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QUANTUM R-MATRIX IDENTITIES AND INTEGRABLE SYSTEMS

Doctoral Thesis

by

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I hereby declare that the work presented in this thesis was carried out by myself at Skolkovo Institute of Science and Technology, Moscow, except where due acknowledgement is made, and has not been submitted for any other degree.

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Abstract.

The thesis is devoted to the applications of the identities on the quantum R-matrices, mainly, the associative Yang–Baxter equation, in the theory of classical and quantum integrable systems.

The new families of classical integrable systems of M non-relativistic and relativistic interacting $\mathfrak{gl}_N$ integrable tops have been constructed in this approach. The obtained integrable systems generalizes both the classical integrable systems of particles (spin Calogero–Moser and spin Ruijsenaars–Schneider models) in the $N = 1$ particular case and the classical integrable tops of Euler–Arnold type in the $M = 1$ case.

The Lax representations with spectral parameter for these systems have been written explicitly, as well as the dynamical classical r-matrix for the generalized interacting integrable tops in the non-relativistic case. In the quantum level, the quantum dynamical RLL-algebra for the quantization of this classical r-matrix is also obtained. In the elliptic case the quadratic quantum algebra based on this RLL-relation generalizing the Sklyanin algebra and the small elliptic quantum group is constructed.

List of publications.

1. A. Grekov, I. Sechin, A. Zotov. *Generalized model of interacting integrable tops*, JHEP 10 (2019) 081; arXiv: 1905.07820 [math-ph].

2. I.A. Sechin, A.V. Zotov, *Integrable system of generalized relativistic interacting tops*. Theoret. and Math. Phys., 205:1 (2020) 1292–1303, arXiv: 2011.09599 [math-ph].

3. I.A. Sechin, A.V. Zotov, *Quadratic algebras based on SL(NM) elliptic quantum R-matrices*, Theoret. and Math. Phys., 208:2 (2021), 1156–1164, arXiv: 2104.04963

Contents

Chapter 1

Introduction

The integrable systems or exactly solvable models are classical and quantum systems with very specific properties, resulting in the fact that there exist a large number of the integrals of motion (conservation laws). The systems of this type are quite rare and therefore finding some new integrable cases is always a very interesting problem. The theory of integrable systems is one of the rapidly developing areas in the modern mathematical physics, new integrable systems are constantly being explored, as well as their connections to the known integrable systems and to the other areas of the mathematics and theoretical physics, such as theory of matrix models, Lie groups theory, quantum algebras, quantum and conformal field theory and string theory. This thesis is also devoted to the introduction of a new family of integrable systems based on the algebraic identities on the quantum R-matrices. Despite the fact that the construction uses the quantum objects, the methods of the thesis allows to define both classical and quantum systems with the very interesting properties. The systems defined can be considered as the generalizations of the well-known mechanical integrable systems, and the quantum algebra corresponding to the defined systems, generalizes two well-known quantum algebra structures: the Sklyanin algebra and the elliptic quatnum group.

There exist two well-known important classes of integrable systems with a finite number of degrees of freedom (mechanical systems), which have representatives both at the classical and at the quantum side. The first class includes the many-body systems of interacting particles on a line, which representative example is a classical Calogero–Moser system with Hamilton function

$$H = \sum_{i=1}^{M} \frac{p_i^2}{2} - \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}}^{M} V(q_i - q_j), \qquad (1.1)$$

where p_i and q_i are the canonical momenta and coordinates with Poisson brackets

$$\{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0.$$

The Calogero–Moser system is known to be integrable for three integrable potentials $V(q)$ — rational, trigonometric and elliptic:

$$V(q) = \begin{cases} \frac{1}{q^2}, & \text{rational case,} \\ \frac{1}{\sin^2(q)}, & \text{trigonometric case,} \\ \wp(q), & \text{elliptic case.} \end{cases} \tag{1.2}$$

In the elliptic case the Weierstrass $\wp$-function on the elliptic curve $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ is used

$$\wp(q) = \frac{1}{q^2} + \sum_{\substack{m,n\in\mathbb{Z}^2 \\ m\neq n}} \left(\frac{1}{(q+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right).$$

The second wide class includes different tops and spins systems (integrable tops, Gaudin models, spin chains). The symplectic manifolds connected with the systems of this type are constructed using the coadjoint orbits of Lie algebras, and the spin variables have Lie-algebraic Poisson brackets. The simplest representatives of this class are known as the integrable $SO(3)$ Euler tops. They have Hamilton functions of the form

$$H = \sum_{\alpha\in\{x,y,z\}} J_\alpha S_\alpha^2, \tag{1.3}$$

where S_α have the Poisson brackets connected with Lie algebra of $SO(3)$

$$\{S_\alpha, S_\beta\} = \epsilon_{\alpha\beta\gamma} S_\gamma$$

and J_α are arbitrary constants corresponding to the components of the inverse of the inertia tensor of the top. There are three integrable Euler top cases, depending on the values of J_α:

$$\begin{cases} J_x = J_y = J_z, & \text{rational case,} \\ J_x = J_y \neq J_z, & \text{trigonometric case,} \\ J_x \neq J_y \neq J_z \neq J_x & \text{elliptic case.} \end{cases} \tag{1.4}$$

The rational, trigonometric and elliptic cases in this classification relate to the type of functions, obtained in the solutions of the equations of motion. In the most general elliptic case, using the conserved quantities H and $S^2 = S_x^2 + S_y^2 + S_z^2$, one can write the equation on S_x

$$\dot{S}_x = \{H, S_x\} = 2(J_y - J_z)S_y S_z = \sqrt{a + bS_x^2 + cS_x^4}, \tag{1.5}$$

where a, b, c are the constant depending on J_x, J_y, J_z, S^2, H. In the general case of nonzero c the solution is elliptic, in the situation $J_x = J_y \neq J_z$ $c = 0$ and the solutions become trigonometric, in the situation $J_x = J_y = J_z$ both $c = 0$ and $b = 0$ and the equation degenerates to the trivial one.

The integrability of these systems in all three cases are based on the set of functional identities satisfying by the rational, trigonometric and elliptic functions. The most fundamental relation from this set, from which one can obtain all other identities, is called the Fay identity

$$\phi(u, z_{12})\phi(v, z_{23}) = \phi(v, z_{13})\phi(u - v, z_{12}) + \phi(v - u, z_{23})\phi(u, z_{13}), \quad z_{ij} = z_i - z_j. \tag{1.6}$$

This identity is satisfied by rational, trigonometric and elliptic functions, which in turn define rational, trigonometric, or elliptic integrable systems. The three series of solutions are

$$\phi(u, z) = \begin{cases} \frac{1}{u} + \frac{1}{z} = \frac{u+z}{uz}, \quad \text{rational case}, \\ \coth(u) + \coth(z) = \frac{\sin(u+z)}{\sin(u)\sin(z)}, \quad \text{trigonometric case}, \\ \frac{\vartheta'(0)\vartheta(z+u)}{\vartheta(z)\vartheta(u)}, \quad \text{elliptic case}, \end{cases} \tag{1.7}$$

where $\vartheta(z)$ is the odd elliptic theta-function

$$\vartheta(z) = \sum_{k \in \mathbb{Z}} \exp\left(\pi i \tau (k + \tfrac{1}{2})^2 + 2\pi i (z + \tfrac{1}{2})(k + \tfrac{1}{2})\right)$$

on the elliptic curve $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ with moduli $\tau : \text{Im}\tau > 0$. The elliptic case is the most general one, and trigonometric and elliptic cases could be obtained from as its limits

$$\vartheta(z) \to \sin z \to z,$$
$$\frac{\vartheta'(0)\vartheta(z+u)}{\vartheta(z)\vartheta(u)} \to \frac{\sin(z+u)}{\sin(z)\sin(u)} = \cot(z) + \cot(u) \to \frac{z+u}{zu} = \frac{1}{z} + \frac{1}{u}.$$

This thesis consists some results of the studying of a family of integrable systems that is in an intermediate position between these two classes of systems and has the properties of both types — interacting integrable tops systems, constructed via a quantum R-matrix satisfying the specific quantum R-matrix identities, generalizing the functional relations which provide the integrability in the known cases.

The main method of investigation of these systems is connected with the extension of the Fay identity on a scalar function ϕ to a noncommutative matrix case. The matrix-valued counterpart of the Fay identity is known as the associative Yang–Baxter equation

$$R^{\hbar}_{12}(z_{12})R^{\eta}_{23}(z_{23}) = R^{\eta}_{13}(z_{13})R^{\hbar-\eta}_{12}(z_{12}) + R^{\eta-\hbar}_{23}(z_{23})R^{\hbar}_{13}(z_{13}). \tag{1.8}$$

This is the relation on the matrix function R taking values in the tensor square of a $\mathrm{Mat}(N,\mathbb{C})$ space, written in a tensor cube of this space, and the lower indices shows the components of the tensor product where this matrix is nontrivial. It is the standard notation using for quantum R-matrices in the Quantum Inverse Scattering Method.

