_\alpha}{2\epsilon_3}\right)\left(n-j+\frac{\epsilon_3-\epsilon_\beta}{2\epsilon_3}\right)(2-\delta_{ij})\partial^2\varphi_i\partial^2\varphi_j, \quad (2.5)
\end{aligned}
$$

where each of the indexes α and β takes the values 1, 2 and 3, corresponding to either B, C or D boundary conditions. And φ_k is the free bosonic field

$$\partial\varphi_k(x) = -i\frac{u_k}{\sqrt{\epsilon_1\epsilon_2}} + \sum_{n\neq 0} a_n^{(k)}e^{-inx}, \qquad [a_m^i, a_n^j] = m\delta_{m,-n}\delta_{i,j}. \quad (2.6)$$

We stress that in general the solution to the commutativity equation (2.4) should be searched in terms of analytically regularized densities rather that Wick ordered ones. In the case of the density of spin 4 these two differ by an amount which is by itself an Integral of Motion. In general this is not the case and starting from the spin 6 one expects to have corrections to the Wick ordered density, which formally correspond to lower spins (see section 2.6 for the example).

2.3 Maulik-Okounkov R-matrix, K-matrix

The Maulik-Okounkov $R-$matrix is related to the Liouville reflection operator [ZZ96] as

$$\mathcal{R}_{ij} = \mathcal{R}[\partial\varphi_i - \partial\varphi_j]. \quad (2.7)$$

We will use both notations (2.7) interchangeably. Sometimes it may also be convenient to use the notation $\mathcal{R}_{i,j}(u_i-u_j)$ in order to emphasise the value of the zero mode (see (2.6)).

This reflection operator can be defined up to a normalisation factor from the condition ($Q = \frac{\epsilon_1+\epsilon_2}{\sqrt{\epsilon_1\epsilon_2}}$)

$$\mathcal{R}[\partial\varphi_i - \partial\varphi_j](Q\partial-\partial\varphi_i)(Q\partial-\partial\varphi_j) = (Q\partial-\partial\varphi_j)(Q\partial-\partial\varphi_i)\mathcal{R}[\partial\varphi_i - \partial\varphi_j], \quad (2.8)$$

In order to introduce the K-operator, we consider rank two W-algebras of BCD type. They can be defined as commutants of screening operators (here $b = \frac{\epsilon_2}{\sqrt{\epsilon_1\epsilon_2}}$)

$$\mathcal{S}_1 = \int e^{b(\varphi_1-\varphi_2)}dz, \qquad \mathcal{S}_2 = \begin{cases} \int e^{b\varphi_2}dz & \text{for B,} \\ \int e^{2b\varphi_2}dz & \text{for C,} \\ \int e^{b(\varphi_1+\varphi_2)}dz & \text{for D.} \end{cases} \quad (2.9)$$

The corresponding holomorphic currents $W^{(2)}$ and $W^{(4)}$ have the explicit form

$$W^{(2)} = (\partial\varphi_1)^2 + (\partial\varphi_2)^2 + \frac{2\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}\partial^2\varphi_1 + \frac{\epsilon_3-\epsilon_\alpha}{\sqrt{\epsilon_1\epsilon_2}}(\partial^2\varphi_2 + \partial^2\varphi_1)$$

and

$$W^{(4)} = (\partial\varphi_1)^2(\partial\varphi_2)^2 + \frac{2\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}\partial\varphi_1\partial\varphi_2\partial^2\varphi_2 + \frac{\epsilon_3-\epsilon_\alpha}{\sqrt{\epsilon_1\epsilon_2}}\left((\partial\varphi_1)^2\partial^2\varphi_2 + (\partial\varphi_2)^2\partial^2\varphi_1\right) -$$
$$- \frac{\epsilon_3\epsilon_\alpha}{\epsilon_1\epsilon_2}(\partial^2\varphi_1)^2 + \frac{(\epsilon_3-\epsilon_\alpha)^2}{\epsilon_1\epsilon_2}\partial^2\varphi_1\partial^2\varphi_2 - \frac{(\epsilon_1-\epsilon_\alpha)(\epsilon_2-\epsilon_\alpha)}{2\epsilon_1\epsilon_2}\left(\partial\varphi_1\partial^3\varphi_1 + \partial\varphi_2\partial^3\varphi_2\right) -$$
$$- \frac{\epsilon_3(\epsilon_3-\epsilon_\alpha)}{\epsilon_1\epsilon_2}\left(\partial\varphi_1\partial^3\varphi_1 - \partial\varphi_1\partial^3\varphi_2\right) + \frac{\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}}\left(\frac{\epsilon_\alpha(\epsilon_3-\epsilon_\alpha)}{2\epsilon_1\epsilon_2} - \frac{\epsilon_3^2}{\epsilon_1\epsilon_2} - \frac{1}{3}\right)\partial^4\varphi_1$$

where $\alpha = 1,2,3$ correspond to B, C and D W-algebras correspondingly.

Each screening operator (2.9) generates the reflection operator according to the rule

$$\mathcal{R}_{1,2}W^{(s)} = W^{(s)}\Big|_{\varphi_1\leftrightarrow\varphi_2}\mathcal{R}_{1,2}, \qquad \mathcal{K}_2W^{(s)} = W^{(s)}\Big|_{\varphi_2\to-\varphi_2}\mathcal{K}_2, \tag{2.10}$$

for $s = 2,4$. We have $\mathcal{R}_{1,2} = \mathcal{R}[\partial\varphi_1 - \partial\varphi_2]$, while $\mathcal{K}_2$ is also equal to the reflection operator of the re-scaled argument

$$\mathcal{K}_2^1 = \mathcal{R}[\sqrt{2}\partial\varphi_2]\Big|_{\epsilon_1\to\sqrt{2}\epsilon_1,\epsilon_2\to\epsilon_2/\sqrt{2}} \quad \text{for B series} \tag{2.11}$$
$$\mathcal{K}_2^2 = \mathcal{R}[\sqrt{2}\partial\varphi_2]\Big|_{\epsilon_1\to\epsilon_1/\sqrt{2},\epsilon_2\to\sqrt{2}\epsilon_2} \quad \text{for C series}$$
$$\mathcal{K}_2^3 = \mathrm{Id} \quad \text{for D series} \tag{2.12}$$

Note that $\mathcal{K}_2^3 = \mathrm{Id}$ is the simplest among the operators, as it does not depend on spectral parameter and has very simple action on bosons.

Now, similar to the argument of Maulik and Okounkov, the K-operator obeys Sklyanin's KRKR equation[2]

$$\mathcal{R}[\partial\varphi_1 - \partial\varphi_2]\mathcal{K}_1^\alpha\mathcal{R}[\partial\varphi_1 + \partial\varphi_2]\mathcal{K}_2^\alpha = \mathcal{K}_2^\alpha\mathcal{R}[\partial\varphi_1 + \partial\varphi_2]\mathcal{K}_1^\alpha\mathcal{R}[\partial\varphi_1 - \partial\varphi_2]. \tag{2.13}$$

It is interesting to note that $\mathcal{K}^1$, $\mathcal{K}^2$ and $\mathcal{K}^3$ seem to exhaust all solutions to KRKR equation (2.13) which preserve the grading operator $\int W_2 dz$. This is an unproven statement, confirmed by explicit calculations on lower levels.

2.3.1 KZ integrals of motion.

Having defined $R-$ and $K-$operators, one can define the important family of IOM's constructed from two solutions of KRKR equation – the so called KZ Integrals of Motion. Let us introduce the following operators:

$$\mathcal{T}_i^+ = \mathcal{R}_{i,\overline{i+1}}\ldots\mathcal{R}_{i,\bar{n}}\mathcal{K}_i^\alpha\mathcal{R}_{i,n}\ldots\mathcal{R}_{i,i+1},$$
$$\mathcal{T}_i^- = \mathcal{R}_{i,1}\ldots\mathcal{R}_{i,i-1}\mathcal{K}_i^\beta\mathcal{R}_{1,\bar{i}}\ldots\mathcal{R}_{i-1,\bar{i}},$$
$$\mathcal{I}_i^{\mathrm{KZ}} = \mathcal{T}_i^-\mathcal{T}_i^+ \tag{2.14}$$

[2] Let us note that originally [Skl88] the KRKR equation was written in a quite different form:

$$\mathcal{R}_{1,2}(u_1-u_2)\tilde{\mathcal{K}}_1(u_1)\mathcal{R}_{2,1}(u_2+u_1)\tilde{\mathcal{K}}_2(u_2) = \tilde{\mathcal{K}}_2(u_2)\mathcal{R}_{1,2}(u_1+u_2)\tilde{\mathcal{K}}_1(u_1)\mathcal{R}_{2,1}(u_1-u_2).$$

The difference is actually insufficient as the two equations are differ by the redefinition of K-operator and overall conjugation by the reflection of bosonic modes $a_n^{1,2} \to -a_n^{1,2}$, $n \neq 0$

where we defined the conjugation operator D_i, and the bared index $\bar{i}$ means the conjugatoin by the D_i

$$D_i f(\boldsymbol{\varphi}) = f(\boldsymbol{\varphi})\Big|_{\varphi_i \to -\varphi_i} D_i,$$
$$\mathcal{R}_{i,\bar{j}} = D_j \mathcal{R}_{i,j} D_j = \mathcal{R}[\partial\varphi_i + \partial\varphi_j],$$
$$\mathcal{R}_{\bar{i},j} = D_i \mathcal{R}_{i,j} D_i = \mathcal{R}[-\partial\varphi_i - \partial\varphi_j],$$

Using KRKR equation (2.13), it is straightforward to check that

$$[\mathcal{I}_i^{\mathrm{KZ}}, \mathcal{I}_j^{\mathrm{KZ}}] = 0.$$

It is also possible to prove the commutativity of KZ Integrals of Motion and local ones. Indeed, any screening operator S_α acts non-trivially only in two (or one at the endpoints) spaces. In order to point it out we will equip it with the label i, such that S_{α_i} acts in the space of two bosons $\mathcal{F}_i \otimes \mathcal{F}_{i+1}$ for $i \neq 0, n$, while S_{α_0} and S_{α_n} acts only on first and the last boson correspondingly (see (2.3)). Now, from the very definition of the reflection operators (2.10) any operator $\mathcal{O}_i$ which commutes with S_{α_i} has a nice intertwining property with the reflection operators

$$\mathcal{R}_{i,i+1}\mathcal{O}_i = \mathcal{O}_i\Big|_{\varphi_i \leftrightarrow \varphi_{i+1}} \mathcal{R}_{i,i+1}\,, \quad i = 1\ldots n-1$$
$$\mathcal{K}_i\mathcal{O}_i = \mathcal{O}_i\Big|_{\varphi_i \to -\varphi_i} \mathcal{K}_i\,, \quad i = 0, n.$$

As local IM's commute with all screening operators, they nicely intertwine with both $\mathcal{T}^-$ and $\mathcal{T}^+$

$$\mathcal{T}_i^+ \mathbf{I}_s = \mathbf{I}_s\Big|_{\varphi_i \to -\varphi_i} \mathcal{T}_i^+, \qquad \mathcal{T}_i^- \,\mathbf{I}_s\Big|_{\varphi_i \to -\varphi_i} = \mathbf{I}_s\, \mathcal{T}_i^-, \tag{2.15}$$

which proves the commutativity $[\mathbf{I}_s, \mathcal{I}_i^{\mathrm{KZ}}] = 0$.

2.3.2 Review of the RLL algebra $\mathrm{YB}(\widehat{\mathfrak{gl}}_1)$

Let us remind the basic properties of RLL algebra and its equivalent description in terms of generating currents h, e and f (for more details see [LV20]).

The Maulik-Okounkov R-matrix defines the Yang-Baxter algebra $(\mathrm{YB}(\widehat{\mathfrak{gl}}(1)))$ in the standard way

$$\mathcal{R}_{ij}(u-v)\mathcal{L}_i(u)\mathcal{L}_j(v) = \mathcal{L}_j(v)\mathcal{L}_i(u)\mathcal{R}_{ij}(u-v).$$

Here $\mathcal{L}_i(u)$ is treated as an operator in some quantum space, a tensor product of n Fock spaces in our case, and as a matrix in the auxiliary Fock space $\mathcal{F}_u$. The algebra (**??**) becomes an infinite set of quadratic relations between the matrix elements labeled by two partitions

$$\mathcal{L}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(u) \overset{\mathrm{def}}{=} \langle u|a_{\boldsymbol{\lambda}}\mathcal{L}(u)a_{-\boldsymbol{\mu}}|u\rangle \quad \text{where} \quad a_{-\boldsymbol{\mu}}|u\rangle = a_{-\mu_1}a_{-\mu_2}\ldots|u\rangle.$$

Let us introduce three basic currents of degree 0, 1 and -1

$$h(u) \overset{\mathrm{def}}{=} \mathcal{L}_{\varnothing,\varnothing}(u), \qquad e(u) \overset{\mathrm{def}}{=} h^{-1}(u)\cdot\mathcal{L}_{\varnothing,\square}(u) \quad \text{and} \quad f(u) \overset{\mathrm{def}}{=} \mathcal{L}_{\square,\varnothing}(u)\cdot h^{-1}(u), \tag{2.16}$$

as well as an auxiliary current (as we will see (2.18a) it also belongs to the Cartan subalgebra of $\mathrm{YB}(\widehat{\mathfrak{gl}}(1))$)

$$\psi(u) \overset{\mathrm{def}}{=} \Big(\mathcal{L}_{\square,\square}(u+\epsilon_3) - \mathcal{L}_{\varnothing,\square}(u+\epsilon_3)h^{-1}(u+\epsilon_3)\mathcal{L}_{\square,\varnothing}(u+\epsilon_3)\Big)h^{-1}(u+\epsilon_3) \tag{2.17}$$

As follows from definition of the R-matrix these currents admit large u expansion

$$h(u) = 1 + \frac{h_0}{u} + \frac{h_1}{u^2} + \dots, \quad e(u) = \frac{e_0}{u} + \frac{e_1}{u^2} + \dots, \quad f(u) = \frac{f_0}{u} + \frac{f_1}{u^2} + \dots, \quad \psi(u) = 1 + \frac{\psi_0}{u} + \frac{\psi_1}{u^2} + \dots$$

Using the definition (2.16) and (2.17) and explicit expression for the R-matrix on first three levels one can prove [LV20] the following relations

$$[h(u), \psi(v)] = 0, \quad [\psi(u), \psi(v)] = 0, \quad [h(u), h(v)] = 0, \tag{2.18a}$$

$$\begin{aligned} (u - v - \epsilon_3) h(u) e(v) &= (u - v) e(v) h(u) - \epsilon_3 h(u) e(u), \\ (u - v - \epsilon_3) f(v) h(u) &= (u - v) h(u) f(v) - \epsilon_3 f(u) h(u), \end{aligned} \tag{2.18b}$$

$$[e(u), f(v)] = \frac{\psi(u) - \psi(v)}{u - v}, \tag{2.18c}$$

as well as ee, ff relations

$$\begin{aligned} g(u-v)\Big[e(u)e(v) - \frac{e_{\square\square}(v)}{u - v + \epsilon_1} - \frac{e_{\substack{\square\\\square}}(v)}{u - v + \epsilon_2} - \frac{e_{\square\!\diagup\!\square}(v)}{u - v + \epsilon_3}\Big] = \\ = \bar{g}(u-v)\Big[e(v)e(u) - \frac{e_{\square\square}(u)}{u - v - \epsilon_1} - \frac{e_{\substack{\square\\\square}}(u)}{u - v - \epsilon_2} - \frac{e_{\square\!\diagup\!\square}(u)}{u - v - \epsilon_3}\Big], \end{aligned}$$

$$\begin{aligned} \bar{g}(u-v)\Big[f(u)f(v) - \frac{f_{\square\square}(v)}{u - v - \epsilon_1} - \frac{f_{\substack{\square\\\square}}(v)}{u - v - \epsilon_2} - \frac{f_{\square\!\diagup\!\square}(v)}{u - v - \epsilon_3}\Big] = \\ = g(u-v)\Big[f(v)f(u) - \frac{f_{\square\square}(u)}{u - v + \epsilon_1} - \frac{f_{\substack{\square\\\square}}(u)}{u - v + \epsilon_2} - \frac{f_{\square\!\diagup\!\square}(u)}{u - v + \epsilon_3}\Big], \end{aligned} \tag{2.18d}$$

ψe, ψf relations

$$\begin{aligned} g(u-v)\psi(u)e(v) &= \bar{g}(u-v)e(v)\psi(u) + \text{locals}, \\ g(u-v)f(v)\psi(u) &= \bar{g}(u-v)\psi(u)f(v) + \text{locals}, \end{aligned} \tag{2.18e}$$

and Serre relations

$$\begin{aligned} \sum_{\sigma \in \mathbb{S}_3} (u_{\sigma_1} - 2u_{\sigma_2} + u_{\sigma_3}) e(u_{\sigma_1}) e(u_{\sigma_2}) e(u_{\sigma_3}) + \sum_{\sigma \in \mathbb{S}_3} [e(u_{\sigma_1}), e_{\square\!\diagup\!\square}(u_{\sigma_2}) + e_{\square\square}(u_{\sigma_2}) + e_{\substack{\square\\\square}}(u_{\sigma_2})] = 0, \\ \sum_{\sigma \in \mathbb{S}_3} (u_{\sigma_1} - 2u_{\sigma_2} + u_{\sigma_3}) f(u_{\sigma_1}) f(u_{\sigma_2}) f(u_{\sigma_3}) + \sum_{\sigma \in \mathbb{S}_3} [f(u_{\sigma_1}), f_{\square\!\diagup\!\square}(u_{\sigma_2}) + f_{\square\square}(u_{\sigma_2}) + f_{\substack{\square\\\square}}(u_{\sigma_2})] = 0. \end{aligned} \tag{2.18f}$$

In the relations above we have used the following notations

$$g(x) \overset{\text{def}}{=} (x + \epsilon_1)(x + \epsilon_2)(x + \epsilon_3), \quad \bar{g}(x) \overset{\text{def}}{=} (x - \epsilon_1)(x - \epsilon_2)(x - \epsilon_3).$$

We note that the terms shown by blue in (2.18c)-(2.18e) depend only on one parameter either u or v (in (2.18e) these terms are so complicated, that we do no write them explicitly) and in (2.18f) they depend only on two parameters instead of three. We call such terms *local*and use shorthand notation *locals* in formulas instead of writing them explicitly. The main idea is that they always can be omitted in actual computations, as we only interested in relations between modes of currents the local terms always will stand inside some contour integral, it turns out that the integration contour always can be chosen in a way to exclude local terms.

Now let us describe the inverse map from the Borel sub-algebra of RLL algebra to the currents. We introduce the modes U_n of $W^{(1)}(z)$ current

$$\langle \varnothing | \mathcal{L}(u) \; a_{-n}^{(0)} | \varnothing \rangle = \frac{U_n}{u} + O\left(\frac{1}{u^2}\right), \quad n > 0$$

$$\langle \varnothing | a_n^{(0)} \; \mathcal{L}(u) | \varnothing \rangle = \frac{U_{-n}}{u} + O\left(\frac{1}{u^2}\right), \quad n > 0$$

It is clear from the RLL relation that the $R-$matrix commutes with the $W^{(1)}$ current:

$$(a_n^{(0)} + U_n)\mathcal{R}_{0,v} = \mathcal{R}_{0,v}(a_n^{(0)} + U_n)$$

Taking the matrix element over the auxiliary space $\langle\varnothing|\ldots|\mu\rangle$ for positive n we will get:

$$[\mathcal{L}_{\boldsymbol{\mu},\varnothing}(u), U_n] = \mathcal{L}_{\boldsymbol{\mu}+n,\varnothing}(u), \tag{2.19}$$

where $\langle\boldsymbol{\mu}+n| \stackrel{\text{def}}{=} \langle\boldsymbol{\mu}|a_n$.

It is also clear, that U_n for $n > 0$ belongs to the subalgebra $\mathfrak{n}^+$. Indeed, explicit calculation of the large u limit of $\mathcal{R}(u)$ (see [LV20] for the details) shows that:

$$U_1 = f_0 \quad U_{-1} = e_0,$$
$$U_{k+1} = -k[f_1, U_k] \quad U_{k-1} = -k[e_1, U_k].$$

Then we get:

$$U_k^{\boldsymbol{x}} = \oint \cdots \oint g_k(\boldsymbol{z}) f(z_1)...f(z_k) d\boldsymbol{z} \quad \text{with} \quad g_{k+1}(\boldsymbol{z}) = -k\Big(z_1 g_k(z_2,\ldots,z_{k+1}) - g_n(z_1,\ldots,z_k)z_{k+1}\Big),$$

and

$$g_k(\boldsymbol{z}) = (-1)^{k-1}(k-1)! \prod_i z_i \left(\sum (-1)^i C_k^i z_i^{-1}\right),$$

where C_n^i are the binomial coefficients.

Finally using (2.19) we may express $\mathcal{L}_{\boldsymbol{\lambda},\varnothing}(u)$ as a multiple commutator of $\mathcal{L}_{\varnothing,\varnothing}(u) = h(u)$ and modes of $f(z)$ currents, or equivalently as contour integral

$$\mathcal{L}_{\boldsymbol{\lambda},\varnothing}(u) = \frac{1}{(2\pi i)^{|\lambda|}} \oint \cdots \oint F_{\boldsymbol{\lambda}}(\boldsymbol{z}|u)\, h(u) f(z_{|\boldsymbol{\lambda}|}) \ldots f(z_1) dz_1 \ldots, dz_{|\boldsymbol{\lambda}|} \tag{2.20}$$

with some explicit function $F_\lambda(\boldsymbol{z}|u)$.

2.3.3 Antipode

As we will see there is an important operation: the reflection of the boson $\varphi(x) \to -\varphi(x)$. Using it we define the antipode of L-operator:

$$(\mathcal{L}_{\boldsymbol{\mu},\boldsymbol{\nu}}(u))^a \stackrel{\text{def}}{=} \bar{\mathcal{L}}_{\boldsymbol{\mu},\boldsymbol{\nu}}(u) = (-1)^{l(\boldsymbol{\mu})+l(\boldsymbol{\nu})}\mathcal{L}(-u)_{\boldsymbol{\mu},\boldsymbol{\nu}},$$
$$\overline{\mathcal{L}(u)\mathcal{L}(v)} \stackrel{\text{def}}{=} \bar{\mathcal{L}}(v)\bar{\mathcal{L}}(u).$$

Here $l(\boldsymbol{\mu})$ is the number of rows in Young diagram $\boldsymbol{\mu}$[3].

It is convenient to write the conjugated L operator as follows:

$$\bar{\mathcal{L}}_{\boldsymbol{\lambda},\varnothing}(u) = \frac{1}{(2\pi i)^{|\boldsymbol{\lambda}|}} \oint \cdots \oint F_{\boldsymbol{\lambda}}(\boldsymbol{z}|u)\, f(-\epsilon_3 - z_{|\lambda|}) \ldots f(-\epsilon_3 - z_1) h(-u) dz_1 \ldots dz_{|\lambda|}$$

[3]Note that if we think of the diagram as of the bosonic state: $|\lambda\rangle = \prod_{i=1}^{l(\lambda)} a_{-\lambda_i}|\varnothing\rangle$, then multiplication by $(-1)^\lambda$ is nothing but the reflection of the bosons $a_{-n} \to -a_{-n}$.

2.4 Off-shell Bethe vector

In order to construct the off-shell Bethe vector we consider the tensor product of $n+N$ Fock spaces

$$\underbrace{\mathcal{F}_{u_n}\otimes\cdots\otimes\mathcal{F}_{u_1}}_{\text{quantum space}}\otimes\underbrace{\mathcal{F}_{x_1}\otimes\cdots\otimes\mathcal{F}_{x_N}}_{\text{auxiliary space}}=\mathcal{F}_{\boldsymbol{u}}\otimes F_{\boldsymbol{x}}$$

generated from the vacuum state

$$|\varnothing\rangle_{\boldsymbol{u}}\otimes|\varnothing\rangle_{\boldsymbol{x}}=|u_n\rangle\otimes\cdots\otimes|u_1\rangle\otimes|x_1\rangle\otimes\cdots\otimes|x_N\rangle.$$

In order not to confuse between the auxiliary and quantum Fock spaces, we will label Fock space not by it's index, but by it's spectral parameter. So that the R-matrix between two Fock spaces will read as $\mathcal{R}_{u_i,u_j}$ while the R-matrix between two auxiliary spaces as $\mathcal{R}_{x_i,x_j}$.

As usual let us introduce $\mathcal{L}_i(u_i)$ operators:

$$\mathcal{L}_i(u_i)=\mathcal{R}_{u_i,\boldsymbol{x}}=\mathcal{R}_{u_i,x_1}\dots\mathcal{R}_{u_i,x_N},\qquad \mathcal{L}_{\boldsymbol{u}}=\mathcal{L}_n(u_n)\dots\mathcal{L}_1(u_1). \tag{2.21}$$

It is also convenient to define opposite $\bar{\mathcal{L}}$ operators:

$$\bar{\mathcal{L}}_i(u_i)=\mathcal{R}_{\bar{u}_i,\boldsymbol{x}}=\mathcal{R}_{\bar{u}_i,x_N}\dots\mathcal{R}_{\bar{u}_i,x_1},\qquad \bar{\mathcal{L}}_{\boldsymbol{u}}=\bar{\mathcal{L}}_1(u_1)\dots\bar{\mathcal{L}}_n(u_n) \tag{2.22}$$

2.4.1 K operators

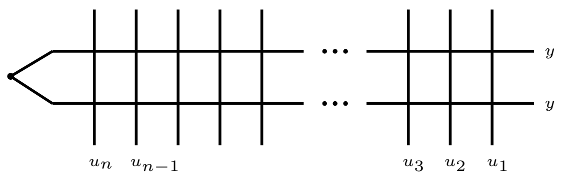
In the previous section we have defined the K-matrix acting on the single Fock space (2.10). It is useful to extend its action to the tensor product of quantum and auxiliary Fock spaces. Let us define

$$\mathcal{K}_{\boldsymbol{u}|y}\overset{\text{def}}{=}\mathcal{R}_{\overline{\boldsymbol{u}},y}\mathcal{K}_y\mathcal{R}_{\boldsymbol{u},y}, \tag{2.23}$$

where

$$\mathcal{R}_{\boldsymbol{u},y}=\mathcal{R}_{u_n,y}\dots\mathcal{R}_{u_1,y}\,,\quad \mathcal{R}_{v,\boldsymbol{x}}=\mathcal{R}_{v,x_N}\dots\mathcal{R}_{v,x_1}$$

and $\mathcal{K}_y$ is the operator defined in (2.10). The definition (2.23) is the direct analog of L-operator to the boundary case. This definition can be conveniently illustrated with the following picture

$\mathcal{K}_{\boldsymbol{u}|y}=$ 

We note that $\mathcal{K}_{\boldsymbol{u}|x_1}$ still enjoys KRKR equation (2.13)

$$\mathcal{R}_{x_1,x_2}\mathcal{K}_{\boldsymbol{u}|x_1}\mathcal{R}_{x_1,\bar{x}_2}\mathcal{K}_{\boldsymbol{u}|x_2}=\mathcal{K}_{\boldsymbol{u}|x_2}\mathcal{R}_{x_1,\bar{x}_2}\mathcal{K}_{\boldsymbol{u}|x_1}\mathcal{R}_{x_1,x_2}.$$

Now let us extend the action of our K-operator to the full auxiliary space $\mathcal{F}_{\boldsymbol{x}}$. The most convenient way to do it is by recurrent formula

$$\mathcal{K}_{\boldsymbol{u}|y,\boldsymbol{x}}=\mathcal{K}_{\boldsymbol{u}|\boldsymbol{x}}\mathcal{R}_{\bar{y},\boldsymbol{x}}\mathcal{K}_y.$$

Here $\mathcal{K}_{\boldsymbol{u}|y}$ is the operator defined in (2.23) acting on a tensor product $\mathcal{F}_{\boldsymbol{u}}\otimes\mathcal{F}_y$, while $\mathcal{K}_{\boldsymbol{u}|\boldsymbol{x}}$ acts on a tensor product of $F_{\boldsymbol{u}}\otimes F_{\boldsymbol{x}}$. The last formula can be illustrated by the following picture (here we consider for simplicity the case of $N=3$)

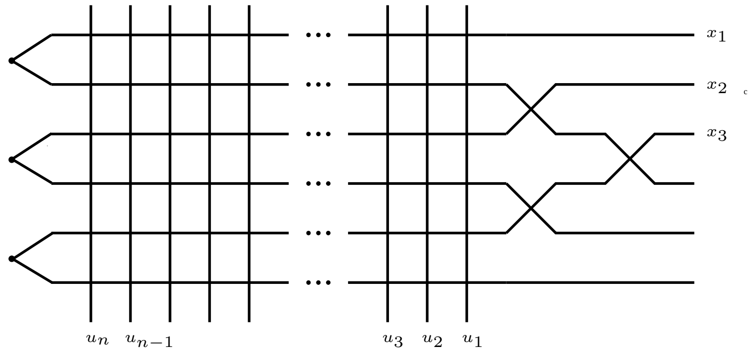
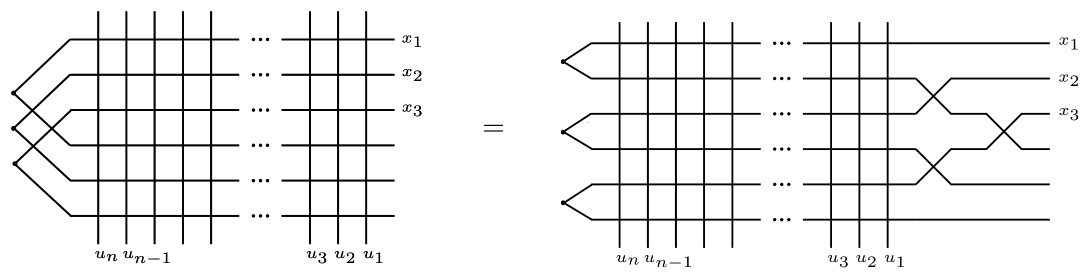
$\mathcal{K}_{\boldsymbol{u}|\boldsymbol{x}} =$ 

Figure 2.1: Explicitly this $\mathcal{K}$ operator reads as $\mathcal{K}_{\boldsymbol{u}|x_3,x_2,x_1} = \mathcal{K}_{\boldsymbol{u}|x_3}\mathcal{R}_{\bar{x}_2,x_3}\mathcal{K}_{\boldsymbol{u}|x_2}\mathcal{R}_{\bar{x}_1,x_3}\mathcal{R}_{\bar{x}_1,x_2}\mathcal{K}_{\boldsymbol{u}|x_1}$

Finally our definition of $\mathcal{K}_{\boldsymbol{v}|\boldsymbol{x}}$ may be summarised in two operations which increase the number of quantum and auxiliary Fock spaces:

$$\Delta^q(\mathcal{K}_{\boldsymbol{u}|\boldsymbol{x}}) \overset{def}{=} \mathcal{K}_{v,\boldsymbol{u}|\boldsymbol{x}} = \bar{\mathcal{L}}_v\mathcal{K}_{\boldsymbol{u}|\boldsymbol{x}}\mathcal{L}_v, \qquad \Delta^a(\mathcal{K}_{\boldsymbol{u}|\boldsymbol{x}}) \overset{def}{=} \mathcal{K}_{\boldsymbol{u}|y,\boldsymbol{x}} = \mathcal{K}_{\boldsymbol{u}|\boldsymbol{x}}\mathcal{R}_{\bar{y},\boldsymbol{x}}\mathcal{K}_y. \tag{2.24}$$

Using the Yang-Baxter equation and KRKR relation, one can show that two operators actually commute $\Delta^q(\Delta^a(\mathcal{K}_{\boldsymbol{v}|\boldsymbol{x}})) = \Delta^a(\Delta^q(\mathcal{K}_{\boldsymbol{v}|\boldsymbol{x}}))$. This property may be illustrated by the following picture

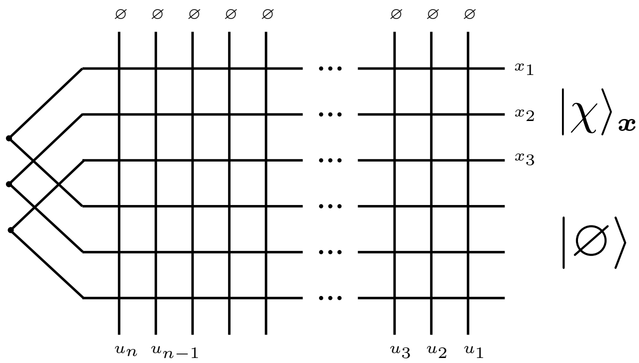


2.4.2 Off-shell Bethe vector

Now we are ready to introduce the off-shell Bethe vector

$$|B(\boldsymbol{x})\rangle = {}_x\langle\varnothing|\bar{\mathcal{L}}_{\boldsymbol{v}}\mathcal{K}_{\boldsymbol{x}}L_{\boldsymbol{v}}|\varnothing\rangle_v|\chi\rangle_x = {}_x\langle\varnothing|\mathcal{K}_{\boldsymbol{v}|\boldsymbol{x}}|\varnothing\rangle_v|\chi\rangle_x \tag{2.25}$$

here $|\chi\rangle_x$ is the same vector as in the A type considerations (1.68). It has grading N with respect to the standard grading operator. The vector $|B(\boldsymbol{x})\rangle$ can be represented by the following picture [4]

$|B(\boldsymbol{x})\rangle =$ 

(2.26)

We will provide more details about the Bethe vector in the next section, for now let us just announce the main property of Bethe vector - .

