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Jury Member Report – Doctor of Philosophy thesis.

Name of Candidate: Ivan Sechin

PhD Program: Mathematics and Mechanics

Title of Thesis: Quantum R-matrix identities and integrable systems

Supervisor: Professor Anton Zabrodin

Co-supervisor: Dr. Andrei Zotov, Steklov Mathematical Institute, RAS

Name of the Reviewer: Nikita Slavnov

I confirm the absence of any conflict of interest	**Date: 28-09-2022**

The purpose of this report is to obtain an independent review from the members of PhD defense Jury before the thesis defense. The members of PhD defense Jury are asked to submit signed copy of the report at least 30 days prior the thesis defense. The Reviewers are asked to bring a copy of the completed report to the thesis defense and to discuss the contents of each report with each other before the thesis defense.

If the reviewers have any queries about the thesis which they wish to raise in advance, please contact the Chair of the Jury.

Reviewer's Report

Reviewers report should contain the following items:

- Brief evaluation of the thesis quality and overall structure of the dissertation.
- The relevance of the topic of dissertation work to its actual content
- The relevance of the methods used in the dissertation
- The scientific significance of the results obtained and their compliance with the international level and current state of the art
- The relevance of the obtained results to applications (if applicable)
- The quality of publications

The summary of issues to be addressed before/during the thesis defense

Ivan Sechin's thesis is devoted to the study of classical and quantum integrable systems. The latter occupy a special place in modern mathematics and physics. The possibility of obtaining exact (non-perturbative) results in such systems is of great importance. It is for this reason that interest in integrable systems does not weaken, and the topic of research carried out in the thesis is extremely relevant.

Ivan Sechin in his thesis studies integrable systems in two directions. The first one is the identification of mathematical structures underlying integrability. The author shows that Fay identity is one of such structures. It is thanks to this identity that it is possible to construct nontrivial solutions to the Yang-Baxter equation, which plays a key role in the theory of integrable systems. The second direction is the construction of new R-matrices, which are generalizations of the previously known solutions to the Yang-Baxter equation. Within this direction, the author significantly expands the class of integrable systems, which is a very interesting result from the point of view of physical applications.

The main text of the thesis is presented in four chapters. The first chapter is essentially an introduction. It briefly summarizes the content of the thesis and introduces the basic concepts.

Chapter 2 is devoted to the classical integrable systems of M interacting GL_{N} tops. These models generalize the classical spin Calogero-Moser models by adding the anisotropy to the interaction of spins. The main result of this chapter is the explicit expression for the $GL_{N,M}$-valued Lax pair with spectral parameter for the system of generalized interacting integrable tops and a Hamiltonian description of this model. The author also considers examples of the interacting tops systems for elliptic, trigonometric and the rational R-matrices. The system of interacting integrable tops can also be considered as an extension of the spin Calogero-Moser systems of interacting particles (in the $N = 1$ case) and as an extension of the integrable top of the Euler-Arnold type (in the $M = 1$ case).

In Chapter 3, the author considers relativistic versions of generalized systems of interacting tops. The main result of this chapter is the description of the Lax structure of a generalized model of relativistic interacting tops constructed using a quantum R-matrix that satisfies the associative Yang-Baxter equation. The obtained $GL_{N,M}$ models are generalizations of the classical Ruijsenaars-Schneider spin systems (special case $N = 1$) and relativistic integrable tops on the Lie group GL_{N} (special case $M = 1$). The generalized model of relativistic interacting tops is obtained using a Lax pair with spectral parameters and equations of motion consistent with a certain constraint. It is noteworthy that the proof of this correspondence is based only on the identities for the quantum R-matrix.

In Chapter 4, the author studies the quantization of the structures obtained in the previous chapters. Here, a generalized quantum dynamical $GL_{N,M}$ R-matrix is constructed based on the GL_{N}-solution of the associative Yang-Baxter equation. This R-matrix is a quantization of the classical r-matrix for generalized systems of interacting integrable tops. The quantum dynamic Yang-Baxter equation in this case can also be considered as a quantum version of the classical dynamic Yang-Baxter equation for models of interacting tops. In the case of $N = 1$, the author reproduces the GL_{M}-valued dynamic Felder R-matrix. In the case $M = 1$, the author obtains a GL_{N} non-dynamic R-matrix of vertex type, which includes the Baxter-Belavin elliptic matrix and its degenerations. This chapter also constructs a quadratic algebra generalizing the elliptic quantum group and the Sklyanin algebra.

I now give a list of the main results obtained in the thesis.

Systems of generalized interacting integrable tops are constructed for any quantum R-matrix which solves the associative Yang-Baxter equation together with skew-symmetry and unitarity conditions.

A relativistic analogue of the system of generalized interacting integrable tops is constructed.

The classical r-matrix structure for the generalized interacting integrable tops is quantized leading to a quantum R-matrix and the corresponding RLL-algebra. In the elliptic case, the quadratic algebra of this RLL-algebra is constructed.

Turning to a brief description of the Ivan Sechin's work, I note that the topic of the thesis fully corresponds to its actual content. The relevance of this topic has already been mentioned above. The obtained results are rigorously substantiated and proved. Their scientific novelty and value are beyond doubt. These results, as well as the methods used in the thesis, completely correspond to the world level. This is confirmed by the publication of the results in leading Russian and foreign journals (including Q1). The methods developed in the thesis can be used to construct new integrable systems, which is relevant for possible physical applications.

At the same time, I must make a remark regarding the presentation of the thesis. This drawback is very typical for texts that are compiled from several papers. The author does not always successfully cope with this work. For example, the Kronecker function and the odd theta-function are introduced three times in the text. The content of Appendix 4.4 largely duplicates Appendix 2.6.1. On the other hand, the author gives many formulas from the main text in Introduction, but often does not explain the meaning of the symbols included in the formulas. I emphasize that this remark refers to the presentation of the thesis, but not to its scientific content.

In summary, Ivan Sechin's thesis shows the role of the associative Yang-Baxter equation in the theory of integrability. The author managed to obtain solutions that define new integrable systems. The results of the thesis are certainly new. The candidate shows a deep understanding of the problem under study and related issues

I have no doubts that this thesis satisfies all the requirements for the degree of the Doctor of Philosophy in Mathematics.

Provisional Recommendation

☐ *I recommend that the candidate should defend the thesis by means of a formal thesis defense*

☒ *I recommend that the candidate should defend the thesis by means of a formal thesis defense only after appropriate changes would be introduced in candidate's thesis according to the recommendations of the present report*

☐ *The thesis is not acceptable and I recommend that the candidate be exempt from the formal thesis defense*

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