There is a natural reason to call the objects in these relations the quantum R-matrices. Namely, any solution of the associative Yang–Baxter equation which additionally satisfies unitarity and skew-symmetry properties

$$R^{\hbar}_{12}(z)R^{\hbar}_{21}(-z) \propto 1\otimes 1, \quad R^{\hbar}_{12}(z) = -R^{-\hbar}_{21}(-z) \tag{1.9}$$

is also a solution of the Yang–Baxter equation.

In order to prove it, one needs to consider the associative Yang–Baxter equations with deformation parameters $\hbar = 2a, \eta = a$

$$R^{2a}_{12}(z_{12})R^{a}_{23}(z_{23}) = R^{a}_{13}(z_{13})R^{a}_{12}(z_{12}) + R^{-a}_{23}(z_{23})R^{2a}_{13}(z_{13}). \tag{1.10}$$

After that, multiply both sides on $R^{a}_{23}(z_{23})$

$$R^{a}_{23}(z_{23})R^{2a}_{12}(z_{12})R^{a}_{23}(z_{23}) = R^{a}_{23}(z_{23})R^{a}_{13}(z_{13})R^{a}_{12}(z_{12}) + R^{a}_{23}(z_{23})R^{-a}_{23}(z_{23})R^{2a}_{13}(z_{13}). \tag{1.11}$$

Using unitarity and skew-symmetry, one obtains

$$R^{a}_{23}(z_{23})R^{-a}_{23}(z_{23}) = -R^{a}_{23}(z_{23})R^{a}_{32}(z_{32}) = -(\wp(a)-\wp(z_{23}))1_N\otimes 1_N, \tag{1.12}$$

therefore, one gets for the r.h.s. of the Yang–Baxter equation

$$R^{a}_{23}(z_{23})R^{a}_{13}(z_{13})R^{a}_{12}(z_{12}) = R^{a}_{23}(z_{23})R^{2a}_{12}(z_{12})R^{a}_{23}(z_{23}) + (\wp(a)-\wp(z_{23}))R^{2a}_{13}(z_{13}). \tag{1.13}$$

Doing the same procedure for the associative Yang–Baxter equation rewritten in the spaces $1-3-2$ instead of $1-2-3$, one obtains the same expression for the l.h.s.

$$\begin{gathered}
R^{2a}_{13}(z_{13})R^{a}_{32}(z_{32}) = R^{a}_{12}(z_{12})R^{a}_{13}(z_{13}) + R^{-a}_{32}(z_{32})R^{2a}_{12}(z_{12}),\\
R^{2a}_{13}(z_{13})R^{a}_{32}(z_{32})R^{a}_{23}(z_{23}) = R^{a}_{12}(z_{12})R^{a}_{13}(z_{13})R^{a}_{23}(z_{23}) + R^{-a}_{32}(z_{32})R^{2a}_{12}(z_{12})R^{a}_{23}(z_{23}),\\
R^{a}_{12}(z_{12})R^{a}_{13}(z_{13})R^{a}_{23}(z_{23}) = (\wp(a)-\wp(z_{23}))R^{2a}_{13}(z_{13}) + R^{a}_{23}(z_{23})R^{2a}_{12}(z_{12})R^{a}_{23}(z_{23}).
\end{gathered} \tag{1.14}$$

The solutions of the associative Yang–Baxter equation are also rational, trigonometric and elliptic quantum R-matrices. All these solutions can be used to construct the generalized integrable models.

Thesis structure and review

The **Chapter 1** is the introduction, where the identities on the quantum R-matrices and their correspondence to the functional relations are discussed, and the short review of the thesis is given.

The **Chapter 2** is devoted to the nonrelativistic generalized interacting integrable tops systems as the main example of models described using the quantum R-matrices and the relations on them. The construction uses $GL(N)$ quantum R-matrices to define M interacting $\mathfrak{gl}_N$ integrable tops and their generalizations. The quantum R-matrices defines the structure of spin variables interaction, they are included into the Hamiltonian and the Lax operators of this system:

$$H = \frac{1}{2}\sum_{i=1}^{M} p_i^2 + \sum_{i=1}^{M} H^{\text{top}}(\mathcal{S}^{ii}) + \frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}^{M} \mathcal{U}(\mathcal{S}^{ij}, \mathcal{S}^{ji}, q_i - q_j), \tag{1.15}$$

$$L(z) = \sum_{i=1}^{M} E_{ii} \otimes \Big(p_i \cdot 1_N + \operatorname{Tr}_2(\mathcal{S}_2^{ii} R_{12}^{z,(0)} P_{12})\Big) + \sum_{\substack{i,j\\ i\neq j}}^{M} E_{ij} \otimes \operatorname{Tr}_2(\mathcal{S}_2^{ij} R_{12}^{z}(q_{ij}) P_{12}). \tag{1.16}$$

Here E_{ij} is the standard matrix basis in $M \times M$ matrices with the matrix elements $(E_{ij})_{ab} = \delta_{ia}\delta_{jb}$, p_i and q_j are canonical momenta and coordinates with the canonical Poisson brackets $\{p_i, q_j\} = \delta_{ij}$, and $\{p_i, p_j\} = \{q_i, q_j\} = 0$ and $\mathcal{S}^{ij}$ are matrices of the spin variables with Lie-algebraic Poisson brackets

$$\{\mathcal{S}_{ab}^{ij}, \mathcal{S}_{cd}^{kl}\} = \mathcal{S}_{cb}^{kj}\delta^{il}\delta_{ad} - \mathcal{S}_{ad}^{il}\delta^{kj}\delta_{cb}, \quad i,j,k,l = 1,\ldots,M, \quad a,b,c,d = 1,\ldots,N. \tag{1.17}$$

The potential function $\mathcal{U}(\mathcal{S}^{ij}, \mathcal{S}^{ji}, q_i - q_j)$, the top Hamiltonian $H^{\text{top}}(\mathcal{S}^{ii})$ and the components of the Lax operator $L(z)$ are defined via the quantum R-matrix satisfying the associative Yang–Baxter equation $R^z(q_{ij})$ and its coefficients in the expansion in the spectral parameter z and coordinates q_{ij}:

$$R_{12}^{z}(q_{ij}) = \frac{1}{z} \cdot 1 + r_{12}(q_{ij}) + z m_{12}(q_{ij}) + \mathcal{O}(z^2), \tag{1.18}$$

$$R_{12}^{z}(q_{ij}) = \frac{1}{q_{ij}} P_{12} + R_{12}^{z,(0)} + \mathcal{O}(q_{ij}), \tag{1.19}$$

$$\mathcal{U}(\mathcal{S}^{ij}, \mathcal{S}^{ji}, q_i - q_j) = \operatorname{Tr}_{12}(\partial_{q_i} r_{21}(q_i - q_j) P_{12} \mathcal{S}_1^{ij} \mathcal{S}_2^{ji}), \tag{1.20}$$

$$H^{\text{top}}(\mathcal{S}) = \frac{1}{2}\operatorname{Tr}_{12}(m_{12}(0)\mathcal{S}_1\mathcal{S}_2). \tag{1.21}$$

where P_{12} denotes the permutation matrix.

The Lax equations are equivalent to the equations of motion under the constraints

$$\mathrm{Tr}\mathcal{S}^{ii} = \mathrm{const} \tag{1.22}$$

and defines the integrable systems on the coadjoint orbits of the group $GL(NM)$. The $N = 1$ case reconstructs the Lax pair for the spin generalization of the many-body Calogero–Moser system, and the $M = 1$ case provides the Lax pair for the Euler–Arnold integrable top — two different systems from two different classes of the integrable mechanical systems.

The Hamiltonian structure of the generalzed interacting integrable tops system is also described in the Chapter 2. It is shown, that the Lax operator for the system satifies the modified dynamical classical r-matrix structure which provides the commutativity of the Hamiltonians in this system under the constraints

$$\begin{aligned}\{L_{1'1}(z_1), L_{2'2}(z_2)\} = [\mathbf{r}_{1'2'12}(z_1,z_2), L_{1'1}(z_1)] - [\mathbf{r}_{2'1'21}(z_2,z_1), L_{2'2}(z_2)] - \\ -\Big(\sum_{i=1}^{M} \mathrm{Tr}\mathcal{S}^{ii}\cdot\partial_{q_i}\Big)\mathbf{r}_{1'2'12}(z_1,z_2),\end{aligned} \tag{1.23}$$

$$\mathbf{r}_{1'2'12}(z,w) = \sum_{i=1}^{M}(E_{ii})_{1'}\otimes(E_{ii})_{2'}\otimes r_{12}(z-w) + \sum_{\substack{i,j\\ i\neq j}}^{M}(E_{ij})_{1'}\otimes(E_{ji})_{2'}\otimes R_{12}^{q_{ij}}(z-w). \tag{1.24}$$

The classical r-matrices for these generalized systems are also defined via the same quantum R-matrices, as in the corrrespoding Lax operators.