2.4.3 Bethe Ansatz equations, eigenvalues of KZ IOMs.

We note that the Yang-Baxter and KRKR relation (2.13) provide the nice intertwining property of the off-shell Bethe vector with the $\mathcal{T}_i^+$ operator (2.14)

$$\mathcal{T}_i^+|B(\boldsymbol{x})\rangle = |B(\boldsymbol{x})\rangle\Big|_{\varphi_i\to-\varphi_i}$$

In the next section we will prove that under the Bethe ansatz equations

$$\begin{gathered} r^\alpha(x_i)r^\beta(x_i)A(x_i)A^{-1}(-x_i)\prod_{j\neq i}G(x_i-x_j)G^{-1}(-x_i-x_j)=1, \\ G(x)=\frac{(x-\epsilon_1)(x-\epsilon_2)(x-\epsilon_3)}{(x+\epsilon_1)(x+\epsilon_2)(x+\epsilon_3)}, \qquad A(x)=\prod_{k=1}^{n}\frac{x-u_k+\frac{\epsilon_3}{2}}{x-u_k-\frac{\epsilon_3}{2}}, \quad r^\alpha(x)=-\frac{x+\epsilon_\alpha/2}{x-\epsilon_\alpha/2}. \end{gathered} \tag{2.27}$$

the Bethe vector with shifted x parameters $|B(\boldsymbol{x}-\frac{\epsilon_3}{2})\rangle$ becomes an eigenvector of KZ IOMs $\mathcal{I}_i^{\mathrm{KZ}}$ (2.14):

$$\mathcal{I}_i^{\mathrm{KZ}}|B(\boldsymbol{x}-\frac{\epsilon_3}{2})\rangle \overset{\mathrm{BAE}(\boldsymbol{x})=1}{=} \prod_a \frac{(u_i+\frac{\epsilon_3}{2})^2-x_a^2}{(u_i-\frac{\epsilon_3}{2})^2-x_a^2}|B(\boldsymbol{x}-\frac{\epsilon_3}{2})\rangle. \tag{2.28}$$

Equations (2.27) and (2.28) together with the explicit form of off-shell Bethe vector (2.25) are the main results of the current chapter.

[4] This formula differs from the one provided by Sklyanin [Skl88] in $\mathfrak{sl}(2)$ case. In his approach $\mathcal{K}_{\boldsymbol{u}|\boldsymbol{x}}$ operator is a product of single space operators $\prod_i \mathcal{K}_{\boldsymbol{u}|x_i}$. It can be shown, that for $\mathfrak{sl}(2)$ case these two approaches coincide. For example on level 3 we have:

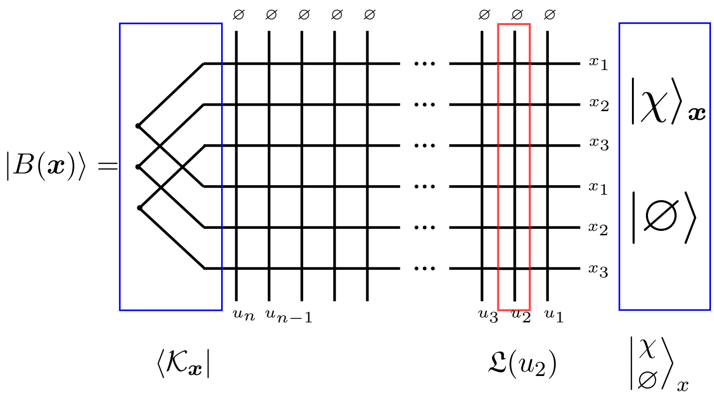
$$|B(\boldsymbol{x})\rangle =_x\langle\downarrow\downarrow\downarrow|\mathcal{K}_{\boldsymbol{u}|x_3,x_2,x_1}|\uparrow\uparrow\uparrow\rangle_x\otimes|\downarrow\rangle_u =_x\langle\downarrow\downarrow\downarrow|\mathcal{K}_{\boldsymbol{u}|x_3}\mathcal{R}_{\bar{x}_2,x_3}\mathcal{K}_{\boldsymbol{u}|x_2}\mathcal{R}_{\bar{x}_1,x_3}\mathcal{R}_{\bar{x}_1,x_2}\mathcal{K}_{\boldsymbol{u}|x_1}|\uparrow\uparrow\uparrow\rangle_x\otimes|\downarrow\rangle_u.$$

In the case of $\mathfrak{sl}(2)$, R-matrices between the auxiliary spaces may be omitted, and we reproduce Sklyanin's formula:

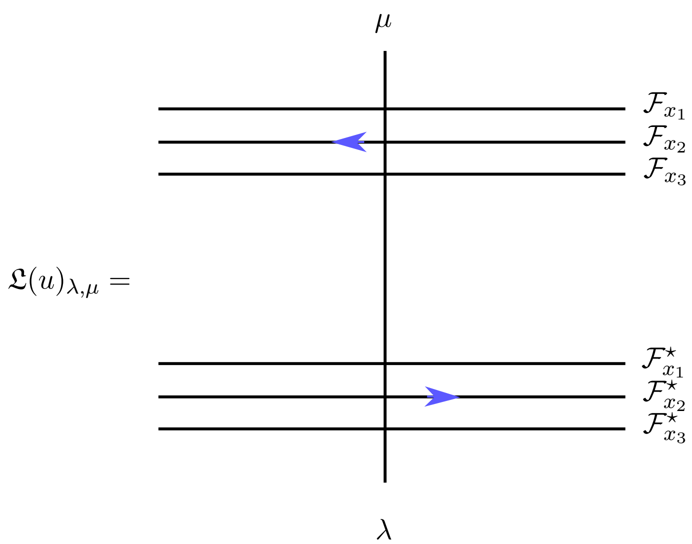
$$|B(\boldsymbol{x})\rangle \overset{\text{for } \mathfrak{sl}(2)}{=}{}_x\langle\downarrow\downarrow\downarrow|\mathcal{K}_{\boldsymbol{u}|x_3}\mathcal{K}_{\boldsymbol{u}|x_2}\mathcal{K}_{\boldsymbol{u}|x_1}|\uparrow\uparrow\uparrow\rangle_x\otimes|\downarrow\rangle_u = \langle\downarrow|\mathcal{K}_{\boldsymbol{u}|x_3}|\uparrow\rangle\langle\downarrow|\mathcal{K}_{\boldsymbol{u}|x_2}|\uparrow\rangle\langle\downarrow|\mathcal{K}_{\boldsymbol{u}|x_1}|\uparrow\rangle|\downarrow\rangle_u.$$

2.5 Diagonalization of KZ integral

Let us revise the formula for the off-shell Bethe vector (2.25). One observes that the definition of (2.25) (as especially seen from the picture (2.26)) suggests that $|B(\boldsymbol{x})\rangle$ can be interpreted as a product of some $L-$operators $\mathfrak{L}(u_n)\dots\mathfrak{L}(u_1)$ sandwiched between bra and ket states $\langle\mathcal{K}_{\boldsymbol{x}}|$ and $\left|\begin{matrix}\chi\\ \varnothing\end{matrix}\right\rangle_x$



This observation can be formalized as follows. Let us define $\mathfrak{L}(u)$ operator by the picture



By definition it acts in the tensor product of Fock module and it's dual $\mathcal{F}_{\boldsymbol{x}} \otimes \mathcal{F}^\star_{\boldsymbol{x}}$ and equals to the infinite sum

$$\mathfrak{L}(u)_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \sum_{\rho} \mathcal{L}(u)_{\boldsymbol{\rho},\boldsymbol{\mu}} \otimes \bar{\mathcal{L}}(u)_{\boldsymbol{\lambda},\boldsymbol{\rho}} \tag{2.29}$$

It is though clear that $\mathfrak{L}(u)$-operators, still enjoys RLL algebra

$$\mathcal{R}(u_1-u_2)\mathfrak{L}(u_1)\mathfrak{L}(u_2) = \mathfrak{L}(u_2)\mathfrak{L}(u_1)\mathcal{R}(u_1-u_2)$$

Using this equation we may define the currents in complete analogy with A case with exactly the same commutation relations (1.40):

$$\mathfrak{h}(u) \overset{\text{def}}{=} \mathfrak{L}_{\varnothing,\varnothing}(u), \qquad \mathfrak{e}(u) \overset{\text{def}}{=} \mathfrak{h}^{-1}(u)\mathfrak{L}_{\varnothing,\square}(u) \quad \text{and} \quad \mathfrak{f}(u) \overset{\text{def}}{=} \mathfrak{L}_{\square,\varnothing}(u)\mathfrak{h}^{-1}(u).$$

Exploiting this picture further, we can consider $K-$operator $\mathcal{K}_{\boldsymbol{x}}$ as a bra vector $\langle\mathcal{K}|$ acting from $\mathcal{F}_{\boldsymbol{x}} \otimes \mathcal{F}^\star_{\boldsymbol{x}}$ to $\mathbb{C}$. We will denote vectors from $\mathcal{F}_{\boldsymbol{x}} \otimes \mathcal{F}^\star_{\boldsymbol{x}}$ by a two rows objects $\left|\begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle$, where $\boldsymbol{\lambda} \in \mathcal{F}_{\boldsymbol{x}}$,

$\boldsymbol{\mu} \in \mathcal{F}^{\star}_{\boldsymbol{x}}$. It allows to rewrite Bethe vector (2.25) as follows:

$$|B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle = {}_{\boldsymbol{x}}\langle\varnothing|\bar{\mathcal{L}}_1 \dots \bar{\mathcal{L}}_n \mathcal{K}^{\alpha}_{\boldsymbol{x}} \mathcal{L}_n \dots \mathcal{L}_1|\chi\rangle_{\boldsymbol{x}}|\varnothing\rangle_{\boldsymbol{u}} \equiv {}_{\boldsymbol{x}}\langle\mathcal{K}^{\alpha}|\mathfrak{L}_n \dots \mathfrak{L}_1\Big|\begin{matrix}\chi\\ \varnothing\end{matrix}\Big\rangle_{\boldsymbol{x}}|\varnothing\rangle_{\boldsymbol{u}}. \tag{2.30}$$

The benefit of this approach is that the structure of the Bethe vector may be analysed by the representation theory of the Affine Yangian in $\mathcal{F}_{\boldsymbol{x}} \otimes \mathcal{F}^{\star}_{\boldsymbol{x}}$. We call the corresponding representation the strange module.

2.5.1 Strange module

Our goal is to describe the action of $\mathfrak{e}$, $\mathfrak{f}$, $\mathfrak{h}$ and ψ currents on this strange module. The first obvious remark is that while there is no highest weight vector, nevertheless Cartan currents $\mathfrak{h}(u)$ and $\psi(u)$ still can be diagonalized. Let us consider the first component of tensor product $\mathcal{F}_{\boldsymbol{x}} \otimes \mathcal{F}^{\star}_{\boldsymbol{x}}$. We already know [LV20] that the eigenbasis is numerated by the collection of Young diagrams $\vec{\boldsymbol{\lambda}} = \{\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(N)}\}$ with the eigenvalues:

$$h(u)|\vec{\boldsymbol{\lambda}}\rangle = \prod_{\square\in\vec{\boldsymbol{\lambda}}} \frac{(u-c_{\square})}{(u-c_{\square}-\epsilon_3)}|\vec{\boldsymbol{\lambda}}\rangle, \qquad \psi(u)|\vec{\boldsymbol{\lambda}}\rangle = \prod_{\alpha=1}^{3}\prod_{\square\in\vec{\boldsymbol{\lambda}}} \frac{(u-c_{\square}-\epsilon_{\alpha})}{(u-c_{\square}+\epsilon_{\alpha})}\prod_{k=1}^{n}\frac{(u-x_k+\epsilon_3)}{(u-x_k)}|\vec{\boldsymbol{\lambda}}\rangle,$$

where by definition the content of the cell with coordinates (i,j) in Young diagram $\boldsymbol{\lambda}^{(k)}$ is

$$c_{\square} = x_k - (i-1)\epsilon_1 - (j-1)\epsilon_2.$$

Now, both $\mathfrak{h}$ and ψ act by a triangle matrices in the tensor product of two eigenbases. Indeed, according to (2.29):

$$\mathfrak{h}(u) = h(u)\otimes h(-u) + \sum_{\boldsymbol{\rho}\neq\varnothing}\mathcal{L}_{\varnothing,\boldsymbol{\rho}}(u)\otimes\bar{\mathcal{L}}_{\boldsymbol{\rho},\varnothing}(u),$$

and hence the eigenbasis of $\mathfrak{h}(u), \psi(u)$ in $\mathcal{F}_{\boldsymbol{x}} \otimes \mathcal{F}^{\star}_{\boldsymbol{x}}$ is enumerated by the collection of $2N$ Young diagrams

$$\mathfrak{h}(u)\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle = \prod_{\square\in\vec{\boldsymbol{\lambda}}}\frac{(u-c_{\square})}{(u-c_{\square}-\epsilon_3)}\prod_{\square\in\vec{\boldsymbol{\mu}}}\frac{(u-c_{\square}-\epsilon_3)}{(u-c_{\square})}\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle,$$

$$\psi(u)\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle = \prod_{\alpha=1}^{3}\left(\prod_{\square\in\vec{\boldsymbol{\lambda}}}\frac{(u-c_{\square}-\epsilon_{\alpha})}{(u-c_{\square}+\epsilon_{\alpha})}\prod_{\square\in\vec{\boldsymbol{\mu}}}\frac{(u-c_{\square}+\epsilon_{\alpha})}{(u-c_{\square}-\epsilon_{\alpha})}\right)\prod_{k=1}^{n}\frac{(u-x_k+\epsilon_3)}{(u-x_k)}\frac{(u+x_k)}{(u+x_k+\epsilon_3)}\Big|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\Big\rangle,$$

where

$$c_{\square} = x_k - (i-1)\epsilon_1 - (j-1)\epsilon_2, \quad \text{for } (i,j)\in\vec{\boldsymbol{\lambda}} \text{ - the contents of the upper diagram,}$$
$$c_{\square} = -\epsilon_3 - x_k + (i-1)\epsilon_1 + (j-1)\epsilon_2, \quad \text{for } (i,j)\in\vec{\boldsymbol{\mu}} \text{ - the contents of the lower diagram.}$$

Moreover from the $\mathfrak{e}, \mathfrak{h}$ commutation relation

$$\mathfrak{h}(u)\mathfrak{e}(v)|\Lambda\rangle = \frac{u-v}{u-v-\epsilon_3}\mathfrak{e}(v)\mathfrak{h}_{\Lambda}(u)|\Lambda\rangle - \frac{\epsilon_3}{u-v-\epsilon_3}\mathcal{L}_{\varnothing,\square}(u)|\Lambda\rangle,$$

it follows that $\mathfrak{e}(u)$ acts on the eigenvectors $\left|\begin{smallmatrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{smallmatrix}\right\rangle$ with the known poles:

$$\mathfrak{e}(u)\left|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle = \sum_{\square\in\text{addable}(\vec{\boldsymbol{\lambda}})} \frac{E\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}+\square\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}\end{pmatrix}}{u-c_\square}\left|\begin{matrix}\vec{\boldsymbol{\lambda}}+\square\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle + \sum_{\square\in\text{removable}(\vec{\boldsymbol{\mu}})} \frac{E\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}-\square\end{pmatrix}}{u-c_\square}\left|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}-\square\end{matrix}\right\rangle,$$

$$\mathfrak{f}(u)\left|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle = \sum_{\square\in\text{removable}(\vec{\boldsymbol{\lambda}})} \frac{F\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}-\square\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}\end{pmatrix}}{u-c_\square}\left|\begin{matrix}\vec{\boldsymbol{\lambda}}-\square\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle + \sum_{\square\in\text{addable}(\vec{\boldsymbol{\mu}})} \frac{F\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}+\square\end{pmatrix}}{u-c_\square}\left|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}+\square\end{matrix}\right\rangle.$$

We have a freedom to change the coefficients F, E by re-scaling the eigenvectors, however their product is fixed by the $\mathfrak{ef}$ commutation relation (2.18c):

$$E\begin{pmatrix}\vec{\boldsymbol{\lambda}}-\square & \to & \vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}\end{pmatrix}F\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}-\square\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}\end{pmatrix} = \operatorname{Res}_{u=c_\square}\frac{\left\langle\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\middle|\psi(u)\middle|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle}{\left\langle\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\middle|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle} \tag{2.31}$$

and

$$E\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}+\square & \to & \vec{\boldsymbol{\mu}}\end{pmatrix}F\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}+\square\end{pmatrix} = \operatorname{Res}_{u=c_\square}\frac{\left\langle\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\middle|\psi(u)\middle|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle}{\left\langle\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\middle|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}}\end{matrix}\right\rangle} \tag{2.32}$$

The choice of coefficients E and F consistent with (2.31)-(2.32) is equivalent to the choice of normalisation for eigenvectors. It is convenient to use the following one

$$E\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}+\square\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}\end{pmatrix} = \frac{\epsilon_1\epsilon_2}{\epsilon_3}\prod_{\square'\in\vec{\boldsymbol{\lambda}}+\square} S^{-1}(c_\square-c_{\square'})\prod_{\square'\in\vec{\boldsymbol{\mu}}} S(c_\square-c_{\square'})\prod_{k=1}^{n}\frac{(c_\square-x_k+\epsilon_3)}{(c_\square-x_k)}\frac{(c_\square+x_k)}{(c_\square+x_k+\epsilon_3)}, \tag{2.33}$$

$$F\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}+\square\end{pmatrix} = \prod_{\square'\in\vec{\boldsymbol{\lambda}}} S(c_{\square'}-c_\square)\prod_{\square'\in\vec{\boldsymbol{\mu}}+\square} S^{-1}(c_{\square'}-c_\square)\prod_{k=1}^{n}\frac{(c_\square-x_k+\epsilon_3)}{(c_\square-x_k)}\frac{(c_\square+x_k)}{(c_\square+x_k+\epsilon_3)}, \tag{2.34}$$

$$E\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}-\square\end{pmatrix} = \frac{\epsilon_1\epsilon_2}{\epsilon_3}\prod_{\square'\in\vec{\boldsymbol{\lambda}}} S^{-1}(c_\square-c_{\square'})\prod_{\square'\in\vec{\boldsymbol{\mu}}-\square} S(c_\square-c_{\square'}),$$

$$F\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}-\square\\ \vec{\boldsymbol{\mu}} & \to & \vec{\boldsymbol{\mu}}\end{pmatrix} = \prod_{\square'\in\vec{\boldsymbol{\lambda}}-\square} S(c_{\square'}-c_\square)\prod_{\square'\in\vec{\boldsymbol{\mu}}} S^{-1}(c_{\square'}-c_\square), \tag{2.35}$$

with

$$S(x) = \frac{(x+\epsilon_1)(x+\epsilon_2)}{x(x-\epsilon_3)}.$$

Let us now define the vector $|\chi\rangle_{\boldsymbol{x}}$ announced in definitions of off-shell Bethe vector (2.25), (2.30). The idea is to choose the vector which will maximally simplify the computation of Bethe vector. The most natural definition is:

$$\left|\begin{matrix}\chi\\ \varnothing\end{matrix}\right\rangle_{\boldsymbol{x}} = \left|\begin{matrix}\square,\ldots,\square\\ \varnothing\end{matrix}\right\rangle_{\boldsymbol{x}}.$$

Alternatively, this vector may be defined (up to proportionality constant) as an eigenvector of $\mathfrak{h}(z)$ with the most natural eigenvalue:

$$\mathfrak{h}(u)\left|\begin{matrix}\chi\\ \varnothing\end{matrix}\right\rangle_{\boldsymbol{x}} = \prod_{i=1}^{N}\frac{u-x_i}{u-x_i-\epsilon_3}\left|\begin{matrix}\chi\\ \varnothing\end{matrix}\right\rangle_{\boldsymbol{x}}.$$

The main advantage of this choice is that it provides an understandable structure of the off-shell Bethe function (2.43), (2.48). Namely the matrix elements

$$_{\boldsymbol{x}}\langle \mathcal{K}^{\alpha}|\mathfrak{h}(u_n)\underbrace{\mathfrak{f}(z_1^{(n)})\mathfrak{f}(z_2^{(n)})\dots}_{|\boldsymbol{\lambda}^{(n)}|}\dots\ \mathfrak{h}(u_2)\underbrace{\mathfrak{f}(z_1^{(2)})\mathfrak{f}(z_2^{(2)})\dots}_{|\boldsymbol{\lambda}^{(2)}|}\mathfrak{h}(u_1)\underbrace{\mathfrak{f}(z_1^{(1)})\mathfrak{f}(z_2^{(1)})\dots}_{|\boldsymbol{\lambda}^{(1)}|}\Big|{\chi \atop \varnothing}\Big\rangle_{\boldsymbol{x}}$$

involved in (2.48) may have poles only at points $z_i = x_j$ or $z_i = -x_j - \epsilon_3$. Then one may compute them explicitly either from formulas (2.34), (2.35), (2.42), or by analysis of $\mathfrak{h}\mathfrak{f}, \mathfrak{f}\mathfrak{f}$ commutation relations (2.18a), (2.18d) and $\langle\mathcal{K}|\mathfrak{f}$ relation (2.39). This logic will be explained in section (2.5.3).

2.5.2 Calculation of K-operator

Our $K-$operator $\mathcal{K}_{\boldsymbol{x}}$ provides the pairing in the space $\mathcal{F}_{\boldsymbol{x}} \otimes \mathcal{F}^{\star}_{\boldsymbol{x}}$. Our goal for this section is the calculation of the matrix elements:

$$\langle\vec{\boldsymbol{\mu}}|\mathcal{K}_{\boldsymbol{x}}|\vec{\boldsymbol{\lambda}}\rangle = \Big\langle\mathcal{K}_{\boldsymbol{x}}\Big|{\vec{\boldsymbol{\lambda}} \atop \vec{\boldsymbol{\mu}}}\Big\rangle$$

In order to do so, we use the reflection equation:

$$K_v\bar{\mathcal{L}}(u)\mathcal{K}_x\mathcal{L}(u) = \mathcal{L}(u)\mathcal{K}_x\bar{\mathcal{L}}(u)K_u \tag{2.36}$$

Being rewritten in terms of $\mathfrak{L}(u)$, equation (2.36) takes the form:

$$\langle\mathcal{K}|\mathfrak{L}(u)_{\boldsymbol{\lambda},\mathcal{K}\boldsymbol{\mu}} = \langle\mathcal{K}|\bar{\mathfrak{L}}(u)_{\mathcal{K}\boldsymbol{\lambda},\boldsymbol{\mu}}$$

Two immediate consequences of these relations are:

$$\langle\mathcal{K}|\mathfrak{h}(u) = \langle\mathcal{K}|\mathfrak{h}(-u), \tag{2.37}$$

$$\langle\mathcal{K}|\mathfrak{L}_{\square,\varnothing}(u) = -\kappa(u)\langle\mathcal{K}|\mathfrak{L}_{\square,\varnothing}(-u), \tag{2.38}$$

where $\mathcal{K}|\square\rangle = \kappa(u)|\square\rangle$. The last equation can be equivalently rewritten in terms of the reflection relation for the $\mathfrak{f}$ current:

$$\langle\mathcal{K}|\mathfrak{f}(u) = r(u)\langle\mathcal{K}|\mathfrak{f}(-\epsilon_3 - u), \tag{2.39}$$

with

$$r(u) = -\frac{2u + \epsilon_3 + \epsilon_3\kappa(u)}{2u\kappa(u)}.$$

This equation immediately follows from (2.37), (2.38) after substitution $\mathfrak{L}_{\square,\varnothing}(u) = \mathfrak{f}(u)\mathfrak{h}(u)$ and the following chain of relations

$$\langle\mathcal{K}|\mathfrak{f}(u)\mathfrak{h}(u) = -\kappa(u)\langle\mathcal{K}|\mathfrak{f}(-u)\mathfrak{h}(-u) = -\kappa(u)\langle\mathcal{K}|\mathfrak{h}(-u)\mathfrak{f}(-u-\epsilon_3) =$$
$$= -\kappa(u)\langle\mathcal{K}|\mathfrak{h}(u)\mathfrak{f}(-u-\epsilon_3) = -\frac{\epsilon_3\kappa(u)}{2u+\epsilon_3}\langle\mathcal{K}|\mathfrak{f}(u)\mathfrak{h}(u) - \frac{2u\kappa(u)}{2u+\epsilon_3}\langle\mathcal{K}|\mathfrak{f}(-u-\epsilon_3)h(u)$$

Finally we have

$$\kappa(u) = 1\,, \quad r(u-\epsilon_3/2) = -\frac{u+\epsilon_3/2}{u-\epsilon_3/2} \quad \text{for the D case,} \tag{2.40}$$

$$\kappa(u) = \frac{u-\epsilon_i-\epsilon_j/2}{u+\epsilon_i+\epsilon_j/2}, \quad r(u-\epsilon_3/2) = -\frac{u+\epsilon_i/2}{u-\epsilon_i/2} \quad \text{for the BC case,} \tag{2.41}$$

where in the last line $\{i,j\} = \{1,2\}$ corresponds to the B case and $\{i,j\} = \{2,1\}$ corresponds to the C case.

Relations (2.37) and (2.40)-(2.41) completely define the matrix elements (2.36). First of all from (2.37) it follows that $\mathcal{K}$ acts diagonally in the eigenbasis of $\mathfrak{h}$ i.e. $\vec{\boldsymbol{\lambda}} = \vec{\boldsymbol{\mu}}$. Then one can find

$$\begin{aligned}\left\langle \mathcal{K}_{\boldsymbol{x}} \middle| \begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\lambda}}\end{matrix}\right\rangle &= F^{-1}\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\lambda}}-\square & \to & \vec{\boldsymbol{\lambda}}\end{pmatrix}\mathrm{res}_{z=c_\square}\left\langle \mathcal{K}_{\boldsymbol{x}} \middle| f(-\epsilon_3 - z) \middle|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\lambda}}-\square\end{matrix}\right\rangle = \\ &= r^{-1}(c_\square)F^{-1}\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\lambda}}-\square & \to & \vec{\boldsymbol{\lambda}}\end{pmatrix}\mathrm{res}_{z=c_\square}\left\langle \mathcal{K}_{\boldsymbol{x}} \middle| f(z) \middle|\begin{matrix}\vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\lambda}}-\square\end{matrix}\right\rangle = \\ &= r^{-1}(c_\square)F^{-1}\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}\\ \vec{\boldsymbol{\lambda}}-\square & \to & \vec{\boldsymbol{\lambda}}\end{pmatrix}F\begin{pmatrix}\vec{\boldsymbol{\lambda}} & \to & \vec{\boldsymbol{\lambda}}-\square\\ \vec{\boldsymbol{\lambda}}-\square & \to & \vec{\boldsymbol{\lambda}}-\square\end{pmatrix}\left\langle \mathcal{K}_{\boldsymbol{x}} \middle| \begin{matrix}\vec{\boldsymbol{\lambda}}-\square\\ \vec{\boldsymbol{\lambda}}-\square\end{matrix}\right\rangle\end{aligned} \tag{2.42}$$

2.5.3 Off-shell Bethe function, diagonalization of KZ integral.

Motivated by the formulas (2.40) and (2.41), it is convenient to shift x variables: $x \to x - \frac{\epsilon_3}{2}$, as well as redefine the operators $\mathfrak{f}$: $\mathfrak{f}(z) \to \mathfrak{f}(z - \frac{\epsilon_3}{2})$.