The **Chapter 3** introduced the relativistic version of the generalized interacting integrable tops systems. These systems are the simultaneous extensions of the spin Ruijsenaars–Schneider particle systems and the relativistic integrable tops. They are described via the Lax pair with a spectral parameter and set of equations of motion equivalent to the Lax equation. There is no known general Hamiltonian structure and classical r-matrix structure in the relativistic case, as well as in the particular case of the elliptic spin version of Ruijsenaars–Schneider model (however, it is known in for the rational and trigonometric spin models).

The Lax operator of the relativistic model has also the construction based on the quantum R-matrix satisfying the set of quantum R-matrix identities and has the form

$$L(z) = \sum_{i,j=1}^{M} E_{ij}\otimes \mathrm{Tr}_2(\mathcal{S}_2^{ij}R_{12}^{z}(q_{ij}+\eta)P_{12}). \tag{1.25}$$

As in the case of the spin Ruijsenaars–Schneider model, the Lax equations are equivalent to the equation of motion on a set of constraints on coordinates q and spin variables S

$$\mu_i = \dot{q}_i - \mathrm{Tr}\mathcal{S}^{ii} = 0. \tag{1.26}$$

These constraints can be presented as the additional term inserted into the Lax equations

$$\dot{L}(z) = [L(z), M(z)] + \sum_{i=1}^{M} (\mu_i - \mu_j) E_{ij} \otimes \mathrm{Tr}_2(\mathcal{S}_2^{ij} F_{12}^z(q_{ij} + \eta) P_{12}), \tag{1.27}$$

where $F^z(q_{ij}) = \partial_{q_i} R^z(q_{ij})$.

In the nonrelativistic $\eta \to 0$ limit on the constraints $\mu_i = 0$ this Lax operator goes to the Lax operator from the Chapter 2 with the identification $p_i = \dot{q}_i/\eta$:

$$L(z) \to \frac{1}{\eta} \sum_{i=1}^{M} E_{ii} \otimes 1 \cdot \mathrm{Tr}\mathcal{S}^{ii} + \sum_{i=1}^{M} E_{ii} \otimes \mathrm{Tr}_2(\mathcal{S}_2^{ii} R_{12}^{z,(0)}(q_{ij}) P_{12}) +$$
$$+ \sum_{\substack{i,j \\ i \neq j}}^{M} E_{ij} \otimes \mathrm{Tr}_2(\mathcal{S}_2^{ij} R_{12}^z(q_{ij}) P_{12}) + O(\eta). \tag{1.28}$$

The **Chapter 4** is devoted to the quantum R-matrix structure for the generalized interacting tops system. It quantizes the modified dynamical classical r-matrix structure defined above. The quantum R-matrix in this case is also "half-dynamical" (like the classical one), depending only on coordinates q. It has a form of a $GL(M)$ block Felder dynamical R-matrix with nondynamical $GL(N)$ quantum R-matrices inside the blocks.

$$\mathbf{R}^{\hbar}_{1'2'12}(z, w \mid q) = \sum_{i=1}^{M} (E_{ii})_{1'} \otimes (E_{ii})_{2'} \otimes R_{12}^{\hbar}(z - w) + \sum_{\substack{i,j \\ i \neq j}}^{M} (E_{ij})_{1'} \otimes (E_{ji})_{2'} \otimes R_{12}^{q_{ij}}(z - w) +$$
$$+ \sum_{\substack{i,j \\ i \neq j}}^{M} (E_{ii})_{1'} \otimes (E_{jj})_{2'} \otimes 1_N \otimes 1_N \ \phi(\hbar, -q_{ij}) = \frac{1}{\hbar} + \mathbf{r}_{1'2'12}(z, w \mid q) + O(\hbar). \tag{1.29}$$

In the case $M = 1$ only the first summand is in the sum, therefore, this R-matrix becomes a nondynamical quantum R-matrix (not depends on q and satisfies the ordinary Yang–Baxter equation), but in the case $N = 1$ it is equivalent to the ordinary dynamical Felder quantum R-matrix, satisfying the dynamical Yang–Baxter.

The modified dynamical Yang–Baxter equation written on this R-matrix has shifts only along $1'$ and $2'$ spaces, corresponding to the Felder-like structures in the R-matrix

$$\mathbf{R}^{\hbar}_{1'2'12}(z_1, z_2 \mid q) \mathbf{R}^{\hbar}_{1'3'13}(z_1, z_3 \mid q - \hbar^{(2')}) \mathbf{R}^{\hbar}_{2'3'23}(z_2, z_3 \mid q) =$$
$$= \mathbf{R}^{\hbar}_{2'3'23}(z_2, z_3 \mid q - \hbar^{(1')}) \mathbf{R}^{\hbar}_{1'3'13}(z_1, z_3 \mid q) \mathbf{R}^{\hbar}_{1'2'12}(z_1, z_2 \mid q - \hbar^{(3')}), \tag{1.30}$$

$$\mathbf{R}^{\hbar}_{1'2'12}(z_1, z_2 \mid q - \hbar^{(3')}) = \left(\sum_{i=1}^{M} (E_{ii})_{3'} \ e^{-\hbar \partial_{q_i}} \right) \mathbf{R}^{\hbar}_{1'2'12}(z_1, z_2 \mid q) \left(\sum_{i=1}^{M} (E_{ii})_{3'} \ e^{\hbar \partial_{q_i}} \right). \tag{1.31}$$

One can also define an L-operator and RLL-algebra corresponding to this quantum R-matrix. Let $h_1, \ldots, h_M$ be commutative elements, then $\hat{L}(z)$ is called an L-operator with Cartan elements h_i if it satisfies the relation

$$\begin{aligned} &\mathbf{R}^{\hbar}_{1'2'12}(z_1, z_2 \mid q)\hat{L}_{1'1}(z_1 \mid q - \hbar^{(2')})\hat{L}_{2'2}(z_2 \mid q) = \\ &= \hat{L}_{2'2}(z_2 \mid q - \hbar^{(1')})\hat{L}_{1'1}(z_1 \mid q)\mathbf{R}^{\hbar}_{1'2'12}(z_1, z_2 \mid q - \hbar h), \end{aligned} \tag{1.32}$$

where the shift along the Cartan algebra is used

$$\mathbf{R}^{\hbar}_{1'2'12}(z_1, z_2 \mid q - \hbar h) = \left(\sum_{i=1}^{M} h_i \; e^{-\hbar\partial_{q_i}}\right) \mathbf{R}^{\hbar}_{1'2'12}(z_1, z_2 \mid q) \left(\sum_{i=1}^{M} h_i \; e^{\hbar\partial_{q_i}}\right). \tag{1.33}$$

The Section 4 is also contains the quadratic algebra defined via generators and relations, which is equivalent to this RLL-algebra in the case of elliptic Baxter–Belavin R-matrix in the blocks. This algebra is generated by the operators t^{α}_{ij}, where $1 < i, j < M$ and $\alpha \in \mathbb{Z}_N \times \mathbb{Z}_N$. For any fixed pair i, j these operators satisfy the Sklyanin algebra relations, and additional relations on the operators generalizes the relations in the small elliptic dynamical quantum group.

Chapter 2

Generalized interacting integrable tops

This chapter is based on our paper [30] and is devoted to the classical integrable systems of M interacting $\mathfrak{gl}_N$ tops. The construction of this integrable system is based on the GL_N quantum R-matrix satisfying additionally the associative Yang–Baxter equation. These models generalize the classical spin Calogero–Moser models by adding the anisotropy to the interaction of spins. The R-matrix data provides the classical analogues of the anisotropic spin exchange operators.

The main results in this chapter are the explicit expression for the $\mathfrak{gl}_{NM}$-valued Lax pair with spectral parameter for the generalized interacting integrable tops system and the Hamiltonian description of this model, including the classical dynamical r-matrix structure associated with the constructed Lax pairs. These results have been obtained via the identities on the quantum R-matrix in the definition of system, therefore, they do not depend on the explicit form of this R-matrix. The chapter also contains the examples of the interacting tops systems for different quantum R-matrix — elliptic, trigonometric and the rational ones, in these cases the potentials and the spin exchanges are written explicitly.

The system of interacting integrable tops can be considered simultaneously as the extension of the spin Calogero–Moser systems of interacting particles (in the $N = 1$ case) and as the extension of the integrable top of the Euler–Arnold type (in the $M = 1$ case). The $\mathfrak{gl}_{NM}$-valued Lax pair for this system also generalizes the $\mathfrak{gl}_N$ Lax structure of the integrable top and the $\mathfrak{gl}_M$ Lax structure of the spin Calogero–Moser system. The Lax operators can be considered as $M \times M$ block matrices with the form of Calogero–Moser Lax operators with $N \times N$ blocks inside, corresponding to the top-like degrees of freedom.

The classical r-matrix for this system generalizes two different classical r-matrix structure — the nondynamical one, typical for the top-like systems, and the dynamical one, typical for the particle-like systems. This means that the classical r-matrix does not depend explicitly on the spin variables by depends only on the coordinates of particles.

2.1 Introduction

In this chapter we describe the classical integrable $\mathfrak{gl}_{NM}$ model given by the Hamiltonian of the following form:

$$\mathcal{H} = \sum_{i=1}^{M} \frac{p_i^2}{2} + \sum_{i=1}^{M} H^{\rm top}(\mathcal{S}^{ii}) + \frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}^{M} \mathcal{U}(\mathcal{S}^{ij},\mathcal{S}^{ji},q_i-q_j), \tag{2.1}$$

where p_i and q_j are the canonical variables:

$$\{p_i,q_j\} = \delta_{ij}, \quad \{p_i,p_j\} = \{q_i,q_j\} = 0, \quad i,j=1,\dots,M. \tag{2.2}$$

For all $i,j=1,\dots,M$ $\mathcal{S}^{ij}$ are $N\times N$ matrices of "classical spin" variables, i.e.

$$\mathcal{S}^{ij} = \sum_{a,b=1}^{N} \mathcal{S}^{ij}_{ab}\, e_{ab} \in \mathrm{Mat}(N,\mathbb{C}), \tag{2.3}$$

where $\{e_{ab},\ a,b=1,\dots,N\}$ is the standard basis in $\mathrm{Mat}(N,\mathbb{C})$. They are naturally arranged into $NM\times NM$ block-matrix $\mathcal{S}$:

$$\mathcal{S} = \sum_{i,j=1}^{M} E_{ij}\otimes \mathcal{S}^{ij} = \sum_{i,j=1}^{M}\sum_{a,b=1}^{N} \mathcal{S}^{ij}_{ab} E_{ij}\otimes e_{ab} \in \mathrm{Mat}(NM,\mathbb{C}), \tag{2.4}$$

where $\{E_{ij},\ i,j=1,\dots,M\}$ is the standard basis in $\mathrm{Mat}(M,\mathbb{C})$. The Poisson structure is given by the Poisson–Lie brackets on $\mathfrak{gl}^*_{NM}$ Lie coalgebra:

$$\{\mathcal{S}^{ij}_{ab},\mathcal{S}^{kl}_{cd}\} = \mathcal{S}^{kj}_{cb}\delta^{il}\delta_{ad} - \mathcal{S}^{il}_{ad}\delta^{kj}\delta_{bc}. \tag{2.5}$$

Integrable tops. In order to clarify the structure of the Hamiltonian (2.1) consider the case $M=1$. Then the last term in (2.1) is absent, and we are left with a free particle (with momenta p_1) and the Hamiltonian $H^{\rm top}(\mathcal{S}^{11})$ of integrable top of Euler–Arnold type [3, 18, 51, 55, 56]. Here we deal with the models admitting the Lax pairs with spectral parameter on elliptic curves [78, 69]. The general form for equations of motion (for the top-like models) is

$$\dot{S} = [S, J(S)], \tag{2.6}$$

where $S\in \mathrm{Mat}(N,\mathbb{C})$ is the matrix of dynamical variables, while the inverse inertia tensor J is a linear map

$$J(S) = \sum_{i,j,k,l=1}^{N} J_{ijkl}\, e_{ij}\, S_{lk} \in \mathrm{Mat}(N,\mathbb{C}) \tag{2.7}$$

In the general case the model (2.6) is not integrable. It is integrable for some special $J(S)$ only. More precisely, here we consider special tops, which were described in [40, 92, ?, 45, 47] for elliptic, trigonometric and rational cases respectively. All of them can be written [45, 47, 49] in the R-matrix form based on a quantum GL_N R-matrix (in the fundamental representation) satisfying the associative Yang–Baxter equation [27, 61]:

$$R^\hbar_{12}(q_{12})R^\eta_{23}(q_{23}) = R^\eta_{13}(q_{13})R^{\hbar-\eta}_{12}(q_{12}) + R^{\eta-\hbar}_{23}(q_{23})R^\hbar_{13}(q_{13}), \quad q_{ab} = q_a - q_b. \tag{2.8}$$

Having solution of (2.8) with some additional properties (see the next Section) the inverse inertia tensor comes from the term $m_{12}(z)$ in the classical limit expansion:

$$R^\hbar_{12}(z) = \frac{1}{\hbar}\, 1_N \otimes 1_N + r_{12}(z) + \hbar m_{12}(z) + O(\hbar^2) \tag{2.9}$$

Namely, for

$$m_{12}(z) = \sum_{i,j,k,l=1}^{N} m_{ijkl}(z)\; e_{ij} \otimes e_{kl} \tag{2.10}$$

the components of J are

$$J_{ijkl} = m_{ijkl}(0), \tag{2.11}$$

that is

$$J(S) = \mathrm{Tr}_2(m_{12}(0)S_2), \qquad S_2 = 1_N \otimes S. \tag{2.12}$$

The Hamiltonian of the model is of the form:

$$H^{\mathrm{top}}(S) = \frac{1}{2}\mathrm{Tr}(SJ(S)) = \frac{1}{2}\mathrm{Tr}_{12}(m_{12}(0)S_1S_2), \qquad S_1 = S \otimes 1_N. \tag{2.13}$$

This expression enters (2.1). The phase space of the model is a coadjoint orbit

$$\mathcal{M}^{\mathrm{top}} = \mathcal{O}_N \tag{2.14}$$

of GL_N Lie group, i.e. the space spanned by S_{ij} with some fixed eigenvalues of matrix S (or the Casimir functions $C_k = \mathrm{Tr}S^k$). Its dimension depends on the eigenvalues. The minimal orbit $\mathcal{O}_N^{\mathrm{min}}$ corresponds to $N-1$ coincident eigenvalues, i.e the matrix S (up to a matrix proportional to identity matrix) is of rank one:

$$\dim \mathcal{O}_N^{\mathrm{min}} = 2(N-1). \tag{2.15}$$

The Lax pair is given in the Appendix.

Spin Calogero-Moser model. In the case $N=1$ the second term in (2.1) is trivial, and the last one boils down to the spin Calogero-Moser model [29, 85, 9, 10, 38]:

$$H^{\rm spin}=\sum_{i=1}^M\frac{p_i^2}{2}-\sum_{\substack{i,j\\ i>j}}^M S_{ij}S_{ji}E_2(q_i-q_j), \tag{2.16}$$

where $E_2(q)$ is the second Eisenstein function (A.4). Some details of the spin Calogero-Moser model are given in the Appendix. Let us only remark here that the model (2.16) is integrable through the Lax representation and the classical r-matrix structure on the constraints

$$S_{ii}=\nu\quad\forall i=1,\dots,M \tag{2.17}$$

supplemented by some gauge fixation conditions generated by the coadjoint action of the Cartan subgroup $\mathfrak{H}_M\subset GL_M$. That is the phase space of the model is given by

$$\mathcal{M}^{\rm spin}=T^*\mathfrak{h}_M\times\mathcal{O}_M//\mathfrak{H}_M, \tag{2.18}$$

where $\mathfrak{h}_M={\rm Lie}(\mathfrak{H}_M)$ is the Lie algebra of $\mathfrak{H}_M$, and $\mathcal{O}_M$ is an orbit of the coadjoint action of GL_M. The first factor in (2.18) describes the many-body degrees of freedom (2.7), and the second factor describes the "classical spin" variables. In the general case the spin variables can be parameterized by the set of canonically conjugated variables:

$$S_{ij}=\sum_{a=1}^N\xi_a^i\eta_a^j, \tag{2.19}$$

$$\{\xi_a^i,\eta_b^j\}=\delta_{ab}\delta_{ij},\quad i,j=1,\dots,M,\quad a,b=1,\dots,N. \tag{2.20}$$

The Poisson structure (2.148) is reproduced in this way. Using these notations it is easy to see that

$$S_{ij}S_{ji}=\sum_{a,b=1}^N\xi_a^i\eta_a^j\xi_b^j\eta_b^i={\rm Tr}(\mathcal{S}^{ii}\mathcal{S}^{jj}), \tag{2.21}$$

and the potential in the Hamiltonian (2.16) takes the form

$$\mathcal{V}^{\rm spin}(\mathcal{S}^{ii},\mathcal{S}^{jj},q_{ij})=-{\rm Tr}(\mathcal{S}^{ii}\mathcal{S}^{jj})E_2(q_i-q_j). \tag{2.22}$$

Below we construct anisotropic (in ${\rm Mat}(N,\mathbb{C})$ space) generalizations of (2.22).