Let us consider the following Bethe vectors:

$$\begin{aligned}|B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle &= {}_{\boldsymbol{x}}\langle\varnothing|\bar{\mathcal{L}}_1\dots\bar{\mathcal{L}}_n\mathcal{K}^\alpha_{\boldsymbol{x}}\mathcal{L}_n\dots\mathcal{L}_1|\varnothing\rangle_{\boldsymbol{u}}|\chi\rangle_{\boldsymbol{x}} \equiv {}_{\boldsymbol{x}}\langle\mathcal{K}^\alpha|\mathfrak{L}_n\dots\mathfrak{L}_1|\varnothing\rangle_{\boldsymbol{u}}\left|\begin{matrix}\chi\\ \varnothing\end{matrix}\right\rangle_{\boldsymbol{x}}\\ |\bar{B}^{\beta,\boldsymbol{u}}(\boldsymbol{x})\rangle &= {}_{\boldsymbol{x}}\langle\varnothing|\mathcal{L}_n\dots\mathcal{L}_1\mathcal{K}^\beta_{\boldsymbol{x}}\bar{\mathcal{L}}_1\dots\bar{\mathcal{L}}_n|\varnothing\rangle_{\boldsymbol{u}}|\bar{\chi}\rangle_{\boldsymbol{x}} \equiv {}_{\boldsymbol{x}}\langle\mathcal{K}^\beta|\bar{\mathfrak{L}}_1\dots\bar{\mathfrak{L}}_n|\varnothing\rangle_{\boldsymbol{u}}\left|\begin{matrix}\bar{\chi}\\ \varnothing\end{matrix}\right\rangle_{\boldsymbol{x}},\end{aligned}$$

where $\alpha = 1, 2, 3$ labels possible $K-$operators.

It is also useful to introduce their matrix elements the so called off-shell Bethe functions:

$$\begin{aligned}\omega_{\alpha,\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) &\stackrel{\text{def}}{=} \langle\vec{\boldsymbol{\lambda}}|B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle = {}_{\boldsymbol{x}}\langle\mathcal{K}^\alpha_x|\mathfrak{L}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)\dots\mathfrak{L}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)\left|\begin{matrix}\chi\\ \varnothing\end{matrix}\right\rangle_{\boldsymbol{x}},\\ \bar{\omega}_{\beta,\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) &\stackrel{\text{def}}{=} \langle\vec{\boldsymbol{\lambda}}|\bar{B}^{\beta,\boldsymbol{u}}(\boldsymbol{x})\rangle = {}_{\boldsymbol{x}}\langle\mathcal{K}^\beta_x|\bar{\mathfrak{L}}_{\boldsymbol{\lambda}^{(n)},\varnothing}(u_n)\dots\bar{\mathfrak{L}}_{\boldsymbol{\lambda}^{(1)},\varnothing}(u_1)\left|\begin{matrix}\bar{\chi}\\ \varnothing\end{matrix}\right\rangle_{\boldsymbol{x}}.\end{aligned} \tag{2.43}$$

The off-shell Bethe vectors and hence the off-shell functions have nice intertwining relations with R-matrix and K-operators:

$$\begin{aligned}\mathcal{R}_{i,i+1}|B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle &= P_{i,i+1}|B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle,\\ \mathcal{R}_{i,i+1}|\bar{B}^{\beta,\boldsymbol{u}}(\boldsymbol{x})\rangle &= P_{i,i+1}|\bar{B}^{\beta,\boldsymbol{u}}(\boldsymbol{x})\rangle,\\ \mathcal{K}^\alpha_n|B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle &= D_n|B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle,\\ \mathcal{K}^\beta_1 D_1|\bar{B}^{\beta,\boldsymbol{u}}(\boldsymbol{x})\rangle &= |\bar{B}^{\beta,\boldsymbol{u}}(\boldsymbol{x})\rangle.\end{aligned}$$

It implies in particular, the simple action of KZ operators:

$$\mathcal{T}^+_i|B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle = D_i|B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle, \tag{2.44}$$

$$\mathcal{T}^-_i D_i|\bar{B}^{\beta,\boldsymbol{u}}(\boldsymbol{x})\rangle = |\bar{B}^{\beta,\boldsymbol{u}}(\boldsymbol{x})\rangle. \tag{2.45}$$

Our goal for this section is to prove that under the Bethe ansatz equations (2.27) two Bethe vectors are proportional to each other:

$$|B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle \stackrel{\mathrm{BAE}(\boldsymbol{x})=1}{=} |\bar{B}^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle \prod_{i,a}\frac{u_i - x_a - \frac{\epsilon_3}{2}}{u_i - x_a + \frac{\epsilon_3}{2}}\ c(\boldsymbol{x}), \tag{2.46}$$

for some $c(\boldsymbol{x})$. Combining this equation and relations (2.44)-(2.45) one can immediately conclude that on-shell Bethe vector is indeed an eigenvector of KZ operator (2.56).

Let us proceed to the proof. It is enough to check (2.46) for any matrix element, or, which is the same, to establish similar relation for off-shell functions (2.49). In order to compute the later, let us remind (2.20) that $\mathfrak{L}_{\boldsymbol{\lambda},\varnothing}$ is generated by the $\mathfrak{h}(u)$ and $\mathfrak{f}(z)$ currents:

$$\mathfrak{L}_{\boldsymbol{\lambda},\varnothing}(u) = \frac{1}{(2\pi i)^{|\boldsymbol{\lambda}|}} \oint_{\mathcal{C}_1} \cdots \oint_{\mathcal{C}_{|\boldsymbol{\lambda}|}} F_{\boldsymbol{\lambda}}(\boldsymbol{z} - \frac{\epsilon_3}{2}|u)\, \mathfrak{h}(u)\mathfrak{f}(z_{|\boldsymbol{\lambda}|}) \dots \mathfrak{f}(z_1) dz_1 \dots dz_{|\boldsymbol{\lambda}|}, \tag{2.47}$$

where each contour $\mathcal{C}_k$ goes clockwise around ∞ and $u-\epsilon_3$, so that it doesn't pick the poles of function $F_{\vec{\boldsymbol{\lambda}}}$. Using (2.47) the weight function (2.43) can be rewritten as

$$\omega_{\alpha,\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) = \frac{1}{(2\pi i)^N} \times$$
$$\times \oint F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}} - \frac{\epsilon_3}{2}|\boldsymbol{u})\ {}_{\boldsymbol{x}}\langle \mathcal{K}^\alpha | \mathfrak{h}(u_n) \underbrace{\mathfrak{f}(z_1^{(n)})\mathfrak{f}(z_2^{(n)}) \dots}_{|\boldsymbol{\lambda}^{(n)}|} \dots\ \mathfrak{h}(u_2) \underbrace{\mathfrak{f}(z_1^{(2)})\mathfrak{f}(z_2^{(2)}) \dots}_{|\boldsymbol{\lambda}^{(2)}|} \mathfrak{h}(u_1) \underbrace{\mathfrak{f}(z_1^{(1)})\mathfrak{f}(z_2^{(1)}) \dots}_{|\boldsymbol{\lambda}^{(1)}|} \Big| {\chi \atop \varnothing} \Big\rangle_{\boldsymbol{x}} d\vec{\boldsymbol{z}}, \tag{2.48}$$

where

$$F_{\vec{\boldsymbol{\lambda}}}(\vec{\boldsymbol{z}}|\boldsymbol{u}) = \prod_{k=1}^{n} F_{\boldsymbol{\lambda}^{(k)}} \left(z_1^{(k)}, \dots, z_{|\boldsymbol{\lambda}^{(k)}|}^{(k)} | u_k \right).$$

The contour integral (2.48) can be computed by residues. In order to do so let us analyse possible poles in z variables. As we already explained any $\mathfrak{f}(z)$ operator either removes one box from the upper Young diagram χ or add a box to the lover Young diagram with a pole equal to the content of the corresponding cell. We also proved that the matrix element $\Big\langle \mathcal{K} \Big| {\vec{\boldsymbol{\lambda}} \atop \vec{\boldsymbol{\mu}}} \Big\rangle$ is nonzero only if $\vec{\boldsymbol{\lambda}} = \vec{\boldsymbol{\mu}}$. Thus we conclude that the only possible poles of the contour integral (2.48) are

$$z_i = x_{\sigma(i)} \quad \text{or} \quad z_i = -x_{\sigma(i)},$$

where σ is some permutation. It is convenient to consider the group spanned by the elements s, which is generated by permutations of all indices S_N and $\mathbb{Z}_2$ reflection of each index $i \to \bar{i}$ with the convention $x_{\bar{i}} = -x_i$. The weight function itself is given by the sum over residues as:

$$\omega_{\alpha,\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) = \sum_s I^\alpha_{s(1,\dots,N)}\ .$$

We are going to prove the proportionality of two weight functions under the Bethe equations:

$$\omega_{\alpha,\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}) \overset{\mathrm{BAE}(\boldsymbol{x})=1}{=} c(\boldsymbol{x}) \bar{\omega}_{\beta,\vec{\boldsymbol{\lambda}}}(\boldsymbol{x}|\boldsymbol{u}). \tag{2.49}$$

Actually, we will prove a stronger statement of proportionality of the corresponding residues:

$$I^\alpha_{s(1,\dots,N)} \overset{\mathrm{BAE}(\boldsymbol{x})=1}{=} c(\boldsymbol{x}) \bar{I}^\beta_{s(1,\dots,N)}. \tag{2.50}$$

Using the fact that the integral (2.47) defined to avoid the poles of $F_{\vec{\boldsymbol{\lambda}}}$, we have explicitly:

$$I^\alpha_{s(1,\dots N)}(\boldsymbol{x}) = F_{\vec{\boldsymbol{\lambda}}}(s(\boldsymbol{x}) - \frac{\epsilon_3}{2}|\boldsymbol{u})\, \mathrm{Res}_{z_i = x_{s(i)}}\ {}_{\boldsymbol{x}}\langle \mathcal{K}^\alpha | \mathfrak{h}(u_n) \underbrace{\mathfrak{f}(z_1^{(n)})\mathfrak{f}(z_2^{(n)}) \dots}_{|\boldsymbol{\lambda}^{(n)}|} \dots$$
$$\dots\ \mathfrak{h}(u_2) \underbrace{\mathfrak{f}(z_1^{(2)})\mathfrak{f}(z_2^{(2)}) \dots}_{|\boldsymbol{\lambda}^{(2)}|} \mathfrak{h}(u_1) \underbrace{\mathfrak{f}(z_1^{(1)})\mathfrak{f}(z_2^{(1)}) \dots}_{|\boldsymbol{\lambda}^{(1)}|} \Big| {\chi \atop \varnothing} \Big\rangle_{\boldsymbol{x}} \tag{2.51}$$

and

$$\bar{I}^{\beta}_{s(1,\dots N)}(\boldsymbol{x}) = F_{\vec{\boldsymbol{\lambda}}}(s(\boldsymbol{x}) - \frac{\epsilon_3}{2}|\boldsymbol{u}) \operatorname{Res}_{z_i = x_{s(i)}} \cdot$$
$$_{\boldsymbol{x}}\langle \mathcal{K}^{\beta}| \underbrace{\mathfrak{f}(-z_1^{(1)})\mathfrak{f}(-z_2^{(1)})\dots}_{|\boldsymbol{\lambda}^{(1)}|} \mathfrak{h}(-u_1) \underbrace{\mathfrak{f}(-z_1^{(2)})\mathfrak{f}(-z_2^{(2)})\dots}_{|\boldsymbol{\lambda}^{(2)}|} \mathfrak{h}(-u_2) \dots \underbrace{\mathfrak{f}(-z_1^{(n)})\mathfrak{f}(-z_2^{(n)})\dots}_{|\boldsymbol{\lambda}^{(n)}|} \mathfrak{h}(-u_n) \Big| \begin{matrix}\chi \\ \varnothing\end{matrix} \Big\rangle_{\boldsymbol{x}}. \tag{2.52}$$

Taking into account formulas (2.33)-(2.35) and (2.42), it is straightforward to compute the matrix elements and check equation (2.50) with

$$c(\boldsymbol{x}) = \prod_{i<j} S(x_i + x_j) \prod_i r^{\beta}(-x_i). \tag{2.53}$$

One can also provide a simpler proof without reference to the explicit formulas for matrix elements, but using the $\mathfrak{f}\mathfrak{h}, \mathfrak{f}\mathfrak{f}$ commutation relation (2.18a),(2.18d) and $\langle \mathcal{K}|\mathfrak{f}$ relation (2.39). The direct consequence of this relations is the formula for the matrix elements:

$$\operatorname{Res}_{z_i = x_i}\ {}_{\boldsymbol{x}}\langle \mathcal{K}^{\alpha}| \dots \mathfrak{f}(z_{i+1})\mathfrak{f}(z_i) \dots \Big| \begin{matrix}\chi \\ \varnothing\end{matrix} \Big\rangle_{\boldsymbol{x}} = G^{-1}(x_{i+1} - x_i) \operatorname{Res}_{z_i = x_i}\ {}_{x}\langle \mathcal{K}^{\alpha}| \dots \mathfrak{f}(z_i)\mathfrak{f}(z_{i+1}) \dots \Big| \begin{matrix}\chi \\ \varnothing\end{matrix} \Big\rangle_{\boldsymbol{x}}, \tag{2.54}$$

$$\operatorname{Res}_{z_i = x_i}\ {}_{\boldsymbol{x}}\langle \mathcal{K}^{\alpha}| \dots \mathfrak{f}(z_i)\mathfrak{h}(u) \dots \Big| \begin{matrix}\chi \\ \varnothing\end{matrix} \Big\rangle_{\boldsymbol{x}} = \frac{u - x_i + \frac{\epsilon_3}{2}}{u - x_i - \frac{\epsilon_3}{2}} \operatorname{Res}_{z_i = x_i}\ {}_{\boldsymbol{x}}\langle \mathcal{K}^{\alpha}| \dots \mathfrak{h}(u)\mathfrak{f}(z_i) \dots \Big| \begin{matrix}\chi \\ \varnothing\end{matrix} \Big\rangle_{\boldsymbol{x}},$$

$$\operatorname{Res}_{z_i = x_i}\ {}_{\boldsymbol{x}}\langle \mathcal{K}^{\alpha}|\mathfrak{f}(-z_N) \dots \Big| \begin{matrix}\chi \\ \varnothing\end{matrix} \Big\rangle_{\boldsymbol{x}} = r(-x_N) \operatorname{Res}_{z_i = x_i}\ {}_{\boldsymbol{x}}\langle \mathcal{K}^{\alpha}|\mathfrak{f}(z_N) \dots \Big| \begin{matrix}\chi \\ \varnothing\end{matrix} \Big\rangle_{\boldsymbol{x}}. \tag{2.55}$$

Note that in the first two relations we additionally used the fact that Bethe roots x_i are not in resonance with each other as well as with evaluation parameters u_k. It allow us to omit local (blue) terms in (2.18b), (2.18d). These three relations are completely define the residues up to a constant.

We immediately observe that both matrix elements (2.51),(2.52) share the same transformation properties under the permutation of x_i variables. At the first glance the transformation under reflection of x_i variables is different. Indeed the reflection of x_N in the first matrix element (2.51) produce a simple factor $r^{\alpha}(x_N)$, while in the opposite matrix element (2.52) we have to move corresponding operator $\mathfrak{f}(-z_N)$ to the left boundary and back which produce the product of many terms:

$$r^{\beta}(-x_N) \prod_k \frac{x_N^2 - (u_k - \frac{\epsilon_3}{2})^2}{x_N^2 - (u_k + \frac{\epsilon_3}{2})^2} \prod_{j \neq N} G^{-1}(x_N - x_j) G^{-1}(x_N + x_j).$$

Two factors coincide under the Bethe equations (2.27). This proves the proportionality of corresponding residues (2.51),(2.52). The proportionality constant (2.53) may be computed along the same lines.

It may be useful to note that the identities (2.54)-(2.55) may be summarized in the following rules for computation of the residues:

$$I^{\alpha}_{1,\dots,N} = F_{\vec{\boldsymbol{\lambda}}}(\boldsymbol{x} + \frac{\epsilon_3}{2}|\boldsymbol{u}) \prod_k \prod_{i=|\lambda_k|+1}^{N} \frac{u_k - x_i + \frac{\epsilon_3}{2}}{u_k - x_i - \frac{\epsilon_3}{2}} \prod_{i<j} S(x_i - x_j),$$

$$I^{\alpha}_{\dots,i+1,i,\dots}(\boldsymbol{x}) = I^{\alpha}_{\dots,i,i+1,\dots}(P_{i,i+1}\boldsymbol{x}),$$

$$I^{\alpha}_{\dots,\bar{N}}(\boldsymbol{x}) = I^{\alpha}_{\dots,N}(\boldsymbol{x})\Big|_{x_N \to -x_N} r^{\alpha}(x_N).$$

Finally, let us compute the action of KZ integral of motion $\mathcal{I}_i^{\mathrm{KZ}} = \mathcal{T}_i^- \mathcal{T}_i^+$ on Bethe vector:

$$\mathcal{T}_i^- \mathcal{T}_i^+ |B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle = \mathcal{T}_i^- D_i |B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle \overset{\mathrm{BAE}(\boldsymbol{x})=1}{=} \mathcal{T}_i^- D_i |\bar{B}^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle \frac{u_i + x_a + \frac{\epsilon_3}{2}}{u_i + x_a - \frac{\epsilon_3}{2}} \prod_{j\neq i,a} \frac{u_i - x_a - \frac{\epsilon_3}{2}}{u_i - x_a + \frac{\epsilon_3}{2}} c(\boldsymbol{x})$$

$$\mathcal{T}_i^- D_i |\bar{B}^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle = |\bar{B}^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle \overset{\mathrm{BAE}(\boldsymbol{x})=1}{=} |B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle \frac{u_i - x_a + \frac{\epsilon_3}{2}}{u_i - x_a - \frac{\epsilon_3}{2}} \prod_{j\neq i,a} \frac{u_j - x_a + \frac{\epsilon_3}{2}}{u_j - x_a - \frac{\epsilon_3}{2}} c^{-1}(\boldsymbol{x}),$$

which finally proves

$$\mathcal{I}_i^{\mathrm{KZ}} |B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle \overset{\mathrm{BAE}(\boldsymbol{x})=1}{=} |B^{\alpha,\boldsymbol{u}}(\boldsymbol{x})\rangle \frac{(u_i + \frac{\epsilon_3}{2})^2 - x_a^2}{(u_i - \frac{\epsilon_3}{2})^2 - x_a^2}. \tag{2.56}$$

2.6 Concluding remarks

In this notes we discovered Bethe ansatz equations for the spectrum of Integrals of Motion in CFT with the W symmetry of BCD type. There many open questions, which we list in random order.

T-operator. We have avoided the construction of the boundary transfer matrix similar to Sklyanin [Skl88]. The reason is that this object is not well defined for $\mathrm{Y}(\widehat{\mathfrak{gl}}(1))$. Its construction requires the corresponding R-matrix to satisfy the property known as crossing unitarity. One can easily show that MO R-matrix satisfies two basic properties of

$$\text{Unitarity}: \qquad R[\partial\varphi_i - \partial\varphi_j] R[\partial\varphi_j - \partial\varphi_i] = 1, \tag{2.57}$$

$$\text{T-symmetry}: \qquad R^t[\partial\varphi_i - \partial\varphi_j] = R[\partial\varphi_i - \partial\varphi_j], \tag{2.58}$$

which follow immediately from the defining relations (2.8). Both (2.57) and (2.58) hold level by level and can be easily verified by explicit calculations for lower levels. However, the crossing unitarity property

$$R^{t_i}[\partial\varphi_i - \partial\varphi_j] R^{t_i}[\partial\varphi_j - \partial\varphi_i] = 1. \tag{2.59}$$

is more subtle, as it mixes different levels and involves infinite sums of matrix elements. It is questionable if one can make it any sense. Even if we believe that (2.59) holds and try to make a step further, we may conjecture the following formula for the generating function of integrals of motion (for the D case)

$$\mathcal{T}(u) = \mathrm{Tr}\Big|_0 \left(\mathcal{R}_{\bar{0},1} \dots \mathcal{R}_{\bar{0},n} \mathcal{R}_{0,n} \dots \mathcal{R}_{0,1}\right)$$

This formula requires more accurate definition as it involves divergent summation over infinite dimensional Fock space. In the A case this divergence has been regularised by an introduction of the twist parameter which preserved the integrability. It is unclear whether such a twist can be introduced in the present case as well. This certainly remains as open interesting question.

Eigenvalues of local Integrals of Motion. Our construction is specially adapted to diagonalization of KZ integral. Diagonalization of local IM's is a separate issue. We note that in the A case [LV20] we provided explicit construction for diagonalization of the simplest non-trivial local IM $\mathbf{I}_2$. In principle, it can be generalized for $\mathbf{I}_s$ with $s > 2$. Despite the fact that direct diagonalization of local IM's is still lacking, it is rather natural to expect that their eigenvalues are some symmetric polynomials in Bethe roots. In the present case, we conjecture the following formula for the integral $\mathbf{I}_3 = \frac{1}{2\pi} \int G_4(x) dx$ corresponding to the local density G_4 given by (2.5). Namely, on level N one has an eigenvalue

$$\mathbf{I}_3^{\mathrm{vac}} + \left(4N - 4\sum_{k=1}^{n} \frac{u_k^2}{\epsilon_1\epsilon_2} + \frac{\epsilon_1^2 + \epsilon_2^2}{3\epsilon_1\epsilon_2}\Big(2n - \frac{\epsilon_\alpha + \epsilon_\beta}{\epsilon_3}\Big)\right) N + \frac{4}{\epsilon_1\epsilon_2}\left(2n - \frac{\epsilon_\alpha + \epsilon_\beta}{\epsilon_3}\right) \sum_{k=1}^{N} x_k^2, \tag{2.60}$$

where x_k's satisfy Bethe ansatz equations

$$r^{(\alpha)}(x_i)r^{(\beta)}(x_i)A(x_i)A^{-1}(-x_i)\prod_{j\neq i}G(x_i-x_j)G^{-1}(-x_i-x_j)=1, \tag{2.61}$$

with

$$r^{(\alpha)}(x)=-\frac{x-\frac{\epsilon_\alpha}{2}}{x+\frac{\epsilon_\alpha}{2}},\qquad A(x)=\prod_{k=1}^{n}\frac{x-u_k+\frac{\epsilon_3}{2}}{x-u_k-\frac{\epsilon_3}{2}}.$$

We have confirmed (2.60) by explicit diagonalization on lower levels and it is interesting to find a proof.

Bullough-Dodd model Integrable systems studied in this notes are already non-trivial for $n=1$. Let us consider $\widehat{\mathrm{BC}}_1$ system, which is known also as Bullough-Dodd model, or Zhiber-Shabat model. This is the theory of one bosonic field φ with the action

$$S=\int\left(\frac{1}{8\pi}(\partial_a\varphi)^2+\Lambda\left(e^{2b\varphi}+e^{-b\varphi}\right)\right)d^2z.$$

According to Zamolodchikov [Zam89], this theory can be interpreted as $\Phi_{1,2}$ integrable perturbation of CFT (or equivalently as $\Phi_{1,5}$ perturbation). The corresponding conformal integrable system has been studied within BLZ approach in [FRS96]. However, as far as we concerned, a system of algebraic equations for the spectrum similar to [BLZ04] has not been derived yet[5].

From the general formula (2.5) we see that $\mathbf{I}_3$ identically vanishes for BD model. It implies the following identity for the Bethe roots

$$\sum_{k=1}^{N}x_k^2=\frac{1}{12}\left(4u^2-4N-Q\right).$$

The first non-trivial integral is $\mathbf{I}_5$ which has the form

$$\mathbf{I}_5=\frac{1}{2\pi}\int\Bigg[(\partial\varphi)^6-\frac{5}{4}(\partial\varphi)^4-\frac{5}{2}(b-b^{-1})(2Q^2+1)(\partial^2\varphi)^3+ \\ +5(3Q^2+1)\left((\partial^2\varphi)^2(\partial\varphi)^2-\frac{1}{12}(\partial^2\varphi)^2\right)+\left(3Q^4+\frac{17Q^2}{4}+\frac{8}{3}\right)(\partial^3\varphi)^2\Bigg]dx. \tag{2.62}$$

Here all densities are Wick ordered. We note that our integral (2.62) differs from the analytically regularized integral by addition of $\mathbf{I}_1$ and a constant. Bethe Ansatz equations follows the general rules (2.61) with $\alpha=1$, $\beta=2$ and $n=1$. We found that the eigenvalues of $\mathbf{I}_5-\mathbf{I}_5^{\text{vac}}$ are given by

$$N\left(\frac{63Q^4}{8}+\left(45N-\frac{63}{2}\right)Q^2+80N^2-95N+27\right)-5N(9Q^2+24N-19)u^2+60Nu^4-270\sum_{k=1}^{N}x_k^4.$$

[5]We note that algebraic equations for quantum KdV system, proposed by Bazhanov, Lukyanov and Zamolodchikov [BLZ04], were given in [Fio05] improved explanation with precise connection to Virasoro symmetry. It will be interesting if the approach considered in [Fio05] will provided new equations for the spectrum of conformal Bullough-Dodd system, which is related to Virasoro algebra as well. It is also worth to mention, that as proposed in the recent paper of Lukyanov and Kotousov [KL20], the desired algebraic equations for BD model should be searched in the Boussinesq integrable system rather than KdV one.

Colored Fock spaces and more general integrable systems. One may wonder that despite affine Yangian commutation relations (1.40) are symmetric with respect to permutations of all ϵ_α, but in the Bethe Ansatz equations (2.27) this symmetry is broken by the source term

$$A(x) = \prod_{k=1}^{n} \frac{x - u_k + \frac{\epsilon_3}{2}}{x - u_k - \frac{\epsilon_3}{2}}.$$

In fact this symmetry is broken by a choice of a particular Fock representation, in order to restore the symmetry back one should introduce three types of Fock modules $\mathcal{F}^\alpha$ (see [FJMM13, BFM18, LS16]), this will provides us with more general integrable systems. In fact, we associate an integrable system to the chain of colored Fock spaces with two colored boundaries $\beta_L \Big| \mathcal{F}_1^{\alpha_1} \otimes \mathcal{F}_2^{\alpha_2} \cdots \otimes \mathcal{F}_n^{\alpha_n} \Big| \beta_R$, $\alpha_i, \beta_{L,R} = 1, 2, 3$. We present the details in Appendix B.1, here we just mention a particular interesting model given as: $1 \Big| \mathcal{F}_1^1 \otimes \mathcal{F}_2^3 \cdots \otimes \mathcal{F}_{2n-1}^1 \otimes \mathcal{F}_{2n}^3 \Big| 3$. This model provides a UV limit for the (dual of) $O(2n+1)$ sigma model considered in [LS18]. Similarly
$3 \Big| \mathcal{F}_1^3 \otimes \mathcal{F}_2^1 \cdots \otimes \mathcal{F}_{2n+1}^3 \Big| 3$ provides the UV limit of $O(2n)$ sigma model.

K**-matrices.** We have mentioned in the main text that there are only three solutions of Sklyanin reflection equation (2.13), which commute with the level. In such a case one can always set the vacuum eigenvalue of $\mathcal{K}$ operator to 1. Then, if we denote

$$\mathcal{K} a_{-1} |u\rangle = g(u) a_{-1} |u\rangle,$$

the reflection relation (2.13) on level 1 is equivalent to the functional relation

$$(u+v)(g(u) - g(v)) = (u-v)(g(u)g(v) - 1) \implies g(u) = \frac{\xi + u}{\xi - u},$$

where ξ is an arbitrary parameter. The reflection relation on level 2 is more restrictive. It is not just fixes the matrix of the K-operator on level 2, but also demands that the parameter ξ takes one of three values

$$\xi = 0, \quad \xi = -\left(b + \frac{1}{2b}\right) \quad \text{or} \quad \xi = -\left(\frac{1}{b} + \frac{b}{2}\right),$$

corresponding to three solutions $\mathcal{K}^{1,2,3}$. In principle, one might go to higher levels and check that there are only three solutions. It would be interesting to prove this statement in general.

Chapter 3

Integrals of motion for the deformed W algebras.

In this chapter we review deformed W algebra. associated to a Lie algebras of B, C, D types. We explicitly construct "local Integrals of Motion" for this W-algebras, which commute with additional affine Screening charge, and mutually commute with each other. Our Integrals of Motion are trigonometric generalisation of integrable structures in non deformed algebras, we argue that their structure is easier in the deformed case. We also present explicit formulas for deformed $R-$matrix and Sklyanin reflection operators.

3.1 Introduction

This chapter is based on my notes which has been never published, but became a part of the joint paper with much more results and details [FJMV21].

From a general philosophy of quantum groups, it is quite natural to consider q-deformations of local IM in search for clarification of the matters. It turns out that after the q-deformation, the local Integral of Motion become non-local, but can be written down explicitly. The q-deformations of W algebras have been provided in [AKOS96] for type A, and in [FR97] for simple Lie algebras, however in this chapter we will rely on [KP18] where the W algebras are defined as a commutant of screenings. The deformations of the local IM associated to the W algebra of type A were constructed in [KOJ06], [FJM17]. In this chapter we provide the construction for the q-deformation of the local Integrals of Motion for the case of BCD type W algebras. We will first define Integrals of Motion as a subset in W algebra which commute with additional affine screening, and then prove their mutual commutativity. We also provide quantum analogs for the reflection and Sklyanin R, K matrices, as well as KZ Integrals of Motion considered in previous chapter (**??**).

As it was noted in [FJM17] the integrable structures of type A can be naturally unified in the quantum toroidal algebra $\hat{\hat{\mathfrak{gl}}}_1$, which may be thought as an analytical continuation of the type A_n W_n-algebra in variable n. Similarly to the A_n case, in [FJMV21] we suggest to unify W currents of type BCD in some other algebra $\mathcal{K}$. It was also shown that algebra $\mathcal{K}$ has a three commutative subalgebras, which after specialisation a concrete representation reproduce the IOMs for W algebras of BCD case. In this chapter we develop more direct and more elementary approach, as I believe that detailed consideration of algebra $\mathcal{K}$ lies beyond the scope of the current thesis, and suggest to an interested reader to read the paper.

This chapter is organised as follows: in section 3.2, we recall a definition of a quiver W-algebra according to the work of Pestun and Kimura [KP18]. In section 3.3, we use a conjecture that the Integrals of Motion could be found as a commutant of affine set of screenings, and provide explicit formulas for them. In section 3.4, we prove that the conjectured Integrals of Motion are indeed commute with each other. In section 3.5 we repeat the arguments of [MO19] and construct the reflection R matrix and Sklyanin's K matrix corresponding to the W algebra, we also provide the formula for the KZ Integrals of Motion which commute with each other and with an infinite series of the local Integrals of Motion built it section 3.3.

3.2 Basic Definitions

In this section we will define q-deformed W algebra following the paper of [KP18]. Our input data will be the numbers $\{q_1, q_2, q_3\}$ such that $q_1 q_2 q_3 = 1$ and a quiver Γ. The rational limit considered in previous chapters may be restored in the limit $q_i = e^{\beta \epsilon_i}$, $\beta \to 0$. For our purposes we will consider quiver Γ to be a Dynkin diagram of an (affine) Lie algebra $\mathfrak{g}$. To the each vertex $i \in \{1, \ldots, rank(\mathfrak{g})\}$ of this quiver we will assign a half integer number d_i which is the squared norm of corresponding root $d_i = (\alpha_i \cdot \alpha_i)$, in our notations roots in the middle of a Dynkin diagram normalised to have a unit lengths. The edges between i-th and j-th vertex will be denoted as $e(i \to j)$, to any oriented edge e it is assigned a number μ_e, in all examples we will put all $\mu_e = q_3^{\frac{1}{2}}$, except the $\hat{gl}(N)$ case, for which $\mu_{i,i+1} = q_3^{1/2}, i = 1, \ldots N-1$, $\mu_{n,1} = \mu q_3^{1/2}$, [1]. We will also define (q_1, q_2) deformed Cartan matrix $c_{i,j}$ and it's symmetrysation $b_{i,j} = c_{i,j} \frac{1-q_1^{d_i}}{1-q_1}$

$$c_{i,j} = (1 + q_2^{-1} q_1^{-d_i}) \delta_{i,j} - \sum_{e(i\to j)} \mu_e \frac{1 - q_1^{-d_i}}{1 - q_1^{-d_{i,j}}} - \sum_{e(j\to i)} \mu_e^{-1} q_1^{-d_{i,j}} q_2^{-1} \frac{1 - q_1^{-d_i}}{1 - q_1^{-d_{i,j}}}, \tag{3.1}$$

here , $d_{i,j} = min(d_i, d_j)$[2]. We will also use a shorthand notation $f^{[n]}(q_1, q_2, \mu) = f(q_1^n, q_2^n, \mu^n)$.
We define a W-algebra as a commutant of a set of screenings $\mathcal{S}_i = \oint \mathcal{S}_i(z) \frac{dz}{2\pi z}$:

$$\mathcal{S}_i(z) =: e^{\sum\limits_{k\neq 0} s_{i,k} z^{-k} + s_{i,0} \log(z) + Q_i} : \tag{3.2}$$

$$[s_{i,n}, s_{j,m}] = -c_{i,j}^{[n]} \frac{1}{n} \frac{1 - q_1^{d_i n}}{1 - q_2^{-n}} \delta_{n+m,0}, \tag{3.3}$$

$$[Q_i, s_{j,0}] = -\frac{\log(q_1)}{\log(q_2^{-1})} c_{i,j}^0.$$

It is convenient to write down a W-currents in terms of Loran series in Y operators:

$$Y_i(z) =: e^{\sum\limits_{k\neq 0} y_{i,k} z^{-k} + y_{i,0}} :$$

$$y_{i,n} = (1 - q_2^{-n})(c^{[n]})^{-1}_{i,j} s_{j,n}. \tag{3.4}$$

Commutation relations of the screening currents (3.7) and definition of the Y-operators (3.4), imply commutation relations:

$$[y_{i,n}, s_{j,m}] = -\frac{1}{n}(1 - q_1^{d_i n}) \delta_{i,j} \delta_{n+m,0},$$

$$[y_{i,n}, y_{j,m}] = -\frac{1}{n}(1 - q_1^{d_i n})(1 - q_2^n)(c^{[-n]})^{-1}_{i,j} \delta_{n+m,0}.$$

[1] In general parameters μ_e are irrelevant, except the cases when quiver has loops, in that case W-algebra depends only on the product of μ_e along the loop.

[2] Originally in paper [KP18], $d_{i,j} = \frac{1}{2} gcd(2d_i, 2d_j)$, however as we are dealing with ABCD series only we may replace in by $d_{i,j} = min(d_i, d_j)$

It follows that the Y operators are almost commute with the screenings, except some contact terms:

$$[Y_i(z), \mathcal{S}_i(w)] = (1 - q_1^{-d_i})\delta(q_1^{d_i}\frac{w}{z}) : Y_i(z)\mathcal{S}_i(w) :$$

Which implies that commutator of a Screening $\mathcal{S}_i$ and a Y- operator is a local operator:

$$[Y_i(z), \mathcal{S}_i] = (1 - q_1^{-d_i}) : Y_i(z)\mathcal{S}_i(zq_1^{-d_i}) :$$

Remarkably , there exist another operator : $Y_i(z)A^{-1}(q_1^{-d}q_2^{-1}z)$: [KP18] , which has the same commutator with the screening charge $\mathcal{S}_i = \oint \mathcal{S}_i(w)\frac{dw}{2\pi w}$.

$$\Big[: Y_i(z)\frac{\mathcal{S}_i(q_1^{-d_i}z)}{\mathcal{S}_i(q_1^{-d_i}q_2^{-1}z)} :, \mathcal{S}_i\Big] = -(1 - q_1^{-d_i}) : Y_i(z)\mathcal{S}_i(q_1^{-d_i}z) :$$

The operator $A_i(z) = \frac{\mathcal{S}_i(z)}{\mathcal{S}_i(q_2 z)}$ can be expressed in terms of the $Y_j(z)$

$$A_i(z) =: Y_i(z)Y_i(q_1^{d_i}q_2 z)\left(\prod_{e(i\to j)}\prod_{r=0}^{\frac{d_j}{d_{i,j}}-1} Y_j(\mu_e q_1^{rd_{i,j}}x)\prod_{e(j\to i)}\prod_{r=0}^{\frac{d_j}{d_{i,j}}-1} Y_j(\mu_e^{-1}q_1^{(r+1)d_{i,j}}q_2 x)\right)^{-1} : \qquad (3.5)$$

Having these operators at hand, we can always complete any expression in a way that it will commute with i-th screening $\mathcal{S}_i$, for example $Y_i(x)$ we will be completed to $Y_i(x)+ : A_i^{-1}(q_1^{-d_i}q_2^{-1}x)Y_i(x)$: which commute with $\mathcal{S}_i$:

$$\begin{aligned} &Y_i(x)+ : A_i^{-1}(q_1^{-d_i}q_2^{-1}x)Y_1(x) := \\ &=Y_i(x)+ : \frac{\left(\prod_{e(i\to j)}\prod_{r=0}^{\frac{d_j}{d_{i,j}}-1} Y_j(\mu_e q q_1^{rd_{i,j}-d_i}x)\prod_{e(j\to i)}\prod_{r=0}^{\frac{d_j}{d_{i,j}}-1} Y_j(\mu_e^{-1}q_1^{(r+1)d_{i,j}-d_i}x)\right)}{Y_i(q_1^{-d_i}q_2^{-1}z)} : \end{aligned}$$

Such combination of operators commute with i-th screening $\mathcal{S}_i$, but doesn't commute with it's nearest neighbours, the strategy is to pick a different screening charge $\mathcal{S}_{i\pm 1}$ and complete the expression to commute with them, and so on. This construction is similar to the one which apeeared in representation theory, namely Y_i is an analog of (exponent of) fundamental weight, while A_i^{-1} is an analog of simple root. For a quivers which are the Dynkin diagrams of a simple Lie algebras, this procedure will have a finite number of terms equal to the dimension of i-th fundamental representation.

So by definition the i-th fundamental W current $T_i(z)$ is a current which starts with $Y_i(z)$ and then completed to commute with all the screenings:

$$T^i(z) = Y_i(z)+ : A_i^{-1}(q_1^{-d_i}q_2^{-1}x)Y_i(x) : + \dots$$

3.2.1 Higher W currents

If there is a product of several $Y_i(z)Y_i(w)$ operators with common i, additional factors S_{d_i} should be added. The combination which starts with $Y_i(z)Y_i(w)$ and commute with $\mathcal{S}_i$ is equal to:

$$\begin{aligned} W_2^i(z,w) =: Y_i(z)Y_i(w) : + S_{d_i}(\frac{w}{z}) : \frac{Y_i(z)Y_i(w)}{A_i(q_1^{-d_i}q_2^{-1}z)} : +S_{d_i}(\frac{z}{w}) : \frac{Y_i(z)Y_i(w)}{A_i(q_1^{-d_i}q_2^{-1}w)} : + \\ + : \frac{Y_i(z)Y_i(w)}{A_i(q_1^{-d_i}q_2^{-1}w)A_i(q_1^{-d_i}q_2^{-1}w)} : \end{aligned} \qquad (3.6)$$

$$S_d(x) = \frac{(1-q_1^d x)(1-q_2 x)}{(1-x)(1-q_1^d q_2 x)} = \prod_{k=0}^{d-1} S(q_1 z). \tag{3.7}$$

Note that the current $T_2^i(z,w)$ should be properly understood: we assume that the first argument is larger than the second one $|z| \gg |w|$ (radial ordering)[3]. Easily to observe, that current $T_2^i(z,w)$ is almost symmetric, except some delta terms arising from pole contribution of $S_{d_i}(\frac{z}{w})$.

$$W_2^i(z,w) - W_2^i(w,z) = \frac{(1-q_2)(1-q_1^{d_i})}{1-q_1^{d_i}q_2}\delta(\frac{zq_1^{d_i}}{qw}) : \frac{Y_i(z)Y_i(\frac{zq_1^{d_i}}{q})}{A_i(z)} : -$$

$$- \frac{(1-q_2)(1-q_1^{d_i})}{1-q_1^{d_i}q_2}\delta(\frac{wq_1^{d_i}q_2}{z}) : \frac{Y_i(z)Y_i(q_1^{-d_i}q_2^{-1})}{A_i(q_1^{-d_i}q_2^{-1}z)} :$$

A higher W-current is defined to start with a product of Y operators, and then completed to commute with all screenings:

$$T_n^i(z_1,\ldots,z_n) =: \prod_{i=1}^n Y_i(z_i) : + \ldots \tag{3.8}$$

This current could also be defined as a product of n fundamental currents:

$$T_n^i(z_1,\ldots,z_n) = \prod_{i<j}\left(f^i(\frac{z_i}{z_j})\right)^{-1} \prod_{i=1}^n T^i(z_i). \tag{3.9}$$

Function $f_i(x)$ could be found by normal ordering of $Y_i(z)$ with $Y_i(w)$.

$$Y_i(z)Y_i(w) = f^i\left(\frac{w}{z}\right) : Y_i(w)Y_i(z) :$$

3.2.2 Example of W_N algebra

In this example, we choose conventions which already adopted for affine system of screenings $\hat{\mathfrak{gl}}(N)$, we will have one extra (affine)screening S_N and one extra Y-current $Y_N(z)$, but will not demand a commutativity with S_N. Cartan matrix of affine $\mathfrak{gl}(N)$ is equal to:

$$c_{i,j} = (1+q_3)\delta_{i,j} - \sum_{i=1}^{N-1}(\delta_{i,i+1}+\delta_{i,i-1})\sqrt{q_3} - \delta_{1,N}\mu\sqrt{q_3} - \delta_{N,1}\mu^{-1}\sqrt{q_3}, \tag{3.11}$$

W_N-algebra corresponds to a linear quiver, and there is $N-1$ currents which corresponds to a $N-1$ fundamental representations of $\mathfrak{gl}(N)$ algebra. For the first fundamental representation we have a sum of N terms:

$$T^1(z) = \frac{Y_1(z)}{Y_N(\mu\sqrt{q_3}z)} + \frac{Y_2(\sqrt{q_3}z)}{Y_1(q_3 z)} + \frac{Y_3(q_3 z)}{Y_2(q_3^{\frac{3}{2}}z)} + \cdots + \frac{Y_{N-1}(q_3^{\frac{N-1}{2}}z)}{Y_{N-2}(q_3^{\frac{N}{2}}z)} + \frac{Y_N(q_3^{\frac{N}{2}}z)}{Y_{N-1}(q_3^{\frac{N+1}{2}}z)}.$$

For the second fundamental representation we have the sum of $\frac{N(N-1)}{2}$ terms which coincides with the dimension of this representation:

$$T^2(z) = \frac{Y_2(z)}{Y_N(\mu q_3)} + \frac{Y_1(\sqrt{q_3}z)Y_3(\sqrt{q_3}z)}{Y_2(q_3 z)Y_N(\mu q_3)} + \cdots + \frac{Y_{N-2}(q_3^{\frac{N-1}{2}}z)Y_N(q_3^{\frac{N-1}{2}}z)}{Y_{N-1}(q_3^{\frac{N}{2}}z)Y_{N-3}(q_3^{\frac{N}{2}}z)} + \frac{Y_N(q_3^{\frac{N-1}{2}}z)}{Y_{N-2}(q_3^{\frac{N+1}{2}}z)},$$

[3] Actually we write $|z| \gg |w|$ here, because there are poles not only at $z = w$, but also at a shifted point $z = (q_1^d q_2)^{\pm 1} w$

...

The last current corresponds to the anti-fundamental representation of $\mathfrak{gl}(N)$:

$$T^{N-1}(z) = \frac{Y_{N-1}(z)}{Y_N(\sqrt{q_3}z)} + \frac{Y_{N-2}(\sqrt{q_3}z)}{Y_{N-1}q_3 z} + \frac{Y_{N-3}(q_3 z)}{Y_{N-4}(q_3^{\frac{3}{2}}z)} + \cdots + \frac{Y_1(q_3^{\frac{N-1}{2}}z)}{Y_2(q_3^{\frac{N}{2}}z)} + \frac{Y_N(\mu q_3^{\frac{N}{2}}z)}{Y_1(q_3^{\frac{N+1}{2}}z)}.$$

Let us find a commutation relations in W_N-algebra. It could be done directly using commutation relations of $Y_i(z)$ currents, but the more convenient way is to use the formula (3.6)

$$T_1(z)T_1(w) = f_\mu\Big(\frac{w}{z}\Big)T_2(z,w) = f_\mu\Big(\frac{w}{z}\Big)\Bigg[: \frac{Y_1(z)}{Y_N(\mu\sqrt{q_3}z)}\frac{Y_1(w)}{Y_N(\mu\sqrt{q_3}w)} : + S(\frac{w}{z}) : \frac{Y_2(\sqrt{q_3}z)}{Y_1(q_3 z)}\frac{Y_1(w)}{Y_n(\mu\sqrt{q_3}w)} : +$$

$$+S\Big(\frac{z}{w}\Big) : \frac{Y_2(\sqrt{q_3}w)}{Y_1(q_3 w)}\frac{Y_1(z)}{Y_n(\mu\sqrt{q_3}z)} : + : \frac{Y_2(\sqrt{q_3}w)}{Y_1(q_3 w)}\frac{Y_2(\sqrt{q_3}z)}{Y_1(q_3 z)} : + \cdots + : \frac{Y_N(q_3^{\frac{N}{2}}z)}{Y_{N-1}(q_3^{\frac{N+1}{2}}z)}\frac{Y_N(q_3^{\frac{N}{2}}w)}{Y_{N-1}(q_3^{\frac{N+1}{2}}w)} : \Bigg],$$

here $S(x) = S_1(x)$ from eq (3.2).
Function $f(x)$ could be found by normal ordering of $\frac{Y_1(z)}{Y_N(\mu\sqrt{q_3}z)}$ with $\frac{Y_1(w)}{Y_N(\mu\sqrt{q_3}w)}$,

$$\frac{Y_1(z)}{Y_N(\mu\sqrt{q_3}z)}\frac{Y_1(w)}{Y_N(\mu\sqrt{q_3}w)} = f_\mu\Big(\frac{w}{z}\Big) : \frac{Y_1(z)}{Y_N(\mu\sqrt{q_3}z)}\frac{Y_1(w)}{Y_N(\mu\sqrt{q_3}w)} :$$

$$f_\mu(z) = exp\left(-\sum_{n>0}\frac{1}{n}(1-q_1^n)(1-q_2^n)\frac{(1-\mu^n q_3^{-n\frac{N-2}{2}})}{(1-\mu^n q_3^{-n\frac{N}{2}})}z^n \right).$$

Analysing the poles of $S(\frac{w}{z})$ function we get:

$$T^1(z)T^1(w)f_\mu^{-1}(\frac{w}{z}) - T^1(w)T^1(z)f_\mu^{-1}\Big(\frac{z}{w}\Big) =$$
$$= \frac{(1-q_2)(1-q_1)}{1-q_3^{-1}}\delta\Big(\frac{z}{q_3 w}\Big) : T^2(\sqrt{q_3}z) - \frac{(1-q_2)(1-q_1)}{1-q_3^{-1}}\delta\Big(\frac{q_3 w}{z}\Big)T^2\Big(\frac{z}{\sqrt{q_3}}\Big).$$

3.3 Commutant of affine set of screenings

In this section we will consider quivers corresponding to affine Dynkin diagrams, we will have one additional affine Screening $\hat{S}(z)$. According to the procedure of building a W-current described in the previous section one would get an infinite number of terms [KP18]. It is not clear what is the meaning of corresponding W-algebras. In this notes we will do quite a different thing: we will search for a subalgebra of W algebra which commute with affine screenings.

$$I_n \subset W(g)\ ,\ [I_n, \hat{\mathcal{S}}] = 0.$$

In this section we will explain how to construct such elements I_n, and consider some examples of type A. In the next section we will consider in details the Integrals of Motion of D type and prove their commutativity.

3.3.1 Example of $\hat{\mathfrak{gl}}(2)$

This section is an illustration on how to construct commutant of screenings in the simplest case of $\mathfrak{gl}(2)$ W alpgebra. According to the section 3.2 we have the q-deformed $\hat{\mathfrak{gl}}(2)$ cartan matrix of the form (3.11) , two screening currents $\mathcal{S}_1(z)$, $\mathcal{S}_2(z) = \hat{S}(z)$ and two Y_i currents. Our strategy will be the following: let us wright explicitly the fundamental current which corresponds to a W-algebra commuting with $\mathcal{S}_1$

$$T^1(z) = \frac{Y_1(z)}{Y_2(\mu\sqrt{q_3}z)} + \frac{Y_2(\sqrt{q_3}z)}{Y_1(q_3z)}.$$

And the fundamental current which commutes with $\mathcal{S}_2$

$$\hat{T}^1(z) = \frac{Y_2(z)}{Y_1(\sqrt{q_3}z)} + \frac{Y_1(\mu^{-1}\sqrt{q_3}z)}{Y_2(q_3z)}$$

This two currents are sums of two terms which are identical up to some shifts of arguments, and we conclude that their zero modes coincides[4]:

$$I_1 = \oint T^1(z)\frac{dz}{2\pi z} = \oint \hat{T}^1(z)\frac{dz}{2\pi z}. \tag{3.19}$$

The operator I_1 coincides with the first Integral of Motion considered in [FJM17], [FKSW07]. Under the limit $\mu \to 0$ we reproduce generalized Macdonald integrable system, current $T^1(z)|_{\mu=0} = e(z)$ where $e(z)$ is a Ding-Iohara-Mikki current at level 2 , [FJM17].
To construct the higher Integrals of Motion, let us consider the current which corresponds to a tensor product of the two fundamental representations (3.6):

$$\begin{aligned} T^1_2(z,w) =&: \frac{Y_1(z)}{Y_2(\mu\sqrt{q_3}z)}\frac{Y_1(w)}{Y_2(\mu\sqrt{q_3}w)} : + S\Big(\frac{z}{w}\Big) : \frac{Y_1(z)}{Y_2(\mu\sqrt{q_3}z)}\frac{Y_2(\sqrt{q_3}w)}{Y_1(q_3w)} : + \\ &+ S\Big(\frac{w}{z}\Big) : \frac{Y_1(w)}{Y_2(\mu\sqrt{q_3}w)}\frac{Y_2(\sqrt{q_3}z)}{Y_1(q_3z)} : + : \frac{Y_2(\sqrt{q_3}z)}{Y_1(q_3z)}\frac{Y_2(\sqrt{q_3}w)}{Y_1(q_3w)} : \end{aligned}$$

Here $S(x) = S_1(x)$ from section 3.2 (3.7)
There is also another current, which commutes with $\hat{S} = \mathcal{S}_2$

$$\begin{aligned} \hat{T}^1_2(z,w) =&: \frac{Y_2(z)}{Y_1(\sqrt{q_3}z)}\frac{Y_2(w)}{Y_1(\sqrt{q_3}w)} : + S\Big(\frac{z}{w}\Big) : \frac{Y_2(z)}{Y_1(\sqrt{q_3}z)}\frac{Y_1(\mu^{-1}\sqrt{q_3}w)}{Y_2(q_3w)} : + \\ &+ S\Big(\frac{w}{z}\Big) : \frac{Y_1(\mu^{-1}\sqrt{q_3}z)}{Y_2(q_3z)}\frac{Y_2(w)}{Y_1(\sqrt{q_3}w)} : + : \frac{Y_1(\mu^{-1}\sqrt{q_3}z)}{Y_2(q_3z)}\frac{Y_1(\mu^{-1}\sqrt{q_3}w)}{Y_2(q_3w)} : \end{aligned}$$

Again these two are the sums of four similar terms, but now after a shifts of arguments current $\hat{T}^2$ will contain shifted rational functions $S(\frac{q_3}{\mu}\frac{w}{z})$. The idea is to find a commutant of two screenings as a convolution which will absorb this difference:

$$I_2 = \oint_C S_{\infty,\tau}(\frac{z}{w})T^1_2(z,w)\frac{dz}{2\pi z}\frac{dw}{2\pi w} = \oint_C S_{\infty,\tau}(\frac{z}{w})\hat{T}^1_2(z,w)\frac{dz}{2\pi z}\frac{dw}{2\pi w}. \tag{3.22}$$

If (3.22) is true, than obviously $[I_2, S_{1,2}] = 0$. Equation (3.22) is equivalent to a system of difference equations on a function $S_{\infty,\tau}(z)$:

$$\begin{aligned} S_{\infty,\tau}\Big(\frac{z}{w}\Big) &= S_{\infty,\tau}\Big(\frac{w}{z}\Big), \\ S_{\infty,\tau}\Big(\frac{z}{w}\Big)S\Big(\frac{z}{w}\Big) &= S_{\infty,\tau}\Big(\frac{\mu}{q_3}\frac{z}{w}\Big)S\Big(\frac{q_3}{\mu}\frac{w}{z}\Big). \end{aligned}$$

[4]Note that the definition of fundamental currents is ambiguous, for example T^1 can be multiplyed by any function of Y_2 operators. However, equation (3.19) uniquely fixes this ambiguity.

Solution to this equations could be conveniently written in terms of the quantum dilogarithm functions [KOJ06]:

$$(z;\tau)_\infty = \prod_{n=0}^{\infty}(1-z\tau^n)\ ;\ \tau = \frac{\mu}{q_3},$$

$$S_{\infty,\tau}(z) = \sigma(z)\sigma(1/z) = \frac{(z;\tau)_\infty (zq_3^{-1};\tau)_\infty}{(q_1 z;\tau)_\infty (q_2 z;\tau)_\infty}\frac{(z^{-1};\tau)_\infty (z^{-1}q_3^{-1};\tau)_\infty}{(q_1 z^{-1};\tau)_\infty (z^{-1}q_2;\tau)_\infty}.$$

Proceeding further, one could construct higher Integrals of Motion:

$$I_n = \oint_C \prod_{1\le j<i\le n} S_{\infty,\tau}\Big(\frac{z_i}{z_j}\Big) T_n^1(z_1,\dots,z_n)\prod_{i=1}^{n}\frac{dz_i}{2\pi z_i},$$

here $T_n^1(z_1,\dots,z_n)$ is a higher W current (3.2.1) :

$$T_n^1(z_1,\dots,z_n) =: \prod_{i=1}^{n}\frac{Y_1(z_i)}{Y_2(\mu\sqrt{q_3}z_i)} : + \dots$$

Here, one should specify the contour of integration: our choice is to do all integration along the unit circle $|z_i| = 1$. Such choice of contour is clearly symmetric under the permutation of variables $z_i \leftrightarrow z_j$, another benefit of this contour is that functions $S_{\infty,\tau}(z)$ could be expanded in a convergent Tailor series, and the whole integral of motion could be computed as a zero mode of some series:

$$I_n = \Bigg[\prod_{1\le j<i\le n} S_{\infty,\tau}\Big(\frac{z_i}{z_j}\Big) T_n^1(z_1,\dots,z_n)\Bigg]_0.$$

Where for function $f(z_1,\dots,z_n) = \sum_{i_k=0}^{\infty} f_{i_1,\dots,i_n} z_1^{i_1} {}_n^{i_n}$ we defined $[f(z)]_0 = f_{0,\dots,0}$. Operators I_n coincides with the ones built in [FKSW07], [FJM17].

3.3.2 Example of $\hat{\mathfrak{gl}}(4) = \hat{\mathfrak{so}}(6)$

In this section we will use the isomorphism between $\hat{\mathfrak{gl}}(4)$ and $\hat{\mathfrak{so}}(6)$ affine Lie algebras, we then expect to have an Integrals of Motion built from the second fundamental current of $\mathfrak{gl}(4)$ W algebra which at the same time is the fundamental current of $\mathfrak{so}(6)$ W algebra. However we will observe that the Integrals of Motion of this type appears only at a spectial parameter of a twist $\mu = 1$.
It is easy to check that similarly to the previous case, zero modes of fundamental current $T^1(z)$ and anti-fundamental current $T^3(z)$ commute with all screenings of $\mathfrak{gl}(4)$

$$I_1^1 = \oint \frac{dz}{2\pi z}T^1(z) = \oint \frac{dz}{2\pi z}\left(\frac{Y_1(z)}{Y_4(\mu q_3^{1/2}z)} + \frac{Y_2(q_3^{1/2}z)}{Y_1(q_3 z)} + \frac{Y_3(q_3 z)}{Y_2(q_3^{3/2}z)} + \frac{Y_4(q_3^{3/2}z)}{Y_3(q_3^2 z)}\right),$$

and

$$I_1^3 = \oint \frac{dz}{2\pi z}T^3(z) = \oint \frac{dz}{2\pi z}\left(\frac{Y_3(z)}{Y_4(q_3^{1/2}z)} + \frac{Y_2(q_3^{1/2}z)}{Y_3(q_3 z)} + \frac{Y_1(q_3 z)}{Y_2(q_3^{3/2}z)} + \frac{Y_4(\mu q_3^{3/2}z)}{Y_1(q_3^2 z)}\right).$$

There also exists another current , which corresponds to the second fundamental representation of $\mathfrak{gl}(4)$ or fundamental representation of $\mathfrak{so}(6)$:

$$T^2(z) = \Big(\frac{Y_2(z)}{Y_4(\mu q_3 z)} + \frac{Y_1(q_3^{1/2}z)Y_3(q_3^{1/2}z)}{Y_4(\mu q_3 z)Y_2(q_3 z)} + \frac{Y_3(q_3^{1/2}z)}{Y_1(q_3^{3/2}z)} + \frac{Y_1(q_3^{1/2}z)Y_4(q_3 z)}{Y_3(q_3^{3/2}z)Y_4(\mu q_3 z)} + \frac{Y_4(q_3 z)Y_2(q_3 z)}{Y_1(q_3^2 z)Y_3(q_3^2 z)} + \frac{Y_4(q_3 z)}{Y_2(q_3^2 z)}\Big) \tag{3.27}$$

And commutes with $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$. One can also construct another current:

$$\hat{T}^2(z) = \left(\frac{Y_4(z)}{Y_2(q_3 z)} + \frac{Y_1(\mu^{-1} q_3^{1/2} z) Y_3(q_3^{1/2} z)}{Y_4(q_3 z) Y_2(q_3 z)} + \frac{Y_3(q_3^{1/2} z)}{Y_1(\mu^{-1} q_3^{3/2} z)} + \frac{Y_1(\mu^{-1} q_3^{1/2} z)}{Y_3(q_3^{3/2} z)} + \frac{Y_4(q_3 z) Y_2(q_3 z)}{Y_1(q_3^2 z) Y_3(q_3^2 z)} + \frac{Y_2(q_3 z)}{Y_4(q_3^2 z)} \right) \tag{3.28}$$

Which commutes with $\mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$.

We see that zero modes of the currents (3.27) and (3.28) don't coincide, except the case $\mu = 1$. We conclude that at the special value of twist parameter ($\mu = 1$): additional current appears, we will see that W-algebras of B, C, D don't have such a parameter and should be thought as $\mu = 1$.