In the special case, when the matrix of spin variables S is of rank 1 (it is the minimal $\mathcal{O}_M^{\rm min}$ orbit (2.15))

$$S_{ij}=\xi_i\eta_j \tag{2.23}$$

the reduction with respect to the action of $\mathfrak{H}_M$ leads to the spinless Calogero-Moser (CM) model [13, 14, 81, 82, 57, 37] since the second factor in (2.18) become trivial. Indeed, plugging (2.23) into (2.16) and using (2.17) we get

$$H^{\text{spin}} = \sum_{i=1}^{M} \frac{p_i^2}{2} - \nu^2 \sum_{\substack{i,j \\ i>j}}^{M} E_2(q_i - q_j). \tag{2.24}$$

The spinless Calogero-Moser models are gauge equivalent to the special top with the minimal orbit (2.15). See [40, 36, 1] for details.

Interacting tops. Turning back to the $\mathfrak{gl}_{NM}$ model (2.1) consider the special case when the matrix $\mathcal{S}$ is of rank 1:

$$\mathcal{S}^{ij}_{ab} = \xi^i_a \eta^j_b. \tag{2.25}$$

We will see that in this case the last term in (2.1) is rewritten in the form

$$\mathcal{U}(\mathcal{S}^{ij}, \mathcal{S}^{ji}, q_i - q_j) = \mathcal{V}(\mathcal{S}^{ii}, \mathcal{S}^{jj}, q_i - q_j), \tag{2.26}$$

and the Hamiltonian (2.1) acquires the form

$$\mathcal{H}^{\text{tops}} = \sum_{i=1}^{M} \frac{p_i^2}{2} + \sum_{i=1}^{M} H^{\text{top}}(\mathcal{S}^{ii}) + \frac{1}{2} \sum_{\substack{i,j \\ i\neq j}}^{M} \mathcal{V}(\mathcal{S}^{ii}, \mathcal{S}^{jj}, q_i - q_j). \tag{2.27}$$

It describes mechanics of M interacting integrable $\mathfrak{gl}_N$ tops. The Hamiltonian of (2.27) type was introduced by A.P. Polychronakos [65, 64, 66] from his study of matrix models. Then the elliptic version of model (2.1) and (2.27) was described as $\mathfrak{gl}_{NM}$ Hitchin system [90, 44, 92] (see some details in Section 2.4), and (2.1) was also generalized for arbitrary complex Lie group [41, 42].

Similarly to the spin Calogero–Moser model the general model (2.1) requires additional constraints (cf. (2.17))

$$\mathrm{Tr}(\mathcal{S}^{ii}) = \nu \quad \forall i = 1, \ldots, M. \tag{2.28}$$

They should be supplied with some gauge fixation conditions generated by the coadjoint action of $\mathfrak{H}'_{NM} \subset \mathfrak{H}_{NM}$ — subgroup of the Cartan subgroup $\mathfrak{H}_{NM} \subset GL_{NM}$ with elements of the form $\sum_{i=1}^{M} h_i E_{ii} \otimes 1_N$. Together with (2.28) the gauge fixation conditions are the second class constraints, and one can perform the Dirac reduction procedure to compute the final Poisson structure starting from the linear one (2.5). The phase space of the general model (2.1) is of the from:

$$\mathcal{M}^{\text{gen}} = T^*\mathfrak{h}'_{NM} \times \mathcal{O}_{NM}//\mathfrak{H}'_{NM}, \qquad \mathfrak{h}'_{NM} = \mathrm{Lie}(\mathfrak{H}'_{NM}). \tag{2.29}$$

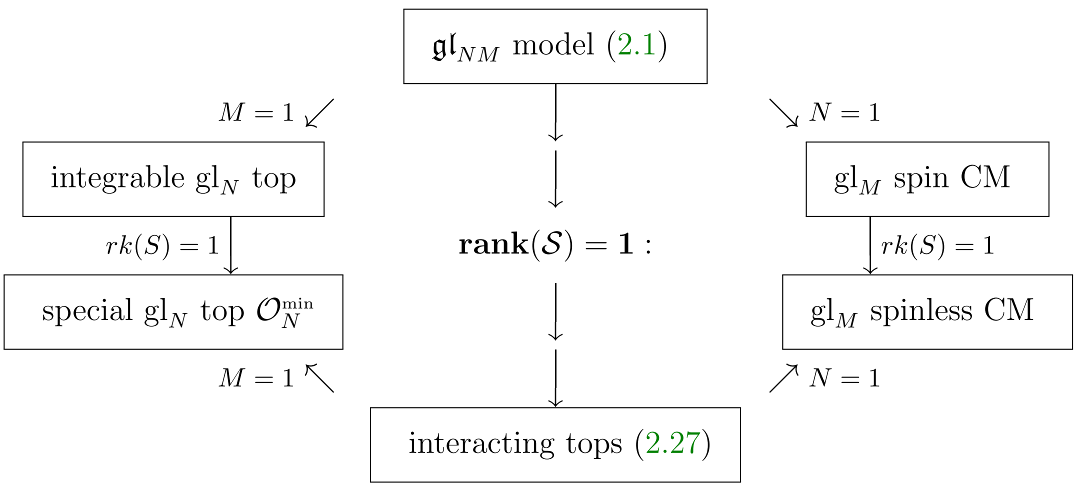
For the interacting tops case (2.25)–(2.27) the orbit $\mathcal{O}_{NM}$ becomes $\mathcal{O}^{\text{min}}_{NM}$. Then the phase space

$$\mathcal{M}^{\text{tops}} = T^*\mathfrak{h}'_{NM} \times \mathcal{O}^{\text{min}}_{NM} // \mathfrak{H}'_{NM} \tag{2.30}$$

has dimension $2NM$, while its "spin part" is of dimension

$$\dim \Big(\mathcal{O}^{\text{min}}_{NM} // \mathfrak{H}'_{NM} \Big) = 2NM - 2M. \tag{2.31}$$

A brief summary of the described models is given in the following scheme:



(2.32)

Purpose of this chapter is to describe a family of the models (2.1) and (2.27) in terms of R-matrices satisfying the associative Yang–Baxter equation (2.8). We give explicit formulae for $NM \times NM$ Lax pair with spectral parameter (see the next Section) and compute the Hamiltonians (2.1) and (2.27). As a result we obtain the potentials

$$\mathcal{U}(\mathcal{S}^{ij}, \mathcal{S}^{ji}, q_i - q_j) = \mathrm{Tr}_{12}\Big(\partial_{q_i} r_{21}(q_{ij}) P_{12} \mathcal{S}^{ij}_1 \mathcal{S}^{ji}_2 \Big) \tag{2.33}$$

for the general model (2.1) and

$$\mathcal{V}(\mathcal{S}^{ii}, \mathcal{S}^{jj}, q_i - q_j) = \mathrm{Tr}_{12}\Big(\partial_{q_i} r_{12}(q_{ij}) \mathcal{S}^{ii}_1 \mathcal{S}^{jj}_2 \Big) \tag{2.34}$$

for the model of interacting tops (2.27). Notice that in the simplest case related to the rational Yang's XXX R-matrix

$$R^z_{12}(q_{ij}) = \frac{1_N \otimes 1_N}{z} + \frac{P_{12}}{q_i - q_j} \tag{2.35}$$

we get just the spin Calogero-Moser model written in terms of matrix variables:

$$\mathcal{V} = -\frac{\mathrm{Tr}(\mathcal{S}^{ii}\mathcal{S}^{jj})}{(q_i - q_j)^2}. \tag{2.36}$$

Next, we proceed to the classical (dynamical) r-matrix. It is similar to the one for the spin Calogero-Moser case [9, 38] but this time its matrix elements are R-matrices themselves. The classical exchange relations are verified directly. This guarantees the Poisson commutativity of the Hamiltonians generated by the Lax matrix.

The answers (2.33) and (2.34) depend on the classical r-matrix, which appears from the quantum one in the limit (2.9). The quantum R-matrix enters the higher Hamiltonians. It should satisfy a set of properties which we discuss in the next Section. The most general R-matrix satisfying all the required properties is the elliptic Baxter–Belavin's one. In this case the the integrable models are known. They were first described by Polychronakos in [65, 64, 66] and later reproduced as Hitchin type systems on the bundles with nontrivial characteristic classes in [90, 44].

The family of the obtained models includes new integrable systems in the trigonometric and rational cases. While the quantization of the potential $\mathcal{V}$ from (2.36) is given by isotropic spin exchange operator $\hat{\mathcal{V}} = -P_{ij}/(q_i - q_j)^2$, the obtained general answer (2.33)–(2.34) leads to the anisotropic potentials. An example of such anisotropic extension to the spin (trigonometric) Calogero–Moser–Sutherland model was first suggested by Hikami and Wadati [33] at quantum level. From the point of view of (2.34) their answer corresponds to the $\mathfrak{gl}_2$ XXZ r-matrix. At the same time the set of trigonometric R-matrices satisfying the required properties is much lager [2, 72, 62], and all these R-matrices can be used for construction of the integrable tops [36]. The results of this chapter are also valid for all these cases. An example based on the $\mathfrak{gl}_2$ 7-th vertex deformation of the XXZ R-matrix is given Section 4. Similarly, in the rational case the admissible R-matrices includes not only the Yang's R-matrix (2.35) but also its deformations such as 11-vertex R-matrix [16] and its higher rank versions [45, 47]. An example related to 11-vertex R-matrix is given in Section 4.