For the end of this section, let us prove that $[I_1^1, I_1^3] = 0$ for any μ. First of all, let us note that due to commutativity of two currents: $[\frac{Y_1(z)}{Y_4(\mu q_3^{1/2} z)}, \frac{Y_3(w)}{Y_4(q_3^{1/2} w)}] = 0$, corresponding currents $T^1(z)$ and $T^3(w)$ commute up to delta terms:

$$\begin{aligned}
[T^1(z), T^3(w)] = \frac{(1-q_2)(1-q_1)}{1-q_3^{-1}} \Big(&\delta(\frac{z}{q_3^2 w}) : \frac{Y_1(z)}{Y_4(\mu q_3^{1/2} z)} \frac{Y_4(\mu q_3^{3/2} w)}{Y_1(q_3^2 w)} : - \delta(\frac{w\mu}{q_3 z}) : \frac{Y_1(z)}{Y_4(\mu q_3^{1/2} z)} \frac{Y_4(\mu q_3^{3/2} w)}{Y_1(q_3^2 w)} : + \\
&+ \delta(\frac{z}{q_3 w}) : \frac{Y_2(q_3^{1/2} z)}{Y_1(q_3 z)} \frac{Y_1(q_3 w)}{Y_2(q_3^{3/2} w)} : - \delta(\frac{w}{z}) : \frac{Y_2(q_3^{1/2} z)}{Y_1(q_3 z)} \frac{Y_1(q_3 w)}{Y_2(q_3^{3/2} w)} : + \\
&+ \delta(\frac{z}{w}) : \frac{Y_3(q_3 z)}{Y_2(q_3^{3/2} z)} \frac{Y_2(q_3^{1/2} w)}{Y_3(q_3 w)} : - \delta(\frac{w}{q_3 z}) : \frac{Y_3(q_3 z)}{Y_2(q_3^{3/2} z)} \frac{Y_2(q_3^{1/2} w)}{Y_3(q_3 w)} : + \\
&+ \delta(\frac{q_3^2 z}{w}) : \frac{Y_4(q_3^{3/2} z)}{Y_3(q_3^2 z)} \frac{Y_3(w)}{Y_4(q_3^{1/2} w)} : - \delta(\frac{w}{q_3 z}) : \frac{Y_4(q_3^{3/2} z)}{Y_3(q_3^2 z)} \frac{Y_3(w)}{Y_4(q_3^{1/2} w)} : \Big)
\end{aligned}$$

Now it is easy to note that under the integral over $\oint \frac{dz}{2\pi z} \frac{dw}{2\pi w}$ all delta terms cancels with each other.

$$[I_1^1, I_1^3] = \oint \frac{dz}{2\pi z} \frac{dw}{2\pi w} [T^1(z), T^3(w)] = 0$$

Now let us look at the OPE of W current corresponding to the first and second fundamental representations [5]

$$T^1(z) T^2(w) = S\Big(\frac{q_3^{1/2} w}{z}\Big) f_\mu\Big(\frac{w}{q_3^{1/2} z}\Big) f_\mu\Big(\frac{q_3^{1/2} w}{z}\Big) \left(: \frac{Y_1(z)}{Y_4(\mu q_3^{1/2} z)} \frac{Y_4(z)}{Y_2(q_3 z)} : + \dots \right)$$

$$f_\mu\Big(\frac{w}{q_3^{1/2} z}\Big) f_\mu\Big(\frac{q_3^{1/2} w}{z}\Big) = exp \left(\sum_{n>0} \frac{1}{n} (1-q_1^n)(1-q_2^n) \frac{(1-\mu^n q_3^{-n})(q_3^{-n/2} + q_3^{n/2})}{(1-\mu^n q_3^{-2n})} \Big(\frac{w}{z}\Big)^n \right)$$

It turns out that at the point $\mu = 1$ the scalar factors exactly cancel each other, and two currents becomes local with respect to each other, and commute up to delta terms. It could be checked that these delta terms cancels under the integral which picks up zero modes, the last statement proofs the commutativity of $[I_1^1, I_1^2] = 0$. [6]
Each of three local currents I_1^i produce a series of integrals:[7]

$$I_n^i = \oint_C \prod_{1 \le j < i \le n} S_{\infty, \tau = q_3^{-1-|i-2|}} \Big(\frac{z_i}{z_j}\Big) T_n^i(z_1, \dots, z_n) \prod_{i=1}^n \frac{dz_i}{2\pi z_i} \,,$$

[5] As explained in [KP18], see also first section of the current manuscript, there is a procedure to uniquely determine "..." by the condition of commutativity with all the screenings, together with some minimality assumption.

[6] Appearance of additional Integrals of Motion for the special values of twist parameter looks similar to ones revealed on Rybnikov Ilyin arXiv:1810.07308

[7] Here $\tau = q_3^{-1-|i-2|}$ is just a short notation to say that $\tau = q_3^{-2}$, for $i = 1, 3$ and $\tau = q_3^{-1}$, for $i = 2$.

here $T_n^i(z_1,\dots,z_n)$ is a higher W current (3.2.1):

$$T_n^1(z_1,\dots,z_n) =: \prod_{i=1}^{n} \frac{Y_1(z_i)}{Y_4(\sqrt{q_3} z_i)} : + \dots$$
$$T_n^2(z_1,\dots,z_n) =: \prod_{i=1}^{n} \frac{Y_2(z_i)}{Y_4(q_3 z_i)} : + \dots$$
$$T_n^3(z_1,\dots,z_n) =: \prod_{i=1}^{n} \frac{Y_3(z)}{Y_4(q_3^{1/2} z)} : + \dots$$

The mutual commutativity of $I_n^{1,3}$ was proven in [FKSW07], the mutual commutativity of I_n^2 and more generally IOMs associated with $\hat{\mathfrak{so}}(2N)$ affine Lie algebra will be proven in the next section.

3.4 Integrals of Motion of $\hat{\mathfrak{so}}(2N)$ type.

In this section we will be concentrated on the $\hat{\mathrm{D}}_N$ Dynkin diagram (fig 3.1). Let us consider $S_{\bar{1}}$

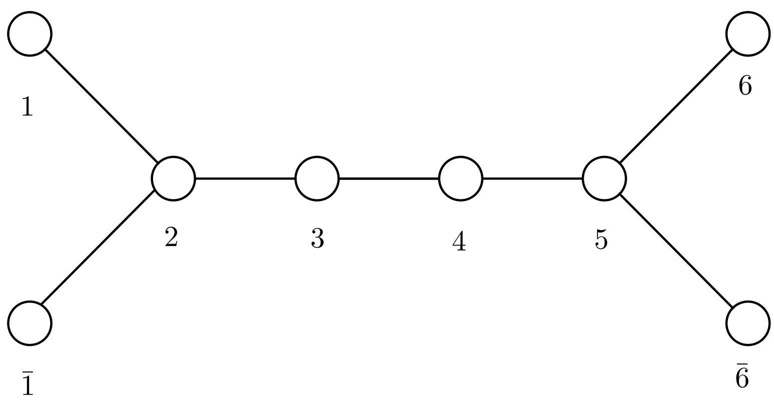


Figure 3.1: $\hat{\mathrm{D}}_7$ Dynkin diagram

screening as an affine one. The definition of fundamental W current $T^1(z)$ is ambiguous, namely it may be multiplied on any operator depending of $Y_{\bar{1}}$. We fix this ambiguity in way that its zero mode $I_1 = \oint \frac{dz}{2\pi z} T^1(z)$ commutes with the affine screening as well:

$$T^1(z) = \frac{Y_1(z)}{Y_{\bar{1}}(q_3 z)} + \frac{Y_2(q_3^{1/2} z)}{Y_{\bar{1}}(q_3 z) Y_1(q_3 z)} + \frac{Y_3(q_3 z)}{Y_2(q_3^{3/2} z)} + \frac{Y_4(q_3^{3/2} z)}{Y_3(q_3^2 z)} + \dots + \frac{Y_{N-1}(q_3^{\frac{N-2}{2}} z) Y_{\overline{N-1}}(q_3^{\frac{N-2}{2}} z)}{Y_{N-2}(q_3^{\frac{N-1}{2}} z)} +$$
$$+ \frac{Y_{N-1}(q_3^{\frac{N-2}{2}} z)}{Y_{\overline{N-1}}(q_3^{\frac{N}{2}} z)} + \frac{Y_{\overline{N-1}}(q_3^{\frac{N-2}{2}} z)}{Y_{N-1}(q_3^{\frac{N}{2}} z)} + \frac{Y_{N-2}(q_3^{\frac{N-1}{2}} z)}{Y_{N-1}(q_3^{\frac{N}{2}} z) Y_{\overline{N-1}}(q_3^{\frac{N}{2}} z)} + \frac{Y_{N-3}(q_3^{\frac{N}{2}} z)}{Y_{N-2}(q_3^{\frac{N+1}{2}} z)} + \dots$$
$$+ \frac{Y_{\bar{1}}(q_3^{N-2} z) Y_1(q_3^{N-2} z)}{Y_2(q_3^{\frac{2N-1}{2}} z)} + \frac{Y_{\bar{1}}(q_3^{N-2} z)}{Y_1(q_3^{N-1} z)}$$

Indeed zero mode $I_1 = \oint \frac{dz}{2\pi z} T^1(z)$ is symmetric with respect to exchange $1 \leftrightarrow \bar{1}$, and so commute with additional affine screening $S_{\bar{1}}$.

Analytical properties of higher W current Before we proceed further let us describe some properties of this current. The fundamental W current is a sum of $2N$ terms:

$$T^1(z) = \sum_1^N \Lambda_i(z) + \sum_{N+1}^{2N} \Lambda_i(z).$$

The contractions of different terms may be computed and written explicitly as:

$$\Lambda_i(z)\Lambda_i(w) = f_\tau\Big(\frac{w}{z}\Big) : \Lambda_i(z)\Lambda_i(w) : \tag{3.39}$$

$$\Lambda_i(z)\Lambda_j(w) = f_\tau\Big(\frac{w}{z}\Big)S\Big(\frac{z}{w}\Big) : \Lambda_i(z)\Lambda_j(w) :, \quad i < j\,, i \neq 2N - j,$$

$$\Lambda_i(z)\Lambda_j(w) = f_\tau\Big(\frac{w}{z}\Big)S\Big(\frac{w}{z}\Big) : \Lambda_i(z)\Lambda_j(w) :, \quad i > j\,, i \neq 2N - j,$$

$$\Lambda_i(z)\Lambda_{2N-i}(w) = f_\tau\Big(\frac{w}{z}\Big)S\Big(\frac{z}{w}\Big)S\Big(\frac{z}{w}q_3^{-N+i+1}\Big) : \Lambda_i(z)\Lambda_{2N-i}(w) :, \quad i < N \tag{3.40}$$

$$\Lambda_{2N-i}(z)\Lambda_i(w) = f_\tau\Big(\frac{w}{z}\Big)S\Big(\frac{w}{z}\Big)S\Big(\frac{w}{z}q_3^{-N+i+1}\Big) : \Lambda_{2N-i}(z)\Lambda_i(w) :, \quad i < N, \tag{3.41}$$

where

$$f_\tau(z) = exp\left(\sum_{n>0}\frac{1}{n}(1-q_1^n)(1-q_2^n)\frac{(1-\tau^n q_3^n)}{(1-\tau^n)}z^n\right), \quad \text{with } \tau = q_3^{2-N}.$$

Now let us consider the higher W current

$$T_2^1(z_1, z_2) = f_\tau^{-1}\Big(\frac{z_2}{z_1}\Big)T^1(z_1)T^1(z_2) =$$
$$=: \Lambda_1(z_1)\Lambda_1(z_2) : +S\Big(\frac{z_1}{z_2}\Big) : \Lambda_1(z_1)\Lambda_2(z_1) : +S\Big(\frac{z_2}{z_1}\Big) : \Lambda_2(z_1)\Lambda_1(z_1) : + \ldots.$$

Naively there is a plenty of poles at points $z_1 = z_2q_3^k$, $|k| \leq N-1$, however all poles except $k = 1, k = N-1$ cancels. Indeed poles at $z_1 = z_2$ cancels because for any term of a form $: \Lambda_i(z_1)\Lambda_j(z_2) :$ there is a term $: \Lambda_j(z_1)\Lambda_i(z_2) :$ which has an opposite residue at point $z_1 = z_2$. Residues at points $z_1 = q_3^k z_2$ cancels due to an identity:

$$: \Lambda_i(z)\Lambda_{2N-i}(q_3^{N-i}z) :=: \Lambda_{i-1}(z)\Lambda_{2N-i+1}(q_3^{N-i}z) :$$

The residue at $z_1 = z_2q_3$ is proportional to a second fundamental current $T^2(q_3^{\frac{1}{2}}z_2)$ (3.9)

$$T^2(z) = \frac{Y_2(z)}{Y_1(q_3^{1/2}z)Y_1(q_3^{3/2}z)} + \frac{Y_3(q_3^{1/2}z)}{Y_2(q_3z)Y_1(q_3^{3/2}z)} + \ldots$$

The residue at $z_2 = z_1q_3^{1-N}$ is proportional to a Heisenberg current $H(z) \overset{def}{=} \frac{Y_{\bar{1}}(q_3^{-1}z)}{Y_{\bar{1}}(q_3z)}$ which commute with all S_i except $S_{\bar{1}}$. These results may be compactly written in a following commutation relation:

$$T_2^1(z_1,z_2) - T_2^1(z_2,z_1) = \frac{(1-q_1^{-1})(1-q_2^{-1})}{1-q_3^{-1}}\left(\delta\Big(q_3\frac{z_1}{z_2}\Big)T^2(q_3^{1/2}z_2) + \delta\Big(q_3\frac{z_2}{z_1}\Big)T^2(q_3^{1/2}z_1)\right) +$$
$$+ S(q_3^{N-3})\frac{(1-q_1^{-1})(1-q_2^{-1})}{1-q_3^{-1}}\left(\delta\Big(q_3^{N-1}\frac{z_2}{z_1}\Big)H(z_2) + \delta\Big(q_3^{N-1}\frac{z_1}{z_2}\Big)H(z_1)\right)$$

Another important observation is that due to zeroes of function $S(z)$ involving into contractions (3.39) – (3.41), we have some zero conditions for the triple product of fundamental currents. let us introduce a function $h(z) = (1-q_3^{N-1}z)(1-q_3^{-N+1}z)(1-q_3z)(1-q_3^{-1}z)$, then the operator:

$$O(z_1,z_2,z_3) = h\Big(\frac{z_1}{z_2}\Big)h\Big(\frac{z_1}{z_3}\Big)h\Big(\frac{z_2}{z_3}\Big)T_3(z_1,z_2,z_3), \tag{3.43}$$

is symmetric and has the following zero conditions:

$$O(z, q_i z, q_i q_j z) = 0 \quad (i \neq j,\ i,j \in \{1,2,3\}), \tag{3.44}$$
$$O(z, q_3^{\pm(N-1)} z, q_{1,2} q_3^{\pm(N-1)} z) = 0 \tag{3.45}$$

which are easily verified by using (3.39)-(3.41). These conditions will be important in the proof of the commutativity of the IOMs.

Commutant of screenings Analogically to $\hat{\mathfrak{gl}}(2)$ case, commutant of screenings could be constructed as a multiple integrals of the higher W-currents:

$$I_n = \oint_C \prod_{1\le j<i\le n} S_{\infty,q_3^{2-N}}\Big(\frac{z_i}{z_j}\Big) T_n^1(z_1,\dots,z_n) \prod_{i=1}^n \frac{dz_i}{2\pi z_i}\ .$$

The integration contour C goes along unit circle $|z_i| = 1$, $T_n^1(z_1,\dots,z_n)$ is a higher W current (3.2.1):

$$T_n^1(z_1,\dots,z_n) =: \prod_{i=1}^n \frac{Y_1(z_i)}{Y_{\bar{1}}(q_3 z_i)} : + \dots$$

Let us show that (3.46) is indeed commute with all the screenings. As we already point out $T^1(z)$ is a sum of vertex operators :

$$T^1(z) = \sum_{i=1}^{2N} \Lambda_i(z), \tag{3.48}$$

If we exchange the argument of first and the last terms: $\Lambda_1(z) \to \Lambda_1(\tau^{-1}z)$, $\Lambda_{2N}(z) \to \Lambda_{2N}(\tau z)$ then corresponding W-current:

$$\hat{T}^1(z) = \Lambda_{2N}(\tau z) + \sum_{i=2}^{2N-1} \Lambda_i(z) + \Lambda_1(\tau^{-1}z) \tag{3.49}$$

will not commute with the first screening charge S_1 but will commute with the affine one $S_{\bar{1}}$. We want to prove that the two currents (3.48),(3.49) are identical under the integral (3.46). Let us for example consider the case of $n = 2$. We want to prove that:

$$\oint_C S_{\infty,\tau}\Big(\frac{z_2}{z_1}\Big) f_\tau^{-1}\Big(\frac{z_2}{z_1}\Big) T^1(z_1)T^1(z_2)\frac{dz_1}{2\pi z_1}\frac{dz_2}{2\pi z_2} \overset{?}{=} \oint_C S_{\infty,\tau}\Big(\frac{z_2}{z_1}\Big) f_\tau^{-1}\Big(\frac{z_2}{z_2}\Big) \hat{T}^1(z_1)\hat{T}^1(z_2)\frac{dz_1}{2\pi z_1}\frac{dz_2}{2\pi z_2}$$

The only problem terms are:

$$\oint_C S_{\infty,\tau}\Big(\frac{z_2}{z_1}\Big) f_\tau^{-1}\Big(\frac{z_2}{z_1}\Big) \Lambda_{1,2N}(z_1)\Lambda_i(z_2)\frac{dz_1}{2\pi z_1}\frac{dz_2}{2\pi z_2} \overset{?}{=} \oint_C S_{\infty,\tau}\Big(\frac{z_2}{z_1}\Big) f_\tau^{-1}\Big(\frac{z_2}{z_1}\Big) \Lambda_{1,2N}(\tau^{\mp 1}z_1)\Lambda_i(z_2)\frac{dz_1}{2\pi z_1}\frac{dz_2}{2\pi z_2}$$

And analogical ones with exchanged order of Λ operators.
The idea is to shift the variable $z_1 \to \tau^{-1}z_1$, and note that the product of two functions $S_{\infty,\tau}\Big(\frac{z_2}{z_1}\Big) f_\tau^{-1}\Big(\frac{z_2}{z_1}\Big)$ remains invariant. The only thing we need to care, is that we don't cross any pole when shifting the integration contour. This can be done using contractions (3.39)-(3.40), the crucial fact is that poles at points $z_1 = z_2 q_3^{\pm 1}$, $z_1 = q_3^{\pm(N-1)} z_2$ are canceled by zeroes of function $S_{\infty,\tau}(\frac{z_1}{z_2})$. The case of general n is completely analogical, the same line of argument leads to the fact that:

$$\oint_C \prod_{1\le j<i\le n} S_{\infty,\tau=q_3^{2-N}}\Big(\frac{z_i}{z_j}\Big) T_n^1(z_1,\dots,z_n) \prod_{i=1}^n \frac{dz_i}{2\pi z_i} = \oint_C \prod_{1\le j<i\le n} S_{\infty,\tau=q_3^{2-N}}\Big(\frac{z_i}{z_j}\Big) \hat{T}_n^1(z_1,\dots,z_n) \prod_{i=1}^n \frac{dz_i}{2\pi z_i}.$$

And so

$$I_n = \oint_C \prod_{1\le j<i\le n} S_{\infty,\tau=q_3^{2-N}}\Big(\frac{z_i}{z_j}\Big) T_n^1(z_1,\dots,z_n) \prod_{i=1}^n \frac{dz_i}{2\pi z_i}$$

commutes with the affine set of screenings. The Integrals of Motion I_n can be rewritten in terms of fundamental current as well:

$$I_n = \oint_C T^1(z_1)\dots T^1(z_n) \prod_{1\le j<i\le n} \omega_{\tau=q_3^{2-N}}\Big(\frac{z_i}{z_j}\Big) \prod_{i=1}^n \frac{dz_i}{2\pi z_i}.$$

Here we used the definition of higher W currents (3.9) and introduce the function:

$$\omega_\tau(z) = f_\tau^{-1}(z) S_{\infty,\tau}(z) = \frac{\Theta_\tau(z)\Theta_\tau(q_3^{-1}z)}{\Theta_\tau(q_1 z)\Theta_\tau(q_2 z)}.$$

Proof of mutual commutativity of I_n In this paragraph we will assume that: $\{q_3 > 1,\ q_1 < 1,\ q_2 < 1\, \tau < 1\}$, the other domains of parameters could be achieved by analytic continuation. Now we want to proof the commutativity $[I_n, I_m] = 0$. We have for the product of two IOMs:

$$I_n I_m = \oint_{C_{n,m}} \prod_{i<j\le n} S_{\infty,\tau}\Big(\frac{z_i}{z_j}\Big) \prod_{n+1\ge i<j\le n+m} S_{\infty,\tau}\Big(\frac{z_i}{z_j}\Big) T_n^1(z_1,\dots,z_n) T_m^1(z_{n+1},\dots,z_{n+m}) \prod_{i=1}^{n+m} \frac{dz_i}{2\pi z_i}.$$

The contour $C_{n,m}$ is understood in the spirit of radial ordering i.e we assume that $z_i \ll z_j$ for $i \le n, j > n$. Now using the identity which follows from pestun's prescription to write a higher W-currents (3.2.1):

$$T_n(z_1,\dots,z_n) T_m(w_1,\dots,w_m) = \prod_{i,j} f_\tau\Big(\frac{w_i}{z_j}\Big) T_{n+m}(z_1,\dots,z_n,w_1,\dots,w_m)$$

we may rewrite the product of two IOMs as:

$$I_n I_m = \oint_{C_{n,m}} \prod_{i=1,j=n+1}^{n,n+m} \frac{1}{\omega_\tau(\frac{z_i}{z_j})} \prod_{1\le j<i\le n+m} S_{\infty,\tau}\Big(\frac{z_i}{z_j}\Big) T_{n+m}(z_1,\dots,z_{n+m}) \prod_{i=1}^{n+m} \frac{dz_i}{2\pi z_i},$$

$$I_m I_n = \oint_{C_{m,n}} \prod_{i=1,j=m+1}^{n,n+m} \frac{1}{\omega_\tau(\frac{z_i}{z_j})} \prod_{1\le j<i\le n+m} S_{\infty,\tau}\Big(\frac{z_i}{z_j}\Big) T_{n+m}(z_1,\dots,z_{n+m}),$$

As follows from the properties of higher W current and the position of zeroes of $S_{\infty,\tau}$ function the operator:

$$S_{\infty,\tau}\Big(\frac{z_i}{z_j}\Big) T_{n+m}(z_1,\dots,z_{n+m})$$

is symmetric with respect to permutation $z_i \leftrightarrow z_j$, and do not have any poles other than the poles of function $S_{\infty,\tau}$. Our strategy is to deform both contours $C_{m,n}$ and $C_{n,m}$ to the unit circle, once it is done the product of theta functions could be symmetrized under the integral, and the commutativity of IM reduced to the following theta identity, which was proven by induction in [FKSW07].

$$Sym\Big(\prod_{i=1,j=n+1}^{n,n+m} \frac{1}{\omega_\tau(\frac{z_i}{z_j})}\Big) = Sym\Big(\prod_{i=1,j=m+1}^{n,n+m} \frac{1}{\omega_\tau(\frac{z_j}{z_i})}\Big)$$

Let us rename the variables $z_{n+i} = w_i$. The function $f\left(\frac{w}{z}\right)$ has its poles outside of the contour so we don't have to cross them when moving contour to the unit circle. The only obstruction to move the contour $C_{n,m}$ to the unit circle are poles of operator $T_{n+m}(z_1, \dots z_n, w_1, \dots w_m)$ at points $w_i = q_3^{-1} z_j, w_i = \tau q_3^{-1} z_j$, while for the contour $C_{m,n}$ we have to cross poles at points $w_i = q_3 z_j, w_i = \tau q_3 z_j$. Now we are going to show that corresponding residues are identical for both integrals which will finish the proof. In this notes we consider only the example of $n = 1, m = 2$, the proof for a general case is analogical and provided in [FJMV21].

Let us consider the residues at $w = z_2 q_3^{-1}$ for the first integral and, $w = z_2 q_3$ for the second one. It is convenient to write the residues in term of $O_3(z_1, z_2, z_3)$ operator (3.43) which doesn't have any poles, and symmetric in its arguments:

$$J_1 = \oint\limits_{|z_1|=|z_2|=1} h^{-1}\Big(\frac{q_3^{-1} z_2}{z_1}\Big) h^{-1}\Big(\frac{z_2}{z_1}\Big) S_{\infty,\tau}\Big(\frac{z_1}{z_2}\Big) f_\tau(q_3^{-1}) f_\tau\Big(\frac{z_2}{z_1} q_3^{-1}\Big) O_3(q_3^{-1} z_2, z_1, z_2) \prod_{i=1}^{3} \frac{dz_i}{2\pi z_i},$$

$$J_2 = \oint\limits_{|z_1|=|z_2|=1} h^{-1}\Big(\frac{q_3 z_2}{z_1}\Big) h^{-1}\Big(\frac{z_2}{z_1}\Big) S_{\infty,\tau}\Big(\frac{z_1}{z_2}\Big) f_\tau(q_3^{-1}) f_\tau\Big(\frac{z_1}{z_2} q_3^{-1}\Big) O_3(z_1, z_2, q_3 z_2) \prod_{i=1}^{3} \frac{dz_i}{2\pi z_i}.$$

Now, after the shift of variables $z_2 \to q_3 z_2$ in the first integral, the integrands become the same due to identities:

$$S_{\infty,\tau}(z) = \sigma_\tau(z)\sigma_\tau(1/z)$$
$$\sigma_\tau(q_3 z) f_\tau(z) = \sigma_\tau(z) S^{-1}(q_3 z)$$
$$S(z^{-1}) = S(q_3 z).$$

And we again have to care that we may perform a shift of argument $z_2 \to q_3 z_2$ without crossing a pole. The only relevant poles at points $z_1 = q_{1,2}^{-1} z_2$ in this case are coming from the $S_{\infty,\tau}\left(\frac{z_1}{z_2}\right)$ function, however thy are cancelled with a zero of $O_3(z_1, q_{1,2}^{-1} z_1, q_{1,2}^{-1} w)$ operator (3.44).

Now consider a residue at $w = \tau q_3^{-1} z_2$ for the first integrals and $w = \tau^{-1} q_3 z_2$ for the second one. We have:

$$J_1' = \oint\limits_{|z_1|=|z_2|=1} h^{-1}\Big(\frac{\tau q_3^{-1} z_2}{z_1}\Big) h^{-1}\Big(\frac{z_2}{z_1}\Big) S_{\infty,\tau}\Big(\frac{z_1}{z_2}\Big) f_\tau(\tau q_3^{-1}) f_\tau\Big(\frac{z_2}{z_1} \tau q_3^{-1}\Big) O_3(\tau q_3^{-1} z_2, z_1, z_2) \prod_{i=1}^{3} \frac{dz_i}{2\pi z_i},$$

$$J_2' = \oint\limits_{|z_1|=|z_2|=1} h^{-1}\Big(\frac{\tau^{-1} q_3 z_2}{z_1}\Big) h^{-1}\Big(\frac{z_2}{z_1}\Big) S_{\infty,\tau}\Big(\frac{z_1}{z_2}\Big) f_\tau(\tau q_3^{-1}) f_\tau\Big(\frac{z_1}{z_2} \tau q_3^{-1}\Big) O_3(z_1, z_2, \tau^{-1} q_3 z_2,) \prod_{i=1}^{3} \frac{dz_i}{2\pi z_i}.$$

Again after a shift $z_2 \to \tau^{-1} q_3 z_2$ in the first integral, integrands become the same due to an identity:

$$\sigma_\tau(q_3 \tau^{-1} z) f(z) = S^{-1}(\tau^{-1} q_3 z) S^{-1}(q_3 z) \sigma_\tau(z).$$

The relevant poles of function $S_{\infty,\tau}\left(\frac{z_1}{z_2}\right)$ at points $z_2 = q_{1,2}^{-1} z_1$ $z_2 = q_{1,2}^{-1} \tau^{-1} z_1$ are cancelled by a zero condition (3.45). Which finishes the proof of commutativity.

3.4.1 Integrals of motion for the q-deformed W algebras of BCD type.

Cases of BCD algebras fits into the same scheme. There are three types of endings of an affine Dynkin diagram (see 3.2). We pick one node as an affine one, and consider W albega of corresponding non affine Lie algebra. We resctricted ourselves to the case when affine node is of type D or C (long root), while the case of type B (short root) is not covered by our construction. We define the fundamental W current $T^1(z)$ such that it commutes with all screenings except affine one and we fix a $U(1)$ ambiguity of this current by a requirement that its zero mode $I_1 = \oint \frac{dz}{2\pi z} T^1(z)$ commutes with all screenings. We define higher W currents according to (3.8),(3.9).

$$T_2^1(z_1,z_2) = f_\tau^{-1}\Big(\frac{z_2}{z_1}\Big) T^1(z_1)T^1(z_2) =$$
$$=: \Lambda_1(z_1)\Lambda_1(z_2): + S\Big(\frac{z_1}{z_2}\Big) : \Lambda_1(z_1)\Lambda_2(z_1): + S\Big(\frac{z_2}{z_1}\Big) : \Lambda_2(z_1)\Lambda_1(z_1): + \dots$$

parameter τ is chosen according to table 3.1.

Affine Lie algebra	Parameter of elliptic deformation τ
A_N	$\mu q_3^{-\frac{N}{2}}$ - arbitrary
$B_N^\vee$	q_1/q_3^{N-1}
C_N	q_1^2/q_3^{N-1}
D_N	q_3^{-N+2}
BC_N	$q_1^{\frac{3}{2}}/q_3^{N-1}$

Table 3.1: Contrary to the A_N case, parameter of elliptic deformation in BCD case should be fixed

It turns out that analytical properties of higher W current are analogical to the ones of D case:

$$T_2^1(z_1,z_2) - T_2^1(z_2,z_1) = \frac{(1-q_1^{-1})(1-q_2^{-1})}{1-q_3^{-1}} \left(\delta\Big(q_3\frac{z_1}{z_2}\Big)T^2(q_3^{1/2}z_2) + \delta\Big(q_3\frac{z_2}{z_1}\Big)T^2(q_3^{1/2}z_1)\right) +$$
$$+ S(C^2)\frac{(1-q_1^{-1})(1-q_2^{-1})}{1-q_3^{-1}} \left(\delta\Big(C^2\frac{z_2}{z_1}\Big)H(z_2) + \delta\Big(C^2\frac{z_1}{z_2}\Big)H(z_1)\right)$$

where $C^2 = \tau^{-1}q_3$ for type D affine node, and $C^2 = \tau^{-1}q_1$ for type C. This similarity was used in [FJMV21] to define $\mathcal{K}$ algebra generated by the current $E(z)$ and $K(z)$ which unifies W algebras of types BCD. We also studied the representation theory of the $\mathcal{K}$ algebra, we found that in concrete representations current $E(z)$ becomes equal to a fundamental current $T^1(z)$ of W algebra. Another result of the paper is a formula for higher Integrals of Motion, let us mention them here without a proof:

$$I_n = \oint_C T^1(z_1)\dots T^1(z_n) \prod_{1\le j<i\le n} \omega_\tau^\alpha\Big(\frac{z_i}{z_j}\Big) \prod_{i=1}^n \frac{dz_i}{2\pi z_i}.$$

here, $\alpha = 3$ for a D type affine node, and $\alpha = 1$ for a C type affine node,

$$\omega_\tau^3(z) = \frac{\Theta_\tau(z)\Theta_\tau(q_3^{-1}z)}{\Theta_\tau(q_1z)\Theta_\tau(q_2z)},$$
$$\omega_\tau^1(z) = \frac{\Theta_\tau(z)\Theta_\tau(q_1^{-1}z)}{\Theta_\tau(q_2z)\Theta_\tau(q_3z)}.$$

In [FJMV21] it is proven that IMs (3.59) commute with each other and with affine set of screenings.

3.5 R - matrix, K-matrix, and associated KZ Integrals of Motion

In this section we provide a q-deformed versions of R and K matrices, build KZ integrals of motion, and show that they commute with the local IMs.

Let us remember that screening operators and Y operators have zero modes, $s_{i,0}$, $y_{i,0}$. In this section we will write them explicitly: $Y_i(z) = e^{y_{i,0}} Y_i^{osc}(z)$, $\mathcal{S}_i(z) = e^{s_{i,0}\log(z)}\mathcal{S}_i^{osc}(z)$, where osc means that operator contains only oscillatoric zero modes $y_{i,k}$,$s_{i,k}$ $k \neq 0$. In order to have a notations convenient for Yang-Baxter equation we introduce spectral parameters $u_i = e^{y_{i,0}}$. Screenings zero modes then equal to:

$$s_{i,0} = \Big(\boldsymbol{\alpha}_i \cdot \log_{q_2}(\boldsymbol{u})\Big),$$

where α_i is a simple roots of a Lie algebra and bold symbols denotes vectors $\log(\boldsymbol{u}) = \{\log(u_1), \dots, \log(u_n)\}$, $\boldsymbol{y}_0 = \{y_{1,0}, \dots, y_{n,0}\}$.
Having at hand the screening operator $\mathcal{S}_i$, one can always define a reflection matrix $\check{R}$, by definition:

- It depends only on current $s_i(z)$, in other words $[\check{R}_i, Y_j(z)] = 0$ for $i \neq j$
- And intertwines the spectral parameter of corresponding W algebra:

$$\check{R}_i \left(e^{y_{i,0}} Y_i^{osc}(z) + e^{y_{i,0}-(\boldsymbol{\alpha}_i \cdot \boldsymbol{y}_0)} : Y_i^{osc}(z) \frac{\mathcal{S}_i^{osc}(zq_1^{-d_i})}{\mathcal{S}_i^{osc}(zq_1^{-d_i}q_2^{-1})} : \right) =$$
$$= \left(e^{y_{i,0}-(\boldsymbol{\alpha}_i \cdot \boldsymbol{y}_0)} Y_i^{osc}(z) + e^{y_{i,0}} : Y_i^{osc}(z) \frac{\mathcal{S}_i^{osc}(zq_1^{-d_i})}{\mathcal{S}_i^{osc}(zq_1^{-d_i}q_2^{-1})} : \right) \check{R}_i \,. \tag{3.62}$$

This last equation means that the result of conjugation by the R-matrix is just an action of Weyl reflection on zero modes of W-current: $y_{i,0} \to y_{i,0} - (\boldsymbol{\alpha}_i \cdot \boldsymbol{y}_0)$. In other words, R-matrix intertwines with the action of a Weyl group on zero modes of W-algebra. We will see that depending on the root system, this reflection operators becomes a q-deformation of R and K matrices considered in previous chapters [8]. $\check{R}$ operator uniquely fixed by these two conditions. It is also clear that $\check{R}_i$ operator doesn't depend on the overal shift of zero modes and so depend only on certain combination $\check{R}_i = \check{R}_i\left(e^{(\boldsymbol{\alpha}_i \cdot \boldsymbol{y}_0)}\right)$

RRR **relation.** Let us consider two roots of equal lengths, connected with a node, corresponding fundamental W current reads:

$$W(z) = u_1 Y_1^{osc}(z) + u_2 \frac{Y_2^{osc}(q_3^{1/2}z)}{Y_1^{osc}(q_3 z)} + u_3 \frac{1}{Y_2^{osc}(q_3^{3/2}z)}.$$

Using the commutation relations (3.62), easy to see that:

$$\check{R}_1(\frac{u_2}{u_1})\check{R}_2(\frac{u_3}{u_1})\check{R}_1(\frac{u_3}{u_2})W(z)\check{R}_1^{-1}(\frac{u_3}{u_2})\check{R}_2^{-1}(\frac{u_3}{u_1})\check{R}_1^{-1}(\frac{u_2}{u_1}) =$$
$$= u_2 Y_1(z) + u_3 \frac{Y_2^{osc}(q_3^{1/2}z)}{Y_1^{osc}(q_3 z)} + u_1 \frac{1}{Y_2^{osc}(q_3^{3/2}z)} =$$
$$= \check{R}_2(\frac{u_3}{u_2})\check{R}_1(\frac{u_3}{u_1})\check{R}_2(\frac{u_2}{u_1})W(z)\check{R}_2^{-1}(\frac{u_2}{u_1})\check{R}_1^{-1}(\frac{u_3}{u_1})\check{R}_2^{-1}(\frac{u_3}{u_2}),$$

this last line is equivalent to a Yang-Baxter equation [MO19]:

$$\check{R}_1(\frac{u_2}{u_1})R_2(\frac{u_3}{u_1})\check{R}_1(\frac{u_3}{u_2}) = \check{R}_2(\frac{u_3}{u_2})\check{R}_1(\frac{u_3}{u_1})\check{R}_2(\frac{u_2}{u_1}).$$

[8]We put a check because rational analogs of reflection operator not only reflects zero modes but also permute Fock spaces.

KRKR relation. Now let us consider two roots of lengths 2 and 1, corresponding fundamental W current reads:

$$W(z) = u_1 Y_1^{osc}(z) + u_2 \frac{Y_2^{osc}(q_3^{1/2} z)}{Y_1^{osc}(q_3 z)} + u_2^{-1} \frac{Y_1^{osc}(q_1^{-2} q_2^{-1} z)}{Y_2^{osc}(\frac{q_2^{-3/2}}{q_1^{5/2}} z)} + u_1^{-1} \frac{1}{Y_1^{osc}(q_1^{-3} q_2^{-2} z)}.$$

Analogically to the previous case, it is clear that:

$$\check{R}_2(u_2^2)\check{R}_1(u_1u_2)\check{R}_2(u_1^2)\check{R}_1(\frac{u_2}{u_1}) = \check{R}_1(\frac{u_2}{u_1})\check{R}_2(u_1^2)\check{R}_1(u_1u_2)\check{R}_2(u_2^2).$$

In this case it is instructive to change a notation $\check{R}_2(u^2) = \check{K}_2(u)$, and recognise a famous Sklyanin KRKR = RKRK relation [Skl88]:

$$\check{K}_2(u_2)\check{R}_1(u_1u_2)\check{K}_2(u_1)\check{R}_1(\frac{u_2}{u_1}) = \check{R}_1(\frac{u_2}{u_1})\check{K}_2(u_1)\check{R}_1(u_1u_2)K_2(u_2).$$

The same relation will be true for two roots of lengths 1 and 2 (clearly this is just a redefinition of (3.66)).

Now let us suppose that we have an affine Lie algebra of $\mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ type it's Dynkin diagram is a strip with three possible endings (see fig(3.2)). One can always define $\check{R}_i(u)$ operators attached to any root in the middle, and $\check{K}_1(u)$, $\check{K}_N(u)$ operators attached to the first and last roots. [9] Such operators will enjoy the relations of affine Weyl group, it is known that there is a commutative N dimensional lattice [I84], we call the corresponding commutative operators - Integrals of Motion of KZ type

$$\begin{aligned} \mathcal{T}_i &= \mathcal{T}_i^+ \mathcal{T}_i^-, \qquad (3.68)\\ \mathcal{T}_i^+ &= \check{R}_i(\frac{u_{i+1}}{u_i}) \dots \check{R}_{N-1}(\frac{u_N}{u_{N-1}})\check{K}_N(u_N)\check{R}_{N-1}(u_N u_{N-1}) \dots \check{R}_i(u_{i+1}u_i),\\ \mathcal{T}_i^- &= \check{R}_{i-1}(u_i u_{i-1}) \dots \check{R}_1(u_2u_1)\check{K}_1(u_1)\check{R}_1(\frac{u_2}{u_1}) \dots \check{R}_{i-1}(\frac{u_i}{u_{i-1}})\\ &[\mathcal{T}_i, \mathcal{T}_j] = 0. \end{aligned}$$

These KZ Integrals of Motion commute with "local" Integrals of motion which uniquely defined as a kernel of affine screening system (see sections (3.3),(3.4))

$$\begin{aligned} &[I_n, \mathcal{S}_i] = 0, \quad for\ i = 1, \dots, rank(\hat{\mathfrak{g}}),\\ &[I_n, \mathcal{T}_i] = 0,\\ &[I_n, I_m] = 0. \end{aligned}$$

The first line is just a definition of integrals I_n, second line follows from the fact that $\hat{R}_{i,i+1}(u_i - u_{i+1})$ acts on the elements of W algebra simply by exchanging $u_i \leftrightarrow u_{i+1}$, as I_n belongs to an intersection of all W algebras, R and K matrices just permute the weights u_i. It is then easy to understand that the result of the KZ operator is the identical permutation. The last line follows from the conjecture that $\mathcal{T}_1$ has a simple spectrum (we checked it numerically in rational limit), and all Integrals of Motion commute with $\mathcal{T}_i$

3.6 Discussion

In this chapter we present (3.59) an explicit formulas for Integtals of motion of q-deformed W algebras of B, C, D types. We explained that contrary to A type, these integrable systems don't have a twist parameter.

[9] The case of D W-algebra should be treated differently, we have two roots 1 and $\bar{1}$ which doesn't connected with a node, which means that $\check{R}_1(u)\check{R}_{\bar{1}}(v) = \check{R}_{\bar{1}}(v)\check{R}_1(u)$. Let us define $\check{K}_1$ operator such that $\check{K}_1 S_1 \check{K}_1 = S_{\bar{1}}$ and $\check{K}_1$ depends only on difference $s_1(z) - s_2(z)$, it is clear that KRKR = RKRK is trivially satisfied. Note that for the D case K operator doesn't depend on spectral parameter u.

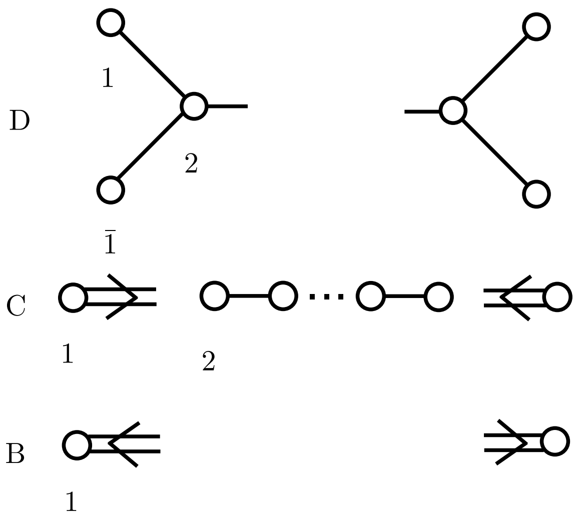


Figure 3.2: Three types of endings of the Dynkin diagram of B, C, D type. The construction of local Integrals of Motion doesn't work for B type node chosen as an affine one.

Generating function of IOMs, KZ IOMs We presented an explicit formula for Sklyanin K matrices (3.66)-(3.67), and found KZ IOMS (3.68) which commute with local ones. It will be nice to build a generating function which contain all these Integrals of Motion under some specialisation, usually this generating function could be found as a trace $T = tr(K_0(u)R_{01}(\frac{u}{u_1})R_{02}(\frac{u}{u_2})_{0n}(\frac{u}{u_n})K_0(u)R^{-1}_{0n}(\frac{u_1}{u})\dots R^{-1}_{01}(\frac{u_1}{u}))$, however we were not able to do that. The deep reason for our failure is that DIM R-matrix doesn't have a crossing unitarity property which is necessarily in Sklyanin's construction [Skl88], more technically we were unable to properly define an auxiliary space which is traced out in Sklyanin's formula.

The case of $\hat{\mathfrak{gl}}(N)$: appearence of additional Integrals of Motion at $\mu = q_3^{\frac{N}{2}-k}$ We found that in $\mathfrak{gl}(N)$ case there are integrals of motion which are built from a fundamental and anti-fundamental W current. Additional integral of motion which corresponds to $k-th$ fundamental representation of $\mathfrak{gl}(N)$:

$$I_1^k = \oint \frac{dz}{2\pi z}\left(\frac{Y_k(z)}{Y_{k+1}(\sqrt{q_3}z)} + \dots\right)$$

appeared at special values of $\mu = q_3^{\frac{N}{2}-k}$ or $\tau = q_3^{-(k-1)}$, for $N > k > 1$.
At this special points two things happens:

- k-th fundamental current become local with respect to a fundamental and anti-fundamental ones. And simple calculation shows that their zero modes commute.
- Zero mode of k-th fundamental current start to commute with affine system of screenings.

It would be interesting to probe the spectrum of IOMs at this special points.

Analytic continuation of W algebras of type BCD In [FJMV21] we introduced a new quantum algebra $\mathcal{K}$ which may be thought as an analytical continuation of BCD W algebras. This algebra posses a comodule structure which allows to multiply the represenations of $\mathcal{K}$ and representations of toroidal algebra $\hat{\hat{\mathfrak{gl}}}(1)$ which corresponds to increasing the rank of a BCD Lie algebra. Each W algebra considered in this chapter corresponds to a particular representation of algebra $\mathcal{K}$. It was observed that algebra $\mathcal{K}$ has three integrable subalgebras, which after specialisation of a concrete representation reproduce Integrals of Motion considered in current notes.

Conclusion

In this thesis we studied the integrable structures associated to an affine Yangian and its q-deformatoin.

- We found that integrable structures of affine Yangian are naturally appears in the context of conformal field theories. Namely we identify the affine Yangian "spin chain" on n sites with integrable systems of W_n algebras of type A.
- We studied the question of integrable boundary conditions for the affine Yangian. We found three solution of Sklyanin $KRKR$ equation (2.11)-(2.12). We identify the boundary affine Yangian "spin chain" with integrable structures of the W algebras of types BCD.
- We studied the spectrum of the integrable structures, and constructed the Bethe vector, and found the correspondig Bethe equations (1.14), (2.27).
- We found that the study of representation theory of the affine Yangian may provide new integrable perturbations of CFT. In particular affine Yangian has Fock representations of three different colors $\mathcal{F}^{\alpha}$. We associate an integrable system to the chain of colored Fock spaces with two colored boundaries $\beta_L \Big| \mathcal{F}_1^{\alpha_1} \otimes \mathcal{F}_2^{\alpha_2} \cdots \otimes \mathcal{F}_n^{\alpha_n} \Big| \beta_R \;\; , \;\; \alpha_i, \beta_{L,R} = 1, 2, 3.$
- We provided explicit formulas for the q-deformed Integrals of Motion of arbitrary high spin for the case of W algebras of type BCD (3.59), we also provide formulas for q-deformed R and K matrices (3.62).

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Appendix A

A.1 Large u expansion of the operator $\mathcal{R}(u)$

We look for the solution to the equation

$$\mathcal{R}(u)\Big(-\big(u+J(x)\big)^2+QJ'(x)\Big)=\Big(-\big(u+J(x)\big)^2-QJ'(x)\Big)\mathcal{R}(u) \tag{A.1}$$

where $J(x)=\sum_{k\neq 0}a_k e^{-ikx}$ in the form [MO19]

$$\mathcal{R}(u)=\exp\left(iQ\Big(2u\log u+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{r_k}{u^k}\Big)\right),\quad \text{where}\quad r_k=\frac{1}{2\pi}\int_0^{2\pi}g_{k+1}(x)dx, \tag{A.2}$$

Solving (A.1) one can find first few densities $g_k(x)$ explicitly:

$$g_2=J^2,\quad g_3=\frac{J^3}{3},\quad g_4=\frac{J^4}{6}+\frac{1-2Q^2}{24}J_x^2,\quad g_5=\frac{J^5}{10}+\frac{1-2Q^2}{8}JJ_x^2,$$
$$g_6=\frac{J^6}{15}+\frac{1-2Q^2}{4}J^2J_x^2+\frac{2-9Q^2+6Q^4}{480}J_{xx}^2,\quad g_7=\frac{J^7}{21}+\frac{5(1-2Q^2)}{12}J^3J_x^2+\frac{2-9Q^2+6Q^4}{96}JJ_{xx}^2, \tag{A.3}$$

and more disgusting expression for g_8

$$g_8=\frac{1}{28}J^8+\frac{5}{8}(1-2Q^2)J^4J_x^2+\frac{1}{32}(2-9Q^2+6Q^4)J^2J_{xx}^2+\frac{1}{576}\left(-9+41Q^2-26Q^4\right)J_x^4+$$
$$+\frac{1}{161280}\left(90-671Q^2+998Q^4-360Q^6\right)J_{xxx}^2.$$

For all densities $g_k(x)$ in (A.3) we used zeta-function regularization. For example

$$\int J^2=\int :J^2:-\frac{1}{24},\quad \int J^4=\int :J^4:-\int\frac{1}{4}:J^2:+\frac{1}{192},\quad \int J_x^2=\int :J_x^2:+\frac{1}{240},$$

where :: stands for the Wick ordering.

Explicit formula (A.2) is useful for us, because it provides a relation between Yangian currents (1.33) and $W^n(z)$ currents. For example for the first few modes:

$$f_0=Qa_1\ ,\quad f_1=Q\sum_n a_{n+1}a_{-n}$$
$$e_0=Qa_{-1}\ ,\quad e_1=Q\sum_n a_{n-1}a_{-n}.$$

Of course these formulas are only true in a bosonic representation, however it is easy to analytically continue them to the arbitrary number of bosons:

$$f_0=QU_1\ ,\quad f_1=QL_1$$
$$e_0=QU_{-1}\ ,\quad e_1=QL_{-1}.$$

Where $L_{\pm 1}$ is a special $\mathcal{W}^{(2)}$ current, such that:

$$[L_{\pm 1}, U_n] = -nU_{n\pm 1}.$$

Let us also note that there is easily established pattern in the densities (A.3). Namely, the first terms in (A.3) can be written as

$$g_n = \frac{2}{n(n-1)}J^n + \frac{(n-2)(n-3)}{48}(1-2Q^2)J^{n-4}J_x^2 + \\ + \frac{(n-2)(n-3)(n-4)(n-5)}{11520}(2-9Q^2+6Q^4)J^{n-6}J_{xx}^2 + \dots$$

Using this observation one can formally do the resumation in (A.2).

$$G(x) \stackrel{\text{def}}{=} 2u\log u + \sum_{k=1}^{\infty}(-1)^{k-1}g_{k+1}(x)/u^k,$$

which admits the derivative expansion

$$G(x) = 2(u+J)\log(u+J) + \frac{1-2Q^2}{24}\frac{J_x^2}{(u+J)^3} + \frac{2-9Q^2+6Q^4}{480}\frac{J_{xx}^2}{(u+J)^5} + \dots \tag{A.4}$$

The expansion (A.4) suggests the following general form

$$G(x) = 2(u+J)\log(u+J) + \sum_{k=1}^{\infty}\frac{U_{2k+2}(J_x, J_{xx}, \dots)}{(u+J)^{2k+1}}, \tag{A.5}$$

where $U_{2k+2}(J_x, J_{xx}, \dots)$ is a homogeneous and even with respect to the transformation $J \to -J$ density of degree $2k+2$. It would be interesting to find the densities $U_{2k+2}(J_x, J_{xx}, \dots)$ exactly.

One can also compute the $\mathcal{R}(u)$ operator in the "free fermion" point $c = -2$. Namely, take $Q = -\frac{i}{\sqrt{2}}$ in (A.2) and represent the current $J(x)$ by the complex fermion $\psi(x)$ as

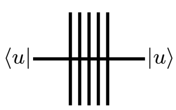
$$J(x) = \frac{1}{\sqrt{2}} : \psi^+(x)\psi(x) : .$$

Then one can check that (see also appendix A.4)

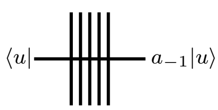
$$\mathcal{R}(u)\Big|_{c=-2} \sim \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\psi^+(x)\log\left(1+\frac{i}{u\sqrt{2}}\partial\right)\psi(x)\,dx\right)$$

A.2 Affine Yangian commutation relations

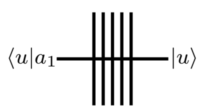
Here we will derive current form of commutation relations of the Yang Baxter algebra (1.40) from the RLL algebra (1.32). Similar analysis has been performed in [Pro19]. We use the following notations



$= \mathcal{L}_{\varnothing,\varnothing}(u) \stackrel{\text{def}}{=} h(u)$



$= \mathcal{L}_{\varnothing,\square}(u)$



$= \mathcal{L}_{\square,\varnothing}(u)$

and admit the convention that the operators acts in "quantum" space from up to down. It is also convenient to define according to (1.33)

$$h(u) \stackrel{\text{def}}{=} \mathcal{L}_{\varnothing,\varnothing}(u), \qquad e(u) \stackrel{\text{def}}{=} h^{-1}(u)\cdot\mathcal{L}_{\varnothing,\square}(u) \quad \text{and} \quad f(u) \stackrel{\text{def}}{=} \mathcal{L}_{\square,\varnothing}(u)\cdot h^{-1}(u).$$

We will introduce currents $e_{\boldsymbol{\lambda}}(u)$ and $f_{\boldsymbol{\lambda}}(u)$ associated to $3D$ partitions. There are 3 currents on level 2 (1.36), 6 currents on level 3 (see (A.17)) etc. Similar expressions one has for $f_{\boldsymbol{\lambda}}(u)$.

All these and other generators of the Yang-Baxter (1.32) admit large u expansion which is inherited from the large u expansion of the $R-$matrix (A.2). In particular,

$$h(u) = 1 + \frac{h_0}{u} + \frac{h_1}{u^2} + \ldots, \quad e(u) = \frac{e_0}{u} + \frac{e_1}{u^2} + \ldots, \quad f(u) = \frac{f_0}{u} + \frac{f_1}{u^2} + \ldots,$$

while the higher currents are expected to behave at $u \to \infty$ as

$$e_{\boldsymbol{\lambda}}(u) \sim \frac{1}{u^{\boldsymbol{\lambda}}}, \quad f_{\boldsymbol{\lambda}}(u) \sim \frac{1}{u^{\boldsymbol{\lambda}}}.$$

The relations of the Yang-Baxter algebra (1.32) appear from the tensor product of two Fock spaces $\mathcal{F}_u$ and $\mathcal{F}_v$. We will use the following notations for the bra and ket highest weight states in $\mathcal{F}_u \otimes \mathcal{F}_v$

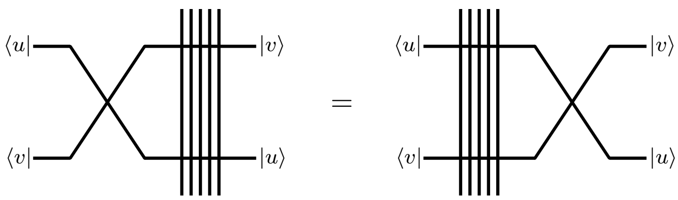
$$\langle \mathrm{vac}| \stackrel{\mathrm{def}}{=} \langle u| \otimes \langle v|, \qquad |\mathrm{vac}\rangle \stackrel{\mathrm{def}}{=} |u\rangle \otimes |v\rangle.$$

The action of the zero-mode a_0 on the vacuum state $|u\rangle$ is

$$a_0|u\rangle = -iu|u\rangle.$$

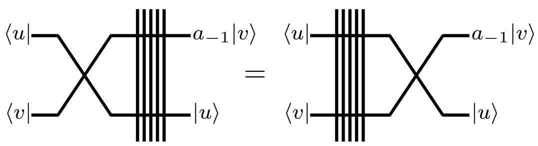
he and hf relations:

Then on level 0 we have



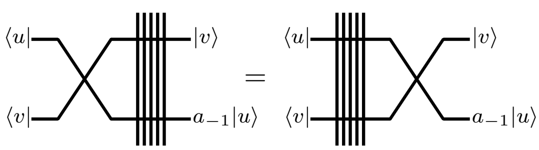
$$\Longrightarrow \quad h(u)h(v) = h(v)h(u) \tag{A.6}$$

On level 1 one has two relations



$$\Longrightarrow \quad h(u)\mathcal{L}_{\varnothing,\square}(v) = \tfrac{u-v}{u-v+Q}\mathcal{L}_{\varnothing,\square}(v)h(u) + \tfrac{Q}{u-v+Q}h(v)\mathcal{L}_{\varnothing,\square}(u) \tag{A.7}$$

and



$$\Longrightarrow \quad \mathcal{L}_{\varnothing,\square}(u)h(v) = \tfrac{Q}{u-v+Q}\mathcal{L}_{\varnothing,\square}(v)h(u) + \tfrac{u-v}{u-v+Q}h(v)\mathcal{L}_{\varnothing,\square}(u) \tag{A.8}$$

In fact (A.7) and (A.8) are not independent. Taking the linear combination $Q \times \text{(A.7)} - (u-v) \times \text{(A.8)}$ one arrives to the equation (A.7) with $u \leftrightarrow v$.

Now, multiplying (A.7) by $(u - v + Q)h^{-1}(v)$ from the left and using (A.6) one get the relation

$$(u-v+Q)h(u)e(v) = (u-v)e(v)h(u)+Q\mathcal{L}_{\varnothing,\square}(u) \implies (u-v+Q)e(v) = (u-v)h^{-1}(u)e(v)h(u)+Qe(u). \tag{A.9}$$

In the leading order in large v expansion one obtains (compare to (1.42))

$$[e_0, h(u)] = Q\mathcal{L}_{\varnothing,\square}(u).$$

We note that the relation (A.6) implies that the product $h(u)e(v)$ could not have poles, which implies that

$$\mathcal{L}_{\varnothing,\square}(u) = h(u)e(u) = e(u+Q)h(u).$$

Formula (A.9) allows one to rewrite

$$e_{j_1}\dots e_{j_n}h(u)e_{j_{n+1}}\dots e_{j_{n+m}} = \oint\limits_{\mathcal{C}_{n+m}}\cdots\oint\limits_{\mathcal{C}_1}\prod_{k=1}^{n+m} z_k^{j_k}\cdot\prod_{k=n+1}^{n+m}\frac{u-z_k}{u-z_k+Q}\,e(z_{n+m})\dots e(z_1)h(u)d\vec{z},$$

where the contours $\mathcal{C}_\infty$ go concentrically around ∞ in such a way that all singularities of the function $F(\boldsymbol{z})$ are kept *inside* of these contours.