Possible applications of the described models are discussed in the end. Namely, we argue that the obtained models can be used for construction of higher Hamiltonians for the anisotropic generalizations of the Haldane–Shastry–Inozemtsev long-range spin chains. The latter is important for the proof of integrability of these chains, which still remains an open problem.

2.2 Lax equations

In this Section we construct the $NM \times NM$ Lax pair $\mathcal{L}(z), \mathcal{M}(z)$ satisfying the Lax equations

$$\dot{\mathcal{L}}(z) = [\mathcal{L}(z), \mathcal{M}(z)] \tag{2.37}$$

for the model (2.1). Our construction is based on GL_N R-matrix — a solution of the associative Yang–Baxter equation (2.8). Besides (2.8) the R-matrix should also satisfy a set of properties.

2.2.1 R-matrix properties.

We consider R-matrices satisfying (2.8) and (2.9). Let us also impose the following set of conditions for GL_N R-matrices under consideration:
Expansion near $z = 0$:

$$R^{\hbar}_{12}(z) = \frac{1}{z} P_{12} + R^{\hbar,(0)}_{12} + z R^{\hbar,(1)}_{12} + O(z^2), \tag{2.38}$$

Also,

$$R^{z,(0)}_{12} = \frac{1}{z} 1_N \otimes 1_N + r^{(0)}_{12} + O(z), \qquad r_{12}(z) = \frac{1}{z} P_{12} + r^{(0)}_{12} + z r^{(1)}_{12} + O(z^2). \tag{2.39}$$

Skew-symmetry:

$$R^{\hbar}_{12}(z) = -R^{-\hbar}_{21}(-z) = -P_{12} R^{-\hbar}_{12}(-z) P_{12}, \qquad P_{12} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}. \tag{2.40}$$

Unitarity:

$$R^{\hbar}_{12}(z) R^{\hbar}_{21}(-z) = f^{\hbar}(z)\ 1_N \otimes 1_N, \qquad f^{\hbar}(z) = \wp(\hbar) - \wp(z). \tag{2.41}$$

We are also going to use the Fourier symmetry:

$$R^{\hbar}_{12}(z) P_{12} = R^{z}_{12}(\hbar). \tag{2.42}$$

It is not necessary but convenient property.

The following relations on the coefficients of expansions (2.9) and (2.38) follow from the skew-symmetry:

$$\begin{gathered} r_{12}(z) = -r_{21}(-z), \qquad m_{12}(z) = m_{21}(-z), \\ R^{\hbar,(0)}_{12} = -R^{-\hbar,(0)}_{21}, \qquad r^{(0)}_{12} = -r^{(0)}_{21}. \end{gathered} \tag{2.43}$$

Similarly, from the Fourier symmetry we have (see details in [87]):

$$R^{z,(0)}_{12} = r_{12}(z)P_{12}, R^{z,(1)}_{12} = m_{12}(z)P_{12}, r^{(0)}_{12} = r^{(0)}_{12}P_{12}.$$

In what follows we use special notation for the R-matrix derivative:

$$F^z_{12}(q) = \partial_q R^z_{12}(q). \tag{2.44}$$

It is the R-matrix analogue of the function (A.5) entering the M-matrix of the spin Calogero-Moser model (2.144) likewise R-matrix itself is a matrix analogue of the Kronecker function (A.1) due to similarity of (A.6) and (2.8). See [46, 48]. Then from the classical limit (2.9) we have

$$F^0_{12}(q) = \partial_q R^z_{12}(q) \mid_{z=0} = \partial_q r_{12}(q). \tag{2.45}$$

The latter is the R-matrix analogue of the function $-E_2(q)$ (A.12) entering the Calogero-Moser potential. Notice also that $F^0_{12}(q) = F^0_{21}(-q)$ due to (2.43). From (2.44) and (2.38) the local expansion near $q = 0$ is as follows

$$F^z_{12}(q) = -\frac{1}{q^2}\, P_{12} + R^{z,(1)}_{12} + O(q) \tag{2.46}$$

and, therefore,

$$F^0_{12} = -\frac{1}{q^2}P_{12} + R^{z,(1)}_{12} \mid_{z=0} + O(q) \overset{(2.44)}{=} -\frac{1}{q^2}\, P_{12} + m_{12}(0)P_{12} + O(q). \tag{2.47}$$

On the other hand,

$$F^0_{12}(q) \overset{(2.45)}{=} \partial_q r_{12}(q) \overset{(2.39)}{=} -\frac{1}{q^2}\, P_{12} + r^{(1)}_{12} + O(q). \tag{2.48}$$

From (2.47) and (2.48) we conclude that

$$r^{(1)}_{12} = m_{12}(0)P_{12}. \tag{2.49}$$

In the elliptic case the set of properties is fulfilled by the Baxter–Belavin [6, 7, 70] R-matrix (2.166). A family of trigonometric R-matrices include the XXZ 6-vertex one, its 7-vertex deformation [16] and GL_N generalizations [2, 72, 62]. See a brief review and applications to integrable tops in [36]. The rational R-matrices possessing the properties are the XXX Yang's R-matrix, its 11-vertex deformation [16] and higher rank analogues obtained from the elliptic case by special limiting procedure [80]. The final answer for such R-matrix was obtained in [45, 47] through the gauge equivalence between the relativistic top with minimal orbit and the rational Ruijsenaars–Schneider model.

2.2.2 Lax pair and equations of motion

Using coefficients of the expansion of the GL_N R-matrix near $z=0$ we define $NM\times NM$ Lax pair

$$L(z)=\sum_{i,j=1}^{M} E_{ij}\otimes L^{ij}(z) \qquad L^{ij}(z)\in \mathrm{Mat}_N \qquad L(z)\in \mathrm{Mat}_{NM}, \tag{2.50}$$

$$L^{ij}(z)=\delta_{ij}\Big(p_i 1_N+\mathrm{Tr}_2(\mathcal{S}_2^{ii}R_{12}^{z,(0)}P_{12})\Big)+(1-\delta_{ij})\ \mathrm{Tr}_2(\mathcal{S}_2^{ij}R_{12}^{z}(q_{ij})P_{12}). \tag{2.51}$$

and similarly for $M^{ij}(z)\in \mathrm{Mat}_N$

$$M^{ij}(z)=\delta_{ij}\mathrm{Tr}_2(\mathcal{S}_2^{ii}R_{12}^{z,(1)}P_{12})+(1-\delta_{ij})\ \mathrm{Tr}_2(\mathcal{S}_2^{ij}F_{12}^{z}(q_{ij})P_{12}). \tag{2.52}$$

where the entries are defined from (2.38) and (2.44). The tensor notations are similar to those used in (2.160)–(2.163).

Theorem.
Consider an R-matrix satisfying the associative Yang–Baxter equation (2.8), the classical limit (2.9) and the set of properties from the previous paragraph. Then the Lax equation (2.37) holds true for the Lax pair (2.50)–(2.52) on the constraints

$$\mathrm{Tr}\Big(\mathcal{S}^{ii}\Big)=\mathrm{const},\quad \forall i \tag{2.53}$$

(cf. (2.113)) and provides the following equations of motion for off-diagonal $N\times N$ blocks of $\mathcal{S}$:

$$\begin{aligned}\dot{\mathcal{S}}^{ij}=\sum_{k\neq i,j}^{M}\Big(&\mathcal{S}^{ik}\mathrm{Tr}_2(\mathcal{S}_2^{kj}F_{12}^{0}(q_{kj})P_{12})-\mathrm{Tr}_2(\mathcal{S}_2^{ik}F_{12}^{0}(q_{ik})P_{12})\mathcal{S}^{kj}\Big)+\\ &\mathcal{S}^{ii}\mathrm{Tr}_2(\mathcal{S}_2^{ij}F_{12}^{0}(q_{ij})P_{12})-\mathrm{Tr}_2(\mathcal{S}_2^{ii}m_{12}(0))\mathcal{S}^{ij}-\\ &-\mathrm{Tr}_2(\mathcal{S}_2^{ij}F_{12}^{0}(q_{ij})P_{12})\mathcal{S}^{jj}+\mathcal{S}^{ij}\mathrm{Tr}_2(\mathcal{S}_2^{jj}m_{12}(0)),\end{aligned} \tag{2.54}$$

for diagonal $N\times N$ blocks of $\mathcal{S}$:

$$\dot{\mathcal{S}}^{ii}=[\mathcal{S}^{ii},\mathrm{Tr}_2(m_{12}(0)\mathcal{S}_2^{ii})]+\sum_{k\neq i}^{M}\Big(\mathcal{S}^{ik}\mathrm{Tr}_2(\mathcal{S}_2^{ki}F_{21}^{0}(q_{ik})P_{12})-\mathrm{Tr}_2(\mathcal{S}_2^{ik}F_{12}^{0}(q_{ik})P_{12})\mathcal{S}^{ki}\Big), \tag{2.55}$$

and for momenta:

$$\dot{p}_i=-\sum_{k\neq i}^{M}\mathrm{Tr}_{23}\Big(\partial_{q_i}F_{32}^{0}(q_{ik})P_{23}\mathcal{S}_2^{ik}\mathcal{S}_3^{ki}\Big). \tag{2.56}$$

Proof: We imply $p_i = \dot{q}_i$ in the formulae above. This follows from the Hamiltonian description, which is given in the next paragraph.

1. Let us begin with the non-diagonal blocks. Consider the one numbered ij ($i \neq j$). The l.h.s. of the Lax equations reads

$$\text{l.h.s.} = \dot{\mathcal{L}}^{ij}(z) = \mathrm{Tr}_2(\dot{\mathcal{S}}_2^{ij} R_{12}^z(q_{ij}) P_{12}) + \mathrm{Tr}_2(\mathcal{S}_2^{ij} F_{12}^z(q_{ij}) P_{12})(\dot{q}_i - \dot{q}_j). \tag{2.57}$$

The r.h.s. of the Lax equation is as follows:

$$\text{r.h.s.} = \mathcal{L}^{ij}\mathcal{M}^{jj} - \mathcal{M}^{ii}\mathcal{L}^{ij} + \mathcal{L}^{ii}\mathcal{M}^{ij} - \mathcal{M}^{ij}\mathcal{L}^{jj} + \sum_{k \neq i,j}^{M} \Big(\mathcal{L}^{ik}\mathcal{M}^{kj} - \mathcal{M}^{ik}\mathcal{L}^{kj} \Big). \tag{2.58}$$

The last sum is computed using identity

$$R_{12}^z(x) F_{23}^z(y) - F_{12}^z(x) R_{23}^z(y) = F_{23}^0(y) R_{13}^z(x+y) - R_{13}^z(x+y) F_{12}^0(x), \tag{2.59}$$

which follows from (2.8). It is the R-matrix analogue of (A.7). In its turn (A.7) is the key tool underlying ansatz for the Lax pairs with spectral parameter [37]. For $k \neq i, j$ we have

$$\begin{gathered}
\mathcal{L}^{ik}\mathcal{M}^{kj} - \mathcal{M}^{ik}\mathcal{L}^{kj} = \\
= \mathrm{Tr}_{23}(R_{12}^z(q_{ik}) P_{12} \mathcal{S}_2^{ik} F_{13}^z(q_{kj}) P_{13} \mathcal{S}_3^{kj}) - \mathrm{Tr}_{23}(F_{12}^z(q_{ik}) P_{12} \mathcal{S}_2^{ik} R_{13}^z(q_{kj}) P_{13} \mathcal{S}_3^{kj}) = \\
= \mathrm{Tr}_{23}\Big(\Big(R_{12}^z(q_{ik}) F_{23}^z(q_{kj}) - F_{12}^z(q_{ik}) R_{23}^z(q_{kj}) \Big) P_{12} P_{13} \mathcal{S}_2^{ik} \mathcal{S}_3^{kj} \Big) \stackrel{(2.59)}{=} \\
= \mathrm{Tr}_{23}\Big(\Big(F_{23}^0(q_{kj}) R_{13}^z(q_{ij}) - R_{13}^z(q_{ij}) F_{12}^0(q_{ik}) \Big) P_{12} P_{13} \mathcal{S}_2^{ik} \mathcal{S}_3^{kj} \Big) = \\
= \mathrm{Tr}_{23}\Big(R_{12}^z(q_{ij}) P_{12} \Big(\mathcal{S}_2^{ik} \mathcal{S}_3^{kj} F_{23}^0(q_{kj}) P_{23} - F_{23}^0(q_{ik}) P_{23} \mathcal{S}_3^{ik} \mathcal{S}_2^{kj} \Big) \Big).
\end{gathered} \tag{2.60}$$

This expression provides the upper line in the equations of motion (2.54). To proceed we need degenerations of the identity (2.59) when $y \to 0$. It comes from the expansions (2.38), (2.46) and (2.48):

$$R_{12}^z(x) R_{23}^{z,(1)} - F_{12}^z(x) R_{23}^{z,(0)} = r_{23}^{(1)} R_{13}^z(x) - R_{13}^z(x) F_{12}^0(x) - \frac{1}{2} P_{23} \partial_x^2 R_{13}^z(x). \tag{2.61}$$

In the same way in the limit $x \to 0$ (2.59) takes the form

$$R_{12}^{z,(0)} F_{23}^z(y) - R_{12}^{z,(1)} R_{23}^z(y) = F_{23}^0(y) R_{13}^z(y) - R_{13}^z(y) r_{12}^{(1)} + \frac{1}{2} \partial_y^2 R_{13}^z(y) P_{12}. \tag{2.62}$$

Similarly to the ordinary (spin) Calogero-Moser case the terms linear in momenta in the r.h.s. (2.58) $(p_i - p_j)\mathcal{M}^{ij}$ are cancelled out by the last term in the l.h.s. of (2.57). Consider the first and the fourth terms from (2.58) without momenta. Using evaluations similar to (2.60) we get

$$\begin{gathered}\mathcal{L}^{ij}\mathcal{M}^{jj} - \mathcal{M}^{ij}(\mathcal{L}^{jj} - p_j 1_N) = \\ = \mathrm{Tr}_{23}\Big(\Big(R^z_{12}(q_{ij})R^{z,(1)}_{23} - F^z_{12}(q_{ij})R^{z,(0)}_{23}\Big)P_{12}P_{13}\mathcal{S}^{ij}_2\mathcal{S}^{jj}_3\Big) \overset{(2.61)}{=} \\ = \mathrm{Tr}_{23}\Big(\Big(r^{(1)}_{23}R^z_{13}(q_{ij}) - R^z_{13}(q_{ij})F^0_{12}(q_{ij}) - \frac{1}{2}P_{23}\partial^2_{q_i}R^z_{13}(q_{ij})\Big)P_{12}P_{13}\mathcal{S}^{ij}_2\mathcal{S}^{jj}_3\Big) = \\ = \mathrm{Tr}_{23}(R^z_{12}(q_{ij})P_{12}\mathcal{S}^{ij}_2\mathcal{S}^{jj}_3 m_{23}(0)) - \mathrm{Tr}_{23}(R^z_{12}(q_{ij})P_{12}F^0_{23}(q_{ij})P_{23}\mathcal{S}^{ij}_3\mathcal{S}^{jj}_2) - \\ - \frac{1}{2}\mathrm{Tr}_{23}(\partial^2_{q_i}R^z_{12}(q_{ij})P_{12}\mathcal{S}^{ij}_2\mathcal{S}^{jj}_3),\end{gathered} \tag{2.63}$$

where the relation (2.49) was also used (for the first term in the answer). The first and the second terms in the obtained answer provide the last line in the equations of motion (2.54), while the last term in (2.54) is the "unwanted term".

In the same way, using (2.62) one gets

$$\begin{gathered}(\mathcal{L}^{ii} - p_i 1_N)\mathcal{M}^{ij} - \mathcal{M}^{ii}\mathcal{L}^{ij} = \\ = \mathrm{Tr}_{23}(R^z_{12}(q_{ij})P_{12}\mathcal{S}^{ii}_2\mathcal{S}^{ij}_3 F^0_{23}(q_{ij})P_{23}) - \mathrm{Tr}_{23}(R^z_{12}(q_{ij})P_{12}m_{23}(0)\mathcal{S}^{ii}_3\mathcal{S}^{ij}_2) + \\ + \frac{1}{2}\mathrm{Tr}_{23}(\partial^2_{q_i}R^z_{12}(q_{ij})P_{12}\mathcal{S}^{ii}_3\mathcal{S}^{ij}_2).\end{gathered} \tag{2.64}$$

Again, the first two terms provide an input to equations of motion — the second line in (2.54). The last term is the "unwanted term". It is cancelled by the one from (2.63) after taking the trace over the third component and imposing the constraints (2.53).