Similarly to (A.9) one also obtains the relation

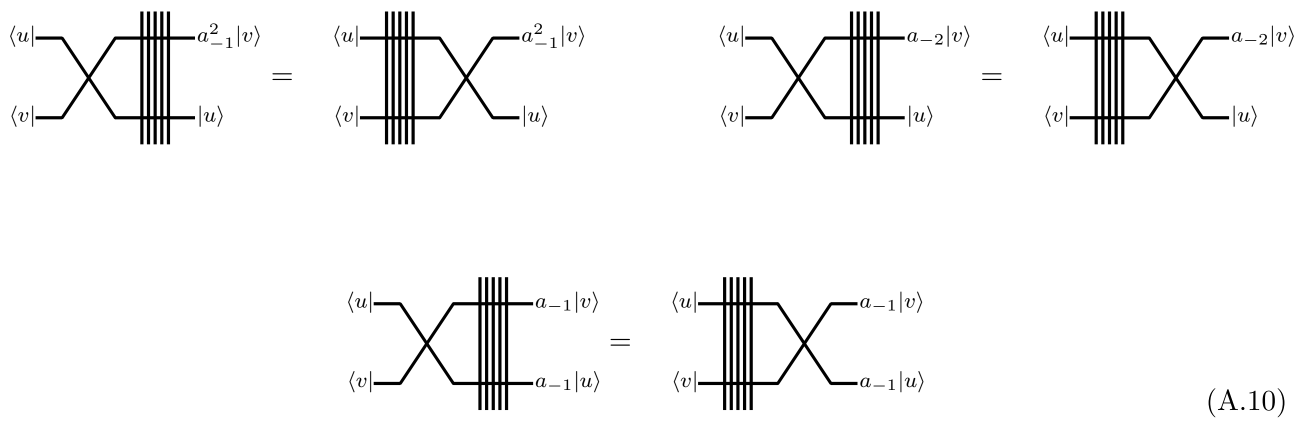
$$(u-v+Q)f(v)h(u) = (u-v)h(u)f(v) + Q\mathcal{L}_{\square,\varnothing}(u),$$

which enables one to rewrite

$$f_{j_{n+m}}\dots f_{j_{n+1}}h(u)f_{j_n}\dots f_{k_1} = \oint\limits_{\mathcal{C}_{n+m}}\cdots\oint\limits_{\mathcal{C}_1}\prod_{k=1}^{n+m} z_k^{j_k}\cdot\prod_{k=1}^{n}\frac{u-z_k}{u-z_k-Q}\,h(u)f(z_{n+m})\dots f(z_1)d\vec{z}.$$

ee and ff relations:

On level 2 we have three independent equations



(A.10)

In the r.h.s of any of these relation one has a linear combination of five terms

$$h(v)\mathcal{L}_{\varnothing,\square\square}(u),\quad h(v)\mathcal{L}_{\varnothing,\substack{\square\\\square}}(u),\quad \mathcal{L}_{\varnothing,\square}(v)\mathcal{L}_{\varnothing,\square}(u),\quad \mathcal{L}_{\varnothing,\square\square}(v)h(u),\quad \mathcal{L}_{\varnothing,\substack{\square\\\square}}(v)h(u).$$

One can always find special linear combination of three equations (A.10) which kills contributions of last two terms. Explicitly, one has

$$\begin{aligned}A_1(u-v)h(u)\mathcal{L}_{\varnothing,\square\square}(v) + A_2(u-v)h(u)\mathcal{L}_{\varnothing,\substack{\square\\\square}}(v) + A_3(u-v)\mathcal{L}_{\varnothing,\square}(u)\mathcal{L}_{\varnothing,\square}(v) =\\ = A_1(v-u)h(v)\mathcal{L}_{\varnothing,\square\square}(u) + A_2(v-u)h(v)\mathcal{L}_{\varnothing,\substack{\square\\\square}}(u) + A_3(v-u)\mathcal{L}_{\varnothing,\square}(v)\mathcal{L}_{\varnothing,\square}(u),\end{aligned}\quad\text{(A.11)}$$

where

$$A_1(u) = -Q(u+Q),\quad A_2(u) = -iQ,\quad A_3(u) = (u+b)(u+b^{-1})$$

We note that using (A.9) one can express[1]

$$\mathcal{L}_{\varnothing,\square}(u)\mathcal{L}_{\varnothing,\square}(v) = \frac{h(u)h(v)}{u-v}\left[(u-v-Q)e(u)e(v)+Qe^2(v)\right],$$

and hence the relation (A.11) can be rewritten as

$$\begin{aligned}(u-v+b)(u-v+b^{-1})(u-v-Q)e(u)e(v) &= (u-v-b)(u-v-b^{-1})(u-v+Q)e(v)e(u)+\\ +\Big((u-v-b^{-1})(u-v+Q)e_{\square\square}(u)+(u-v-b)(u-v+Q)&e_{\substack{\square\\\square}}(u)+(u-v-b)(u-v-b^{-1})e_{\boxslash\square}(u)+(u\longleftrightarrow v)\Big),\end{aligned} \tag{A.12}$$

where the higher currents $e_{\boldsymbol\lambda}(u)$ are given by (1.36).

Other two relations from (A.10) are equivalent to commutation relations between $h(u)$ end $e_{\boldsymbol\lambda}(v)$ (similar to (A.9))

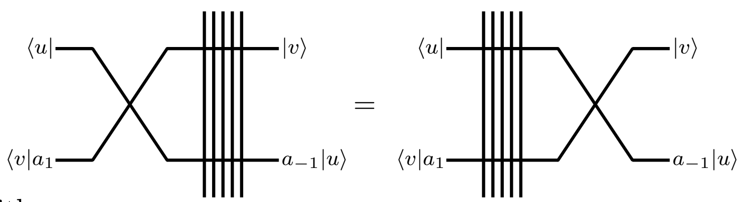
$$\begin{aligned}(u-v+Q)\Big(u-v+Q+\frac{1}{b}\Big)e_{\square\square}(v) &= (u-v)\Big(u-v+\frac{1}{b}\Big)h^{-1}(u)e_{\square\square}(v)h(u)+\\ +\left((u-v+Q)\Big(u-v+Q+\frac{1}{b}\Big)-(u-v)\Big(u-v+\frac{1}{b}\Big)\right)&e_{\square\square}(u)-\frac{2bQ(u-v)}{b-b^{-1}}\Big(e_{\square\square}(u)-e_{\substack{\square\\\square}}(u)\Big)-\\ &-2ibQ\Big((u-v+Q)e(v)e(u)-Qe^2(u)\Big)\end{aligned}$$

and similar for $e_{\substack{\square\\\square}}(v)$

$$\begin{aligned}(u-v+Q)(u-v+Q+b)e_{\substack{\square\\\square}}(v) &= (u-v)(u-v+b)h^{-1}(u)e_{\substack{\square\\\square}}(v)h(u)+\\ +\big((u-v+Q)(u-v+Q+b)-(u-v)(u-v+b)\big)e_{\substack{\square\\\square}}(u)&-\frac{2b^{-1}Q(u-v)}{b-b^{-1}}\Big(e_{\square\square}(u)-e_{\substack{\square\\\square}}(u)\Big)-\\ &-2ib^{-1}Q\Big((u-v+Q)e(v)e(u)-Qe^2(u)\Big)\end{aligned}$$

ef relation

In order to obtain the relation (1.40d), we consider matrix element



which reads explicitly as

$$\frac{u-v}{u-v+Q}\mathcal{L}_{\varnothing,\square}(u)\mathcal{L}_{\square,\varnothing}(v)+\frac{Q}{u-v+Q}\mathcal{L}_{\square,\square}(u)h(v) = \frac{u-v}{u-v+Q}\mathcal{L}_{\square,\varnothing}(v)\mathcal{L}_{\varnothing,\square}(u)+\frac{Q}{u-v+Q}\mathcal{L}_{\square,\square}(v)h(u).$$

Using (A.8) and similar relation

$$\frac{u-v}{u-v+Q}h(u)\mathcal{L}_{\square,\varnothing}(v)+\frac{Q}{u-v+Q}\mathcal{L}_{\square,\varnothing}(u)h(v) = \mathcal{L}_{\square,\varnothing}(v)h(u)$$

[1] We also have more general relation

$$\mathcal{L}_{\varnothing,\square}(u)\mathcal{L}_{\varnothing,\boldsymbol\lambda}(v) = \frac{h(u)h(v)}{u-v}\Big[(u-v-Q)e(u)e_{\boldsymbol\lambda}(v)+Qe(v)e_{\boldsymbol\lambda}(v)\Big].$$

nothing that

$$e(u+Q) = \mathcal{L}_{\varnothing,\square}(u)h^{-1}(u),$$

and multiplying by $h^{-1}(u)h^{-1}(v)$ from the right one obtains

$$e(u+Q)f(v) + \frac{Q}{u-v+Q}\left(\mathcal{L}_{\square,\square}(u)h^{-1}(u) - \mathcal{L}_{\varnothing,\square}(u)h^{-1}(u)\mathcal{L}_{\square,\varnothing}(u)h^{-1}(u)\right) =$$
$$= f(v)e(u+Q) + \frac{Q}{u-v+Q}\left(\mathcal{L}_{\square,\square}(v)h^{-1}(v) - \mathcal{L}_{\square,\varnothing}(v)h^{-1}(v)\mathcal{L}_{\varnothing,\square}(v)h^{-1}(v)\right). \quad \text{(A.13)}$$

After shifting $u \to u - Q$, equation (A.13) reads

$$[e(u), f(v)] = -Q\frac{\psi(u)-\psi(v)}{u-v} \quad \text{where} \quad \psi(u+Q) = \mathcal{L}_{\square,\square}(u)h^{-1}(u) - \mathcal{L}_{\varnothing,\square}(u)h^{-1}(u)\mathcal{L}_{\square,\varnothing}(u)h^{-1}(u).$$

Serre relations

Formulas at level 3 becomes tough, however they are straightforward. For example for $e(u)e_{\square\square}(v)$ we will have:

$$e(u)e_{\square\square}(v) = \frac{\bar{g}(u-v)\bar{g}(u-v+b)}{g(u-v)g(u-v+b)}e_{\square\square}(v)e(u) + \frac{2Q}{(b-b^{-1})(Q+b)}\frac{\bar{g}(u-v)\bar{g}(u-v+b)}{g(u-v)g(u-v+b)}\times$$
$$\times e(v)\left(\frac{1}{v-u-Q-b}e_{\square\!\!\!\square}(u) + \frac{1}{v-u}e_{\square\square}(u) + \frac{1}{v-u-b-b^{-1}}e_{\substack{\square\\\square}}(u)\right) + \text{locals}$$

We will specify "locals" terms later (see (A.16)), here we just want to point out that l.h.s by definition doesn't have any poles and so does the r.h.s. This condition will imply some additional relations, most of them will be non local, and we are not gonna discuss them. However we note that the multiplier

$$\frac{\bar{g}(u-v)\bar{g}(u-v+Q)}{g(u-v)g(u-v+Q)}$$

doesn't have pole at $u = v + Q$. Surprisingly, there is a "local" term with pole at this point. Setting the residue to zero, one finds a relation

$$(1+b^2)(1+2b^2)e(u)e_{\square\square}(u) + Q(1+2b^2)e(u)e_{\substack{\square\\\square}}(u) - b(b^4-b^2-4)e_{\square\square}(u)e(u)+$$
$$+ 2b(1+b^2)e_{\substack{\square\\\square}}(u)e(u) + 2b(1+b^2)e_{\square\!\!\!\square}(u)e(u) - (b^2+1)^2e_{\square\square\square}(u)) \stackrel{\text{def}}{=} e_{\substack{\ \ \square\\\square\square}}(u) = 0 \quad \text{(A.14)}$$

We will have similar relation for $e(u)e_{\substack{\square\\\square}}(v)$. And also one trivial relation which follows from the fact that $e(u)e(u)^2 = e(u)^2e(u) = e(u)^3$. As a result we will have 6 independent currents at level 3, which is equal to the number of 3d young diagrams with 3 boxes. In practise we used this three relations in order to exclude composite currents $(e_{\square\!\!\!\square}(u)e(u), e_{\square\square}(u)e(u), e_{\substack{\square\\\square}}(u)e(u))$. This three relations may look mysterious, however, after explicit calculation we found that this three relations are equivalent to Serre relations, the later could be written down in terms of currents as

$$\sum_{\sigma\in\mathbb{S}_3}(x_{\sigma_1} - 2x_{\sigma_2} + x_{\sigma_3})e(x_{\sigma_1})e(x_{\sigma_2})e(x_{\sigma_3}) + \sum_{\sigma\in\mathbb{S}_3}[e(x_{\sigma_1}), e_{\square\!\!\!\square}(x_{\sigma_2}) + e_{\square\square}(x_{\sigma_2}) + e_{\substack{\square\\\square}}(x_{\sigma_2})] = 0 \quad \text{(A.15)}$$

Namely, using the quadratic relations, we may reorder any polynomial in $e(u_i)$ in a way that it will contain only monomials with ordered arguments: $e(u_{i_1})e(u_{i_2})...e(u_{i_n})$, $i_1 < i_2... < i_n$, this could be done explicitly with the formulas (A.16). After doing this procedure with Serre relations we found that they proportional to linear combination of three currents, and so equal to zero in Yangian algebra.

Finally, after imposing three relations (A.14) we will have explicitly:

$$g(u-v)g(u-v+b)\Big[e(u)e_{\square\square}(v)\Big]_d = \bar{g}(u-v)\bar{g}(u-v+b)\Big[e_{\square\square}(v)e(u)\Big]_d + \\ + \frac{2Q}{(b-b^{-1})(Q+b)}\left(\frac{\bar{g}(u-v)\bar{g}(u-v+b)}{v-u-Q-b}\Big[e(v)e_{\square\diagup\square}(u)\Big]_d + \frac{\bar{g}(u-v)\bar{g}(u-v+b)}{v-u}\Big[e(v)e_{\square\square}(u)\Big]_d + \right. \\ \left. + \frac{\bar{g}(u-v)\bar{g}(u-v+b)}{v-u-b-b^{-1}}\Big[e(v)e_{\substack{\square\\\square}}(u)\Big]_d\right), \quad \text{(A.16)}$$

where

$$\Big[e(u)e_{\square\square}(v)\Big]_d \stackrel{\text{def}}{=} e(u)e_{\square\square}(v) - \frac{1}{u-v+2b}e_{\square\square\square}(v) - \frac{b-2b^{-1}}{u-v+b^{-1}}e_{\substack{\square\\\square\square}}(v) - \frac{b+2Q}{u-v-Q}e_{\square\diagup\square\square}(v),$$

$$\Big[e(u)e_{\substack{\square\\\square}}(v)\Big]_d \stackrel{\text{def}}{=} e(u)e_{\substack{\square\\\square}}(v) - \frac{1}{u-v+2b^{-1}}e_{\substack{\square\\\square\\\square}}(v) - \frac{2b-b^{-1}}{u-v+b}e_{\substack{\square\\\square\square}}(v) - \frac{2b+b^{-1}}{u-v-Q}e_{\square\diagup\substack{\square\\\square}}(v),$$

$$\Big[e(u)e_{\square\diagup\square}(v)\Big]_d \stackrel{\text{def}}{=} e(u)e_{\square\diagup\square}(v) - \frac{1}{u-v-2Q}e_{\square\diagup\square\diagup\square}(v) - \frac{Q+2b}{u-v+b}e_{\square\diagup\square\square}(v) - \frac{(Q+2b^{-1})(Q+b)}{u-v+b^{-1}}\frac{e_{\square\diagup\substack{\square\\\square}}(v)}{2Q+b^{-1}},$$

and

$$\Big[e_{\square\square}(v)e(u)\Big]_d \stackrel{\text{def}}{=} e_{\square\square}(v)e(u) - \frac{2}{(b-b^{-1})(b+Q)}\frac{1}{u-v-2b^{-1}}e_{\substack{\square\\\square\\\square}}(u) - \frac{2bQ}{Q+b}\frac{u-v-b+b^{-1}}{(u-v-b)(u-v-b^{-1})}e_{\substack{\square\\\square\square}}(u) - \\ - \frac{2}{b-b^{-1}}\frac{u-v-Q-b}{(u-v-b)(u-v-Q)}e_{\square\diagup\square\square}(u) - \frac{2Q}{(b-b^{-1})(Q+b)}\frac{1}{(u-v-2Q)}e_{\square\diagup\square\diagup\square}(u) + \\ + \frac{b^2}{(b-b^{-1})(b+Q)}\frac{2v-2u-4b+b^{-1}+b^{-3}}{(u-v-b)(u-v-2b)}e_{\square\square\square}(u) - \frac{(Q+b^{-1})b^{-1}}{(2Q+b^{-1})(b-b^{-1})}\frac{2v-2u-3Qb^{-2}}{(u-v-b^{-1})(u-v+Q)}e_{\square\diagup\substack{\square\\\square}}(u)$$

with the higher currents given by

$$e_{\square\square}(v) = \frac{Q}{(b^{-1}-b)(2b+b^{-1})}h^{-1}(v)\left(\mathcal{L}_{\varnothing,\square\square}(v) + ib\mathcal{L}_{\varnothing,\substack{\square\\\square}}(v)\right)$$

$$e_{\substack{\square\\\square}}(v) = \frac{Q}{(b-b^{-1})(b+2b^{-1})}h^{-1}(v)\left(\mathcal{L}_{\varnothing,\square\square}(v) + ib^{-1}\mathcal{L}_{\varnothing,\substack{\square\\\square}}(v)\right)$$

$$e_{\square\diagup\square}(v) = \frac{Q}{(2b+b^{-1})(b+2b^{-1})}h^{-1}(v)\left(2Q^2\mathcal{L}_{\varnothing,\square\square}(v) + iQ\mathcal{L}_{\varnothing,\substack{\square\\\square}}(v) - (2b+b^{-1})(b+2b^{-1})e^2(v)\right)$$

$$e_{\square\square\square}(v) = \frac{2Q^2}{(b-b^{-1})(2b-b^{-1})(Q+b)(Q+2b)}h^{-1}(v)\big(\mathcal{L}_{\varnothing,\square\,\square\,\square}(v) + 3ib\mathcal{L}_{\varnothing,\ \square\!\!\square}(v) - 2b^2\mathcal{L}_{\varnothing,\square\square\square}(v)\big),$$

$$e_{\substack{\square\\\square\\\square}} = \frac{2Q^2}{(b-b^{-1})(b-2b^{-1})(2b^{-1}+b)(3b^{-1}+b)}h^{-1}(v)\big(\mathcal{L}_{\varnothing,\square\,\square\,\square}(v) + 3ib^{-1}\mathcal{L}_{\varnothing,\ \square\!\!\square}(v) - 2b^{-2}\mathcal{L}_{\varnothing,\square\square\square}(v)\big),$$

$$e_{\square\diagup\square\diagup\square} = \frac{2Q^4}{(Q+b^{-1})(Q+b)}h^{-1}(v)\big(\mathcal{L}_{\varnothing,\square\,\square\,\square}(v) + \frac{12iQ}{(2Q+b)(2Q+b^{-1})}\mathcal{L}_{\varnothing,\ \square\!\!\square}(v) - \frac{4}{(2Q+b)(2Q+b^{-1})}\mathcal{L}_{\varnothing,\square\square\square}(v)\big) - \\ - \frac{2Q^2(Q+2b^{-1})}{2Q+b^{-1}}e(v)e_{\substack{\square\\\square}}(v) - \frac{2Q^2(Q+2b)}{2Q+b}e(v)e_{\square\square}(v) - 2Qe(v)e_{\square\diagup\square}(v),$$

$$e_{\substack{\square\\\square\square}} = -\frac{2Q^2}{(b-b^{-1})(b-2b^{-1})(Q+b^{-1})(2b-b^{-1})(Q+b)}h^{-1}(v)\big(\mathcal{L}_{\varnothing,\square\,\square\,\square}(v) + iQ\mathcal{L}_{\varnothing,\ \square\!\!\square}(v) - \mathcal{L}_{\varnothing,\square\square\square}(v)\big),$$

$$e_{\boxplus} = \frac{Q^3 b^{-1}}{(b-b^{-1})(Q+b)^2(Q+b^{-1})(Q+2b^{-1})} \times$$
$$\times\, h^{-1}(v)\Big((2Q+b^{-1})\mathcal{L}_{\varnothing,\square\square\square}(v) + i(3Q+b)b^{-1}\mathcal{L}_{\varnothing,\,\square\square}(v) - 2b^{-1}\mathcal{L}_{\varnothing,\square\square\square}(v)\Big) - \frac{Q}{Q+b^{-1}}e(v)e_{\boxplus}(v),$$

and

$$e_{\square\square\square} = \frac{Q^3 b}{(b-b^{-1})(Q+b)(Q+b^{-1})(Q+2b)(2Q+b)} \times$$
$$\times\, h^{-1}(v)\big((2Q+b)\mathcal{L}_{\varnothing,\square\square\square}(v) + i(3Q+b^{-1})b\mathcal{L}_{\varnothing,\,\square\square}(v) - 2b\mathcal{L}_{\varnothing,\square\square\square}(v)\big) - \frac{Q}{2Q+b}e(v)e_{\square\square\square}(v),$$

In principle we may go further, and calculate quadratic relations at next levels, however as we already have shown, the algebra is generated by the $h(u)$, $e(u)$ and $f(u)$ currents, so in principle we don't need to use higher currents. The only problem is to prove that quadratic and Serre relations are the only ones which currents $e(u)$ obeys.

Relations in ϵ notations

We see that there is an S_3 symmetry associated to permutation of the triple $(b, b^{-1}, -Q)$. In fact it is more convenient to go to epsilon notations:

$$b = \frac{\epsilon_1}{\sqrt{\epsilon_1\epsilon_2}}, \quad b^{-1} = \frac{\epsilon_2}{\sqrt{\epsilon_1\epsilon_2}}, \quad Q = -\frac{\epsilon_3}{\sqrt{\epsilon_1\epsilon_2}} \implies \epsilon_1+\epsilon_2+\epsilon_3 = 0.$$

It is also convenient to change a normalization of the highest weight/spectral parameters, together with the normalization of the bosonic zero mode:

$$\varphi(x) \to \phi(x) = -i\frac{\varphi(x)}{\sqrt{\epsilon_1\epsilon_2}}.$$

Then the relation (A.12) takes apparently symmetric form

$$g(u-v)\Big[e(u)e(v) + \frac{e_{\square\square\square}(v)}{u-v+\epsilon_1} + \frac{e_{\boxplus}(v)}{u-v+\epsilon_2} + \frac{e_{\boxminus}(v)}{u-v+\epsilon_3}\Big] =$$
$$= \bar{g}(u-v)\Big[e(v)e(u) + \frac{e_{\square\square\square}(u)}{u-v-\epsilon_1} + \frac{e_{\boxplus}(u)}{u-v-\epsilon_2} + \frac{e_{\boxminus}(u)}{u-v-\epsilon_3}\Big],$$

where

$$g(x) = (x+\epsilon_1)(x+\epsilon_2)(x+\epsilon_3), \quad \bar{g}(x) = (x-\epsilon_1)(x-\epsilon_2)(x-\epsilon_3).$$

Our conventions about relation between ϵ and b, Q notations are summarized in the table below

	fields normalisation	Current e	commutator $[e,f]$
b notations	$\partial\varphi(x)\partial\varphi(y) = -\frac{1}{\sin^2(x-y)} + \text{reg}$	$e(u) = h^{-1}(u)\mathcal{L}_{\varnothing,\square}(u)$	$[e(u_1), f(u_2)] = -Q\frac{\psi(u_1)-\psi(u_2)}{u_1-u_2}$
ϵ notations	$\partial\phi(x)\partial\phi(y) = \frac{1}{\sin^2(x-y)} + \text{reg}$	$e(v) = \sqrt{\epsilon_3}h^{-1}(v)\mathcal{L}_{\varnothing,\square}(v)$	$[e(v_1), f(v_2)] = \frac{\psi(v_1)-\psi(v_2)}{v_1-v_2}$

In definition of matrix elements $\mathcal{L}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(u)$ we define the state $|\square\rangle$, as well as any state $|\boldsymbol{\lambda}\rangle$ to be normalized as $\langle\boldsymbol{\lambda}|\boldsymbol{\lambda}\rangle = 1$ in any notation.

Note that here we used a Maulik-Okounkov $\mathcal{R}$ matrix, which breaks the symmetry between $\epsilon_1, \epsilon_2, \epsilon_3$, so that we have a selected ϵ_3. In fact, there exist additional $\mathcal{R}_f^{(1,2)}$ matrices with either ϵ_1 or ϵ_2 selected (see appendix A.4).

A.3 Special vector $|\chi\rangle$ and shuffle functions

In the later we will need a more detailed description of a subalgebra $\mathfrak{n}^+$ generated by currents $f(z)$. Easy to understand that the subspace of the form $\mathcal{L}_{\mu,\varnothing_1}(v_1)\dots\mathcal{L}_{\mu,\varnothing_n}(v_n)$ may be identified with the subspace $\mathfrak{n}^+(\boldsymbol{v}) = h(v_1)\dots h(v_n)\mathfrak{n}^+$. A particular result of this section is an explicit realization of this mapping, which plays an essential role in sections 1.4.1-1.4.5.

First of all let us note that both spaces are graded by the number of $f(\xi_i)$ currents in the monomial, let us note each graded component of corresponding algebras by $\mathfrak{n}^+_N$, $\mathfrak{n}^+_N(\boldsymbol{v})$.

It is a natural idea to identify elements of $\mathfrak{n}^+$ and $\mathfrak{n}^+(\boldsymbol{v})$ by their matrix elements in some representation:

$$\mathfrak{n}^+_N \to \langle\varnothing|\mathfrak{n}^+_N|\chi\rangle \tag{A.18}$$

In order to unambiguously characterize the elements of $\mathfrak{n}^+_N$, $\mathfrak{n}^+_N(\boldsymbol{v})$ we need a big enough set of representations and vectors $|\chi\rangle$. Our choice is the following: let us pick an N Fock spaces: $\mathcal{F}_{x_1}\otimes\dots\mathcal{F}_{x_N}$, and consider simplest vector of grade N :

$$|\chi\rangle_{\boldsymbol{x}} \overset{\text{def}}{=} |\underbrace{\square,\dots,\square}_{N}\rangle = \lim_{\xi_i\to x_i}\prod_{i,k}\frac{\xi_i - x_k}{\xi_i - x_k - \epsilon_3}\prod_{i<j}S(\xi_i-\xi_j)e(\xi_N)...e(\xi_1)|0\rangle$$

Then, our mapping (A.18) maps an element of $\mathfrak{n}^+_N$, $\mathfrak{n}^+_N(\boldsymbol{v})$ to a rational function of N variables $f(x_1,\dots,x_N)$ obeying the so called "wheel" condition [FJMM15]:

$$f(x_1, x_1+\epsilon_i, x_1+\epsilon_i+\epsilon_j, x_4,\dots) = 0$$

For $\mathfrak{n}^+_N$ and additional condition:

$$f(v, v+\epsilon_3, x_3,\dots) = 0$$

For $\mathfrak{n}^+_N(\boldsymbol{v})$. This functions is a rational limits of Sh_0 and Sh_1 functions from [FJMM15]. The multiplication in algebra, implies the multiplication of Shuffle functions

$$\begin{gathered} S_0 : \mathfrak{n}^+_N \times \mathfrak{n}^+_M \to \mathfrak{n}^+_{N+M} \\ f(\boldsymbol{x})\star g(\boldsymbol{y}) \equiv \mathrm{Sym}_{x,y}\Big(f(\boldsymbol{x})g(\boldsymbol{y})\prod_{i,j}S(x_i-y_j)\Big) \end{gathered} \tag{A.19}$$

For $\mathfrak{n}^+$, And

$$\begin{gathered} S_1 : \mathfrak{n}^+_N(\boldsymbol{v}) \times \mathfrak{n}^+_M(\boldsymbol{u}) \to \mathfrak{n}^+_{N+M}(\boldsymbol{u},\boldsymbol{v}) \\ f(\boldsymbol{x})\star g(\boldsymbol{y}) \equiv \mathrm{Sym}_{x,y}\Big(f(\boldsymbol{x})g(\boldsymbol{y})\prod_{n,i}\frac{u_n - x_i}{u_n - x_i - \epsilon_3}\prod_{i,j}S(x_i-y_j)\Big) \end{gathered}$$

For $\mathfrak{n}^+(\boldsymbol{v})$.

Let us introduce, $W^{(1)}(z)$ current U_n

$$\begin{aligned} \langle\varnothing|\mathcal{L}(u)\ a^{(0)}_{-n}|\varnothing\rangle &= \frac{U_n}{u} + o\left(\frac{1}{u^2}\right) \quad , n>0 \\ \langle\varnothing|a^{(0)}_{n}\ \mathcal{L}(u)|\varnothing\rangle &= \frac{U_{-n}}{u} + o\left(\frac{1}{u^2}\right) \quad , n>0 \end{aligned} \tag{A.20}$$

It is clear from the $\mathcal{RLL}$ relation that $R(u)$ matrix commute with $W^{(1)}$ current:

$$(a^{(0)}_n + U_n)R^{0,v} = R^{0,v}(a^{(0)}_n + U_n)$$

Taking the matrix element over the auxiliary space $\langle\varnothing|\ldots|\mu\rangle$ for positive n we will get:

$$[\mathcal{L}_{\boldsymbol{\mu},\varnothing}(u), U_n] = \mathcal{L}_{\boldsymbol{\mu}+n,\varnothing}(u), \tag{A.21}$$

where $\langle\boldsymbol{\mu}+n| \overset{\text{def}}{=} \langle\boldsymbol{\mu}|a_n$.

It is also clear, that J_n for $n>0$ belongs to the subalgebra $\mathfrak{n}^+$. Indeed, explicit calculation of the large u limit of $R(u)$ matrix (A.2) shows that:

$$\begin{aligned} U_1 = f_0 \quad U_{-1} = e_0, \\ U_{n+1} = -n[f_1, U_n] \quad U_{n-1} = -n[e_1, U_n]. \end{aligned} \tag{A.22}$$

Then we get:

$$\begin{aligned} U_k^{\boldsymbol{x}} &= \oint \cdots \oint g_k(\boldsymbol{\xi}) f(\xi_1)...f(\xi_k) d\boldsymbol{\xi} \quad \text{with,} \\ g_{n+1}(\vec{\xi}) &= -k\Big(\xi_1 g_n(\xi_2\ldots\xi_{n+1}) - g_n(\xi_1\ldots\xi_n)\xi_{n+1}\Big), \end{aligned} \tag{A.23}$$

and

$$g_n(\boldsymbol{\xi}) = (-1)^{n-1}(n-1)! \prod_i \xi_i \left(\sum (-1)^i C_n^i \xi_i^{-1}\right),$$

where C_n^i are the binomial coefficients.

Note that the function $g(\boldsymbol{\xi})$ defined ambiguously, indeed algebra $Y(\widehat{\mathfrak{gl}}(1))$ enjoys Serre relations (A.15)

$$\mathrm{Sym}_{i,j,k}[f_i,[f_j[f_{k+1}]]] = 0$$

Indeed such an element lies in the kernel of the Shuffle map (A.19)

$$\mathrm{Sym}_{i,j,k}\Big(\xi_1^i\xi_2^k\xi_3^k(\xi_1 - 2\xi_2 + \xi_3)S(\xi_1-\xi_2)S(\xi_1-\xi_3)S(\xi_2-\xi_3)\Big) = 0$$

In particular, commutativity of J_n may be thought as a consequence of Serre relation, for example choosing $i=j=k=0$

$$[U_1, U_2] = [f_0, [f_1, f_0]] \overset{\text{Serre}}{=} 0$$

We should consider functions $g_n(\boldsymbol{\xi})$ modulo equivalence:

$$g_n^{(1)}(\boldsymbol{\xi}) \sim g_n^{(2)}(\boldsymbol{\xi}) + \mathrm{Ker}(S_0) \tag{A.24}$$

It is easy to understand that modulo this equivalence function $g_n(\boldsymbol{\xi})$ is invariant under the simultaneous shift of all variables $\xi \to \xi + \hbar$ we will use this fact in section 1.4.5.

As we announced, operators $\mathcal{L}(u)_{\mu,\varnothing}$ belongs to the subspace $\mathfrak{n}^+(u)_{|\mu|}$:

$$\mathcal{L}_{\boldsymbol{\lambda},\varnothing}(u) = \frac{1}{(2\pi i)^{|\boldsymbol{\lambda}|}} \oint \cdots \oint F_\lambda(\boldsymbol{z}|u)\, h(u) f(z_{|\boldsymbol{\lambda}|})\ldots f(z_1) dz_1 \ldots dz_{|\boldsymbol{\lambda}|} \tag{A.25}$$

where contours go clockwise around ∞ and all poles of $F_\lambda(\boldsymbol{z})$.