2. Consider a diagonal $N \times N$ block (numbered ii) of the Lax equation. The l.h.s. of the Lax equations is

$$\text{l.h.s.} = \dot{\mathcal{L}}^{ii}(z) = \dot{p}_i 1_N + \mathrm{Tr}_2(\dot{\mathcal{S}}^{ii}_2 R^{z,(0)}_{12}P_{12}) \overset{(2.44)}{=} \dot{p}_i 1_N + \mathrm{Tr}_2(\dot{\mathcal{S}}^{ii}_2 r_{12}(z)). \tag{2.65}$$

The r.h.s. of the Lax equation is as follows:

$$\text{r.h.s.} = [\mathcal{L}^{ii}, \mathcal{M}^{ii}] + \sum_{k\neq i}^{M}\Big(\mathcal{L}^{ik}\mathcal{M}^{ki} - \mathcal{M}^{ik}\mathcal{L}^{ki}\Big). \tag{2.66}$$

The commutator term in (2.66) provides the commutator term in the equations of motion (2.55) since it is the input from the internal ii-th top's ynamics, and this was derived in [49]. See

(2.159)–(2.161). In order to simplify expression in the sum we need the following degeneration of (2.8):

$$R^z_{12}(x)R^z_{23}(y) = R^z_{13}(x+y)r_{12}(x) + r_{23}(y)R^z_{13}(x+y) - \partial_z R^z_{13}(x+y), \tag{2.67}$$

It corresponds to $\hbar = \eta = z$. In the scalar case it is the identity (A.9). In the limit $x = q = -y$ from (2.67) we get

$$R^z_{12}(q)R^z_{23}(-q) = R^{z,(0)}_{13} r_{12}(q) - r_{32}(q)R^{z,(0)}_{13} - \partial_z R^{z,(0)}_{13} + F^0_{32}(q)P_{13}, \tag{2.68}$$

or, using (2.44)

$$R^z_{12}(q)R^z_{23}(-q) = (r_{13}(z)r_{32}(q) - r_{32}(q)r_{13}(z))P_{13} - F^0_{13}(z)P_{13} + F^0_{32}(q)P_{13}. \tag{2.69}$$

By differentiating (2.69) with respect to q we obtain

$$R^z_{12}(q)F^z_{23}(-q) - F^z_{12}(q)R^z_{23}(-q) = [F^0_{32}(q), r_{13}(z)]P_{13} - \partial_q F^0_{32}(q)P_{13}. \tag{2.70}$$

For $k \neq i$ consider

$$\begin{gathered}
\mathcal{L}^{ik}\mathcal{M}^{ki} - \mathcal{M}^{ik}\mathcal{L}^{ki} = \\
= \mathrm{Tr}_{23}\Big(\Big(R^z_{12}(q_{ik})F^z_{23}(q_{ki}) - F^z_{12}(q_{ik})R^z_{23}(q_{ki})\Big)P_{12}P_{13}\mathcal{S}^{ik}_2\mathcal{S}^{ki}_3\Big) \stackrel{(2.70)}{=} \\
= \mathrm{Tr}_{23}\Big(\Big([F^0_{32}(q_{ik}), r_{13}(z)]P_{13} - \partial_{q_i}F^0_{32}(q_{ik})P_{13}\Big)P_{12}P_{13}\mathcal{S}^{ik}_2\mathcal{S}^{ki}_3\Big).
\end{gathered} \tag{2.71}$$

The commutator term in the obtained expression yields the sum term in the equations of motion (2.55), while the last term in (2.71) provides equations of motion (2.56). Indeed,

$$\mathrm{Tr}_{23}\Big(\Big(\partial_{q_i}F^0_{32}(q_{ik})P_{13}\Big)P_{12}P_{13}\mathcal{S}^{ik}_2\mathcal{S}^{ki}_3\Big) = 1_N \mathrm{Tr}_{23}\Big(\partial_{q_i}F^0_{32}(q_{ik})P_{23}\mathcal{S}^{ik}_2\mathcal{S}^{ki}_3\Big), \tag{2.72}$$

and the momenta is the scalar component in the l.h.s. (2.65).

2.2.3 Hamiltonian description

The Hamiltonian function. Let us compute the Hamiltonian for the model (2.50)–(2.56). It comes from the generating function

$$\frac{1}{2N}\mathrm{Tr}(\mathcal{L}^2(z)) = \frac{1}{2N}\sum_{i=1}^{M}\mathrm{Tr}\Big(\mathcal{L}^{ii}(z)\Big)^2 + \frac{1}{2N}\sum_{\substack{i,j\\ i\neq j}}^{M}\mathrm{Tr}\Big(\mathcal{L}^{ij}(z)\mathcal{L}^{ji}(z)\Big). \tag{2.73}$$

Consider

$$\mathrm{Tr}\Big(\mathcal{L}^{ii}(z)\Big)^2 = Np_i^2 + 2p_i\mathrm{Tr}_{12}\Big(r_{12}(z)\mathcal{S}_2^{ii}\Big) + \mathrm{Tr}_{123}\Big(r_{12}(z)r_{13}(z)\mathcal{S}_2^{ii}\mathcal{S}_3^{ii}\Big). \tag{2.74}$$

As before, the numbered tensor components are $\mathrm{Mat}(N,\mathbb{C})$–valued. In order to simplify (2.74) we use the identity (see [46, 48])

$$r_{12}(z)r_{13}(z+w) - r_{23}(w)r_{12}(z) + r_{13}(z+w)r_{23}(w) = m_{12}(z) + m_{23}(w) + m_{13}(z+w), \tag{2.75}$$

which can be treated as a half of the classical Yang–Baxter equation [1]. In the limit $w \to 0$ (2.75) yields

$$r_{12}(z)r_{13}(z) = r_{23}^{(0)}r_{12}(z) - r_{13}(z)r_{23}^{(0)} - F_{13}^0(z)P_{23} + m_{12}(z) + m_{23}(0) + m_{13}(z). \tag{2.76}$$

Also, we are going to use the following R-matrix property:

$$\mathrm{Tr}_1 R_{12}^q(z) = \mathrm{Tr}_2 R_{12}^q(z) = \tilde{\phi}(z,q)1_N, \tag{2.77}$$

where $\tilde{\phi}(z,q)$ is the Kronecker function (A.1) but with possibly different normalization factor and normalization of arguments. The property (2.77) holds true in the elliptic case (2.167) as well as for its trigonometric and rational degenerations. From (2.77), expansion (2.9) and (A.10) we also have similar properties for $\mathrm{Tr}_1 r_{12}(z) = \tilde{E}_1(z)$ and $\mathrm{Tr}_1 m_{12}(z)$ — they are scalar operators:

$$\mathrm{Tr}_1 R_{12}^q(z) = q^{-1}1_N + \mathrm{Tr}_1 r_{12}(z) + q\mathrm{Tr}_1 m_{12}(z) + O(q^2). \tag{2.78}$$

Return now to (2.74). On the constraints (2.53) the second term is equal to $2p_i\tilde{E}_1(z)$const. After summation over i it provides the Hamiltonian proportional to $\sum_{i=1}^M p_i$. Plugging (2.76) into the last term of (2.74) we get

$$\mathrm{Tr}_{123}\Big(r_{12}(z)r_{13}(z)\mathcal{S}_2^{ii}\mathcal{S}_3^{ii}\Big) =$$
$$= \mathrm{Tr}_{123}\Big(\Big(r_{23}^{(0)}r_{12}(z) - r_{13}(z)r_{23}^{(0)} - F_{13}^0(z)P_{23} + m_{23}(0) + m_{12}(z) + m_{13}(z)\Big)\mathcal{S}_2^{ii}\mathcal{S}_3^{ii}\Big). \tag{2.79}$$

Due to (2.77) the first two terms are cancelled out after taking the trace over the component 1. By the same reason the last two terms in (2.79) provide $2\mathrm{Tr}_1(m_{12}(z))\mathrm{Tr}_{23}(\mathcal{S}_2^{ii}\mathcal{S}_3^{ii})$. These are constants on the constraints (2.53). The rest of the terms are

$$\mathrm{Tr}_{123}\Big(\Big(-F_{13}(z)P_{23} + m_{23}(0)\Big)\mathcal{S}_2^{ii}\mathcal{S}_3^{ii}\Big) \overset{(2.77)}{=} \tilde{E}_2(z)\mathrm{Tr}\Big(\mathcal{S}^{ii}\Big)^2 + N\mathrm{Tr}_{23}\Big(m_{23}(0)\mathcal{S}_2^{ii}\mathcal{S}_3^{ii}\Big), \tag{2.80}$$

[1]The difference of two such equations gives the classical Yang–Baxter equation for the classical r-matrix.

where $\tilde{E}_2(z)1_N = -\mathrm{Tr}_1(F^0_{13}(z)) = -\partial_z \mathrm{Tr}_1(r_{13}(z)) = -\partial_z \tilde{E}_1(z)1_N$. It is a scalar function coming from (2.78) and similar to $E_2(z)$ (A.4). The factor N in the last term comes from Tr_1. The first term in (2.80) is a part of the Casimir function $\mathrm{Tr}\mathcal{S}^2$, and the second one is $H^{\mathrm{top}}(\mathcal{S}^{ii})$ from (2.1):

$$H^{\mathrm{top}}(\mathcal{S}^{ii}) = \frac{1}{2}\mathrm{Tr}_{12}\Big(m_{12}(0)\mathcal{S}^{ii}_1\mathcal{S}^{ii}_2\Big). \tag{2.81}$$

Next, consider