Let us prove this statement, and find recurrence relations for the rational function $F_{\boldsymbol{\lambda}}(\boldsymbol{z}|u)$. Now in order to recover formula (A.25) we have to use relation (A.21) together with the formula (A.23). In order to reproduce (A.25) we have to reorder h and f current, in order to move h to the left, this can be done with the simple fact

$$\begin{aligned} \oint_\infty \xi^n f(\xi)h(u)\frac{d\xi}{2\pi i} &= \oint_\infty \Big[\frac{(u-\xi)}{(u-\xi-\epsilon_3)}h(u)f(\xi) - \frac{\epsilon_3}{(u-\xi-\epsilon_3)}f(u)h(u)\Big]\xi^n\frac{d\xi}{2\pi i} = \\ &= \oint_{\infty+\{u-\epsilon_3\}} \frac{(u-\xi)}{(u-\xi-\epsilon_3)}h(u)f(\xi)\xi^n\frac{d\xi}{2\pi i} \end{aligned} \tag{A.26}$$

Here in the first equality we used equation (1.40c), while in the second we used a simple fact that l.h.s of (1.40c) doesn't have pole at $u = v + \epsilon_3$, and so r.h.s does $(h(u)f(u+\epsilon_3) = f(u)h(u))$, thus we may deform integration contour.

Equation (A.21) together with (A.26) implies integral formula (A.25) together with recurrence representation for $F_{\boldsymbol{\lambda}}(\boldsymbol{z}|u)$

$$F_{\boldsymbol{\lambda}+n}(\boldsymbol{z},\boldsymbol{w}|u) = F_{\boldsymbol{\lambda}}(\boldsymbol{z}|u) g_n(\boldsymbol{w}) \left(1 - \prod_{i,j} G(z_i - w_j) \prod_i \frac{u - w_i}{u - w_i - \epsilon_3}\right) \tag{A.27}$$

A.4 Other representations of $\mathbf{YB}(\widehat{\mathfrak{gl}}(1))$

In this notes we were concentrated on an examples of "spin chain" with n sites and periodic boundary conditions, this setup corresponds to an affine A_n Toda field theory. At the each site of our "spin chain" we should place a representation of RLL algebra. The generating function of IM's is equal to

$$T(u) = \mathrm{Tr}_{\mathcal{F}_0}\left(q^{\sum_n a^{(0)}_{-n} a^{(0)}_n} \mathcal{R}_{0,1}(u-u_1)\dots\mathcal{R}_{0,n}(u-u_n)\right)$$

One possibility is to choose $\mathcal{R}_{0,k}(u-u_k)$ to be the Maulik-Okounkov $R-$matrix. However we have already seen that RLL algebra in current realization is symmetric under permutation of three parameters ϵ_α, in terms of usual parameters b, Q, b^{-1} this means a symmetry between b and $Q = b + \frac{1}{b}$ where $b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}$.

In order to see two additional representations of RLL algebra let us realize representation of W algebra in the space of two bosons as commutant of Screening charge, according to [BFM18, LS16] there are three choices of screening currents. Our notation is that there exist three different types of representation of $YB(\widehat{\mathfrak{gl}}(1)$: we call them $\mathcal{F}^{(1)}_u$, $\mathcal{F}^{(2)}_u$ and $\mathcal{F}^{(3)}_u$. We assign the screening charge S_k to a tensor product of two Fock spaces of the same type $\mathcal{F}^{(k)}_u \otimes \mathcal{F}^{(k)}_v$, and we assign "fermionic" screening charge $S_{i,f}$ to the tensor product of different Fock spaces $\mathcal{F}^{(j)}_u \otimes \mathcal{F}^{(k)}_v$ with $(i,j,k) = \mathrm{cycl}(1,2,3)$. Fixing one of the Fock spaces to be of the type 3, we will have three options for the other one

$$S_{f,1} = \oint e^{b\phi_0(x) - \beta\phi_1(x)} dx, \quad S_{f,2} = \oint e^{b^{-1}\phi_0(x) - \bar{\beta}\phi_1(x)} dx, \quad S_3^{\pm} = \oint e^{b^{\pm 1}(\phi_0(x) - \phi_1(x))} dz,$$

where $\beta = i\sqrt{b^2+1}$ and $\bar{\beta} = i\sqrt{1+b^{-2}}$.

While the third screening charge S_3 leads to the MO $R-$matrix

$$\mathcal{R}^{(3)}_{0,1} = \mathcal{R}^{MO}_{0,1} = e^{iQ\int\limits_{x=0}^{2\pi}\left[\frac{1}{2u}(\partial\phi_0(x) - \partial\phi_1(x))^2 - \frac{1}{6u^2}(\partial\phi_0(x) - \partial\phi_1(x))^3\right] + o(\frac{1}{u^2})]\frac{dx}{2\pi}},$$

the first and the second screenings have dimension $\frac{1}{2}$ and the corresponding W algebra admits free fermion representation. For example for the first screening, let us introduce two fermionic currents

$$\psi(x) = e^{-ibux} e^{b\phi_0(x) - \beta\phi_1(x)} \tag{A.28}$$
$$\psi^\dagger(x) = e^{ibux} e^{-b\phi_0(x) + \beta\phi_1(x)}, \tag{A.29}$$

where iu is the zero mode of $\phi_0(x)$. It is easy to check that they obeys free fermionic OPE's

$$\psi(x)\psi^\dagger(y) = \frac{1}{\sin(x-y)} + \mathrm{reg}, \quad \psi(x)\psi(y) = \mathrm{reg}, \quad \psi^\dagger(x)\psi^\dagger(y) = \mathrm{reg}.$$

Correspondingly, $W^{(2)}(x)$ current which commutes with S_1 is simply

$$W^{(2)}(x) = \psi^\dagger(x)(i\partial + ub)\psi(x)$$

Intertwining relation implies

$$\mathcal{R}_f^{(1)}\psi^\dagger(x)(i\partial + ub)\psi(x) = \psi(x)(i\partial - ub)\psi^\dagger(x)\mathcal{R}_f^{(1)} \tag{A.30}$$

One can find that the $\mathcal{R}_f^{(1)}$ matrix is given by the explicit formula

$$\mathcal{R}_f^{(1)}(u) = \exp\left[\frac{1}{2\pi}\int_0^{2\pi}\psi^\dagger(x)\log\left(1+\frac{i\partial}{ub}\right)\psi(x)dx\right] \tag{A.31}$$

Indeed under the adjoint action of $\mathcal{R}$ matrix fermions transform as:

$$\mathcal{R}_f^{(1)}\psi(z)\left(\mathcal{R}_f^{(1)}\right)^{-1} = \frac{1}{1+\frac{i\partial}{ub}}\,\psi(z)$$
$$\mathcal{R}_f^{(1)}\psi^\dagger(z)\left(\mathcal{R}_f^{(1)}\right)^{-1} = (1-\frac{i\partial}{ub})\psi^\dagger(z)$$

Such that (A.30) holds. Although formula for $\mathcal{R}$ matrix looks pretty simple, it's structure is quite complicated because one should remember that $\psi(z)$ operator is nontrivial in terms of individual bosons (A.28),(A.29).

In order to find local integrals of motion we have to expand $\mathcal{R}$ matrix in powers of $\frac{1}{u}$. Let us introduce a shorthand notation

$$\Phi(x) = b\phi_0(x) - \beta\phi_1(x)$$

It is easy to find that

$$:\psi^\dagger(x)\psi(x) := \partial\Phi(x) \tag{A.32}$$
$$:\psi^\dagger(x)\partial\psi(x) := \frac{1}{2}\big(\partial\Phi(x)\big)^2 + \frac{1}{2}\partial^2\Phi(x) \tag{A.33}$$
$$:\psi^\dagger(x)\partial^2\psi(x) := \frac{1}{3}(\partial\Phi(x))^3 + \partial\Phi(x)\partial^2\Phi(x) + \frac{1}{3}\partial^3\Phi(x)$$
$$\cdots$$

Using the formulas (A.31), (A.32)-(A.33), it is easy to find first non trivial integral of motion in the space of one boson F_2:

$$T(u) = \mathrm{Tr}'_{\mathrm{aux}}(q^{\sum_n a_{-n}a_n}R_f^{(1)}) = e^{\frac{I_1}{u}+\frac{I_2}{u^2}+\dots},$$

where

$$I_1 = \frac{iQ}{2\pi}\int_{x=0}^{2\pi}\partial\phi^2dx, \qquad I_2 = -\frac{iQ}{b}\int_{x=0}^{2\pi}\left[\frac{1}{3}\beta(\partial\phi)^3 - \frac{1}{2}b^2\partial\phi D\partial\phi\right]\frac{dx}{2\pi}$$

In general, representation may contain Fock modules of different types. Let us consider the following one

$$\mathcal{F}_q = \left(\mathcal{F}_u^{(1)}\right)^{\otimes n_1}\left(\mathcal{F}_u^{(2)}\right)^{\otimes n_2}\left(\mathcal{F}_u^{(3)}\right)^{\otimes n_3}, \quad \mathcal{F}_{\mathrm{aux}} = F_3$$

Where $\mathcal{F}_q$ is our quantum space, and $\mathcal{F}_{\mathrm{aux}}$ is an auxiliary space. As usual the generating function of Integrals of Motion is

$$T(u) = \mathrm{Tr}_{\mathrm{aux}}\left(q^{\sum_n a_na_{-n}}R_{aux,q}\right) = tr_{aux}\left(q^{\sum_n a_na_{-n}}\prod_{j=1}^{n_1}R_{2,f}(u-v_j)\prod_{i=n_1+1}^{n_1+n_2}R_{1,f}(u-v_i)\prod_{k=n_1+n_2}^{n_1+n_2+n_3}R^{MO}(u-v_k)\right)$$

Expanding at large spectral parameter, it is easy to find first non trivial integral of motion:

$$I_2 = iQ\int_{x=0}^{2\pi}\Big[\frac{\bar\beta}{3}\sum_{i=1}^{n_1}(\partial\phi_i)^3+\frac{\beta}{3}\sum_{i=n_1+1}^{n_1+n_2}(\partial\phi_i)^3-\frac{1}{3}\sum_{i=n_1+n_2+1}^{n_1+n_2+n_3}(\partial\phi_i)^3-\left(\frac{1}{2}\sum_{i,j}B_{i,j}\partial\phi_i D\partial\phi_j+\sum_{i<j}B_{i,j}\partial\phi_i\partial^2\phi_j\right)\Big]\frac{dx}{2\pi}$$

Where B is a $n_1 \times n_2 \times n_3$ block matrix:

$$B = \begin{pmatrix} b & 1 & \beta \\ 1 & b^{-1} & \bar\beta \\ \beta & \bar\beta & -Q \end{pmatrix}$$

Alternatively, after switching to the epsilon notations:

$$I_2 = -\epsilon_3\int_{x=0}^{2\pi}\Big[\frac{1}{3}\frac{\epsilon_1}{\sigma_3}\sum_{i=1}^{n_1}(\partial\phi_i)^3+\frac{1}{3}\frac{\epsilon_2}{\sigma_3}\sum_{i=n_1+1}^{n_1+n_2}(\partial\phi_i)^3+\frac{1}{3}\frac{\epsilon_3}{\sigma_3}\sum_{i=n_1+n_2+1}^{n_1+n_2+n_3}(\partial\phi_i)^3-$$
$$-\left(\frac{1}{2}\sum_{i,j}\frac{\epsilon_i\epsilon_j}{\sigma_3}\partial\phi_i D\partial\phi_j+\sum_{i<j}\frac{\epsilon_i\epsilon_j}{\sigma_3}\partial\phi_i\partial^2\phi_j\right)\Big]\frac{dx}{2\pi}$$

And basic fields normalized as follows:

$$\partial_i\phi(x)\partial_j\phi(y) = -\delta_{i,j}\frac{\sigma_3}{\epsilon_i}\frac{1}{\sin^2(x-y)}$$

Where $\sigma_3 = \epsilon_1\epsilon_2\epsilon_3$.

Bethe ansatz for the models considered in this section could be derived along the same lines, the only difference is in the action of $\psi(u)$ generators on vacuum. For a Fock space representation $\mathcal{F} = \prod_k \otimes\mathcal{F}_{u_k}^{(\alpha_k)}$ we have:

$$\psi(u)|\varnothing\rangle = \prod_{k=1}^{n}\frac{u-u_k-\epsilon_{\alpha_k}}{u-u_k}|\varnothing\rangle$$

So that we will have the same Bethe equations as in (1.81), but with different source function

$$A(u) = \prod_{k=1}^{n}\frac{u-u_k-\epsilon_{\alpha_k}}{u-u_k}.$$

Appendix B

B.1 Restoring the symmetry between ϵ_α

One may note that affine Yangian commutation relations (1.40) are symmetric with respect to permutations of all ϵ_α. Nevertheless Bethe Ansatz equations (2.27) are not symmetric in all ϵ_α, because of the source term $A(x) = \prod_{k=1}^{n} \frac{x-u_k+\frac{\epsilon_3}{2}}{x-u_k-\frac{\epsilon_3}{2}}$. We are now in a position to restore the symmetry, which will help us to build more general integrable systems. The resolution of the paradox is the following: there actually exists three types of Fock modules $\mathcal{F}^\alpha_x$. In order to describe Integrable systems, we have to define an $\mathcal{R}$-matrix acting between different Fock spaces $\mathcal{F}^\alpha_{x_1} \otimes \mathcal{F}^\beta_{x_2}$. In the following we will use the results of [BFM18, LS16] and also [FJMV21] where various integrable systems of this type considered in details for the q-deformed case. To the Fock module $\mathcal{F}^\alpha_v$ we assign a free bosonic field (2.6):

$$\partial\varphi(x) = -i\frac{u}{\sqrt{\epsilon_\beta\epsilon_\gamma}} + \sum_{n\neq 0} a_n e^{-inx}, \qquad [a_m, a_n] = m\delta_{m,-n},$$

here $(\alpha,\beta,\gamma) = \mathrm{perm}(1,2,3)$. To the tensor product of two Fock modules we have to assign a W-algebra and an R-matrix. If the both Fock modules are of the same type $\mathcal{F}^\alpha_{x_1} \otimes \mathcal{F}^\alpha_{x_2}$ then we assign to them two Screening currents:

$$S^\pm_\alpha = \oint e^{\left(\frac{\epsilon_\beta}{\epsilon_\gamma}\right)^{\pm\frac{1}{2}}(\varphi_1(x)-\varphi_2(x))} dx,$$

where $(\alpha,\beta,\gamma) = \mathrm{perm}(1,2,3)$. The W-algebra which commutes with these Screenings consists of two currents of spin 1 and 2

$$W^\alpha_1 = \partial\varphi_1(x) + \partial\varphi_2(x)$$
$$W^\alpha_2 = \frac{1}{2}(\partial\varphi_1(x) - \partial\varphi_2(x))^2 + \frac{\epsilon_\alpha}{\sqrt{\epsilon_\beta\epsilon_\gamma}}(\partial^2\varphi_1(x) - \partial^2\varphi_2(x))$$

defines the R-matrix in the usual way:

$$\mathcal{R}_{1,2}(\varphi_1-\varphi_2|\epsilon_\alpha,\epsilon_\beta,\epsilon_\gamma)W^\alpha_{1,2} = W^\alpha_{1,2}\Big|_{\varphi_1\leftrightarrow\varphi_2}\mathcal{R}_{1,2}(\varphi_1-\varphi_2|\epsilon_\alpha,\epsilon_\beta,\epsilon_\gamma).$$

Here $\mathcal{R}^{\alpha,\alpha}_{1,2}(\varphi_1-\varphi_2|\epsilon_\beta,\epsilon_\gamma,\epsilon_\alpha)$ is our old Maulik-Okounkov $\mathcal{R}$-matrix (2.8).

Now to Fock modules of different types: $\mathcal{F}^\alpha_{x_1} \otimes \mathcal{F}^\beta_{x_2}$ we assign a single "fermionic" screening charge

$$S_{f,\gamma} = \oint e^{\sqrt{\frac{\epsilon_\alpha}{\epsilon_\gamma}}\varphi_1(x) - \sqrt{\frac{\epsilon_\beta}{\epsilon_\gamma}}\varphi_2(x)} dx.$$

This screening called "fermionic" because it is a zero mode of a free fermion:

$$\psi(x) = e^{\sqrt{\frac{\epsilon_\alpha}{\epsilon_\gamma}}\varphi_1(x) - \sqrt{\frac{\epsilon_\beta}{\epsilon_\gamma}}\varphi_2(x)}, \quad \psi^\dagger(x) = e^{-\sqrt{\frac{\epsilon_\alpha}{\epsilon_\gamma}}\varphi_1(x) + \sqrt{\frac{\epsilon_\beta}{\epsilon_\gamma}}\varphi_2(x)}$$
$$\psi(x)\psi^\dagger(y) = \frac{1}{\sin(x-y)} + reg$$
$$S_{f,\gamma} = \psi_0$$

Corresponding W algebra which commutes with screening charges consists of two currents of spin 1, 2:

$$W_{f;1} = \frac{1}{\sqrt{\epsilon_\alpha}}\partial\varphi_1(x) + \frac{1}{\sqrt{\epsilon_\beta}}\partial\varphi_2(x)$$
$$W_{f;2} = (\partial\Phi(x))^2 + \partial^2\Phi(x),$$

where $\Phi(x) = \sqrt{\frac{\epsilon_\alpha}{\epsilon_\gamma}}\varphi_1(x) - \sqrt{\frac{\epsilon_\beta}{\epsilon_\gamma}}\varphi_2(x)$ and also an auxiliary current of spin 3. Again the R-matrix can be found from the condition:

$$\mathcal{R}^{\alpha,\beta}_{1,2}\, W_{f;1,2} = P_{1,2}(W_{f;1,2})\,\mathcal{R}^{\alpha,\beta}_{1,2},$$

here $P_{1,2}$ is a permutation operator $P_{1,2} : \mathcal{F}^\alpha_{x_1}\otimes\mathcal{F}^\beta_{x_2} \to \mathcal{F}^\beta_{x_2}\otimes\mathcal{F}^\alpha_{x_1}$. Note that now we have to permute not only the bosonic field $\varphi_1 \leftrightarrow \varphi_2$, but also have to exchange $\epsilon_\alpha \leftrightarrow \epsilon_\beta$ (we use $(\alpha,\beta,\gamma) = \mathrm{perm}(1,2,3)$).

Fermionic R-matrix has very similar form in terms of free fermions:

$$\mathcal{R}^{\alpha,\beta}_{1,2} = \exp\Big[\frac{1}{2\pi}\int_0^{2\pi} :\psi^\dagger(x)\log(\partial)\,\psi(x): dx\Big].$$

Boundaries and K-matrices. We already seen (2.11)-(2.12) that there is three types of boundaries, which produce three types of $\mathcal{K}$-matrices. First let us consider the case of the right boundary. We will use the following notation $\mathcal{F}^{\alpha_1}_1\otimes\mathcal{F}^{\alpha_2}_2\cdots\otimes\mathcal{F}^{\alpha_n}_n\Big|\beta_R$ for n Fock spaces and the right boundary. The case of a left boundary is completely similar and can be obtained by the following isomorphism

$$(\beta_L=\beta_R)\Big|\mathcal{F}^{\alpha_n}_n\otimes\mathcal{F}^{\alpha_{n-1}}_{n-1}\cdots\otimes\mathcal{F}^{\alpha_1}_1 \simeq D_1\dots D_n\left(\mathcal{F}^{\alpha_1}_1\otimes\mathcal{F}^{\alpha_2}_2\cdots\otimes\mathcal{F}^{\alpha_n}_n\Big|\beta_R\right),$$

where D_i is the operator of reflection of the bosonic fields $\varphi_i \to -\varphi_i$.

For the Fock module of type α and the boundary of type β: $\mathcal{F}^\alpha_n\Big|\beta$ we assign two screenings charges:

$$S^\pm_\gamma = \oint e^{\left(\frac{2\epsilon_\beta}{\epsilon_\gamma}\right)^{\pm\frac{1}{2}}\sqrt{2}\varphi_n(x)}dx,$$

where $(\alpha,\beta,\gamma) = \mathrm{perm}(1,2,3)$. The corresponding K-matrix is equal to:

$$\mathcal{K}_{\alpha|\beta} = \mathcal{R}(\sqrt{2}\varphi_n|\frac{\epsilon_\beta}{\sqrt{2}}, \sqrt{2}\epsilon_\gamma, -\frac{\epsilon_\beta}{\sqrt{2}} - \sqrt{2}\epsilon_\gamma)$$

If the Fock module and the boundary are of the same color, then the K-matrix is equal to the identity matrix:

$$\mathcal{K}_{\alpha|\alpha} = \mathrm{Id}.$$

Additional screening charges depend not only on the last Fock module $\mathcal{F}^\alpha_{x_n}$, but on the previous one $\mathcal{F}^\beta_{x_{n-1}}$. For $\mathcal{F}^\alpha_{n-1}\otimes\mathcal{F}^\alpha_n\Big|\alpha$ we have

$$S^\pm_\alpha = \oint e^{\left(\frac{\epsilon_\beta}{\epsilon_\gamma}\right)^{\pm\frac{1}{2}}(\varphi_{n-1}(x)-\varphi_n(x))}dx, \quad \bar{S}^\pm_\alpha = \oint e^{\left(\frac{\epsilon_\beta}{\epsilon_\gamma}\right)^{\pm\frac{1}{2}}(\varphi_{n-1}(x)+\varphi_n(x))}dx.$$

And for $\mathcal{F}^{\alpha}_{n-1}\otimes\mathcal{F}^{\beta}_{n}\Big|\beta$ we have:

$$S_{f,\gamma}=\oint e^{\sqrt{\frac{\epsilon_\alpha}{\epsilon_\gamma}}\varphi_{n-1}(x)-\sqrt{\frac{\epsilon_\beta}{\epsilon_\gamma}}\varphi_n(x)}dx\,,\quad \bar{S}_{f,\gamma}=\oint e^{\sqrt{\frac{\epsilon_\alpha}{\epsilon_\gamma}}\varphi_{n-1}(x)+\sqrt{\frac{\epsilon_\beta}{\epsilon_\gamma}}\varphi_n(x)}dx.$$

Equivalently corresponding W-algebra may be found from the condition of symmetry under reflection of the last boson $\varphi_n\to-\varphi_n$. These rules may be summarised by the following picture:

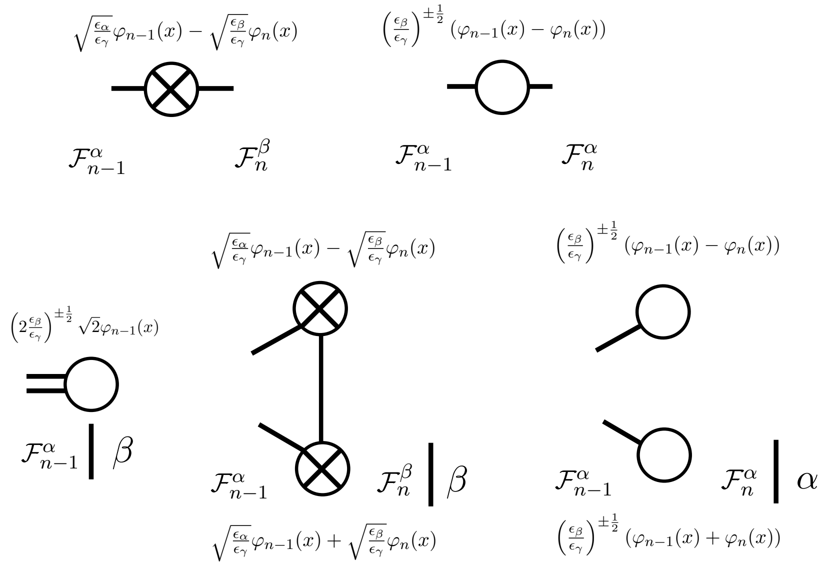


Figure B.1: The exponent of corresponding screenings, for different choices of Fock spaces and boundary conditions.

Finally we may assign an integrable system to the chain of colored Fock spaces with two colored boundaries $\beta_L\Big|\mathcal{F}^{\alpha_1}_1\otimes\mathcal{F}^{\alpha_2}_2\cdots\otimes\mathcal{F}^{\alpha_n}_n\Big|\beta_R$, using the corresponding $R-$ and K-matrices. We may construct KZ Integrals of Motion (2.14) and off-shell Bethe vectors (2.25) in precisely the same way s described in the main text. We may also find local Integrals of Motion as commutant of screenings charges. The corresponding Bethe equations read as:

$$r^{\beta_L}(x_i)r^{\beta_R}(x_i)A(x_i)A^{-1}(-x_i)\prod_{j\neq i}G(x_i-x_j)G^{-1}(-x_i-x_j)=1,$$

$$G(x)=\frac{(x-\epsilon_1)(x-\epsilon_2)(x-\epsilon_3)}{(x+\epsilon_1)(x+\epsilon_2)(x+\epsilon_3)},\qquad A(x)=\prod_{k=1}^{n}\frac{x-u_k+\frac{\epsilon_{\alpha_k}}{2}}{x-u_k-\frac{\epsilon_{\alpha_k}}{2}},\quad r^{\alpha}(x)=-\frac{x+\epsilon_\alpha/2}{x-\epsilon_\alpha/2}.$$

Appendix C

C.1 Sklyanin's K matrices

In this appendix we will present some formulas which relate the $K_{B,C,D}(u)$ matrices and R-matrices of some shifted parameters, u, q_1, q_2. R(u)-matrix itself could be computed explicitly at each level, or order by order in $\frac{1}{u}$ expansion [FHMZ17].
In the D case, there is an explicit formula in terms of modes of screening current

$$\check{K}_D = e^{i\pi \sum_n \frac{n(1-q_2^{-n})}{(1-q_1^n)(1+q_3^n)}(s_{1,n}-s_{\bar{1},n})(s_{1,-n}-s_{\bar{1},-n})}$$

Straightforward property of this operator is:

$$\check{K}_D \mathcal{S}_1(z) \check{K}_D = \mathcal{S}_{\bar{1}}(z)$$
$$\check{K}_D R_1(u) \check{K}_D = \check{K}_D \check{R}_{\bar{1}}(u) \check{K}_D.$$

Using the fact that nodes 1 and $\bar{1}$ doesn't connected with a node : $[\mathcal{S}_1(z), \mathcal{S}_{\bar{1}}(w)] = 0$ we have

$$\check{R}_1(u)\check{R}_{\bar{1}}(v) = \check{R}_{\bar{1}}(v)\check{R}_1(u).$$

The later immediately leads to the $KRKR$ relation:

$$\check{K}_D R_1(u_1u_2)\check{K}_D \check{R}_1(\frac{u_2}{u_1}) = \check{R}_1(\frac{u_2}{u_1})\check{K}_D \check{R}_1(u_1u_2)\check{K}_D$$

C and $\mathcal{B}$ cases are more complicated, here we will concentrated on a C case, while the case of B is completely analogical. As was defined in previous section $\check{K}(u)$ depends only on the longest screening $S_1(z)$, and obey the relation

$$\check{K}(u)\left(uY_1(z) + \frac{1}{u} : Y_1(z)\frac{S_1(q_1^{-d_1}z)}{S_1(q_1^{-d_1q_2^{-1}}z)} :\right) = \left(\frac{1}{u}Y_1(z) + u : Y_1(z)\frac{S_1(q_1^{-d_1}z)}{S_1(q_1^{-d_1}q_2^{-1}z)} :\right)\check{K}(u)$$

Here $d_1 = 2$, but let us stay with a general d_1 for a moment. K-matrix could be found as a series in $\frac{1}{u^2}$:

$$\check{K}(u) = (1 + \sum_{n=1}^{\infty} \check{K}_n \frac{1}{u^{2n}})\check{K}_0$$

For $u \to \infty$ one has:

$$\check{K}_0 Y_1(z) =: Y_1(z)\frac{\mathcal{S}_1(q_1^{-d_1}q_2^{-1}z)}{\mathcal{S}_1(q_1^{d_1}z)} : \check{K}_0$$

With a straightforward solution:

$$\check{K}_0 = exp\left((\log(q_1^{-d_1}q_2^{-1}) + \frac{i\pi}{n})\frac{1-q_2^{-n}}{\left(1-q_1^n\right)\left(1+q_1^{-d_1}q_2^{-1}\right)}s_{-n}s_n\right)$$

Proceeding further, K_i could be found as a solution of equations [FHMZ17]:

$$[\check{K}_i, : Y_1(z)A^{-1}(q_1^{-d_1}q_2^{-1}z) :] = Y_1(z)\check{K}_{i-1} - \check{K}_{i-1} : Y_1(z)A^{-1}(q_1^{-d_1}q_2^{-1}z)A(q_1^{-2d_1}q_2^{-2}z) : \tag{C.2}$$

For $i > 1$, and:

$$[\check{K}_1, : Y_1(z)A_1^{-1}(q_1^{-d_1}q_2^{-1}z) :] = Y_1(z) - : Y_1(q_1^{-2d_1}q_2^{-2}z)A_1^{-1}(q_1^{-d_1}q_2^{-1}z)A_1(q_1^{-2d_1}q_2^{-2}z) :$$

For $i = 1$

Here we introduced shorthand notation: $A_i(z) =: \frac{\mathcal{S}_i(z)}{\mathcal{S}_i(q_2 z)}$: see (3.5) , [KP18] for details. General solutions of equations (C.2) are not known, however they could be solved order by order, for example:

$$\check{K}_1 = \frac{1 - q_1^{-d_1}q_2^{-1}}{(1 - q_2^{-1})(1 - q_1^{-d_1})} \oint A_1(z)\frac{dz}{2\pi z}$$

In general, K_n is a multiple integral, containing n integrations.

Easily to observe that structure of the formula doesn't depend on the value of parameter d_1 and so we have an identity:

$$\check{K}_C(u, q_1, q_2) = \check{R}(u^2, q_1^2, q_2)$$
$$\check{K}_B(u, q_1, q_2) = \check{R}(u, q_1^{\frac{1}{2}}, q_2)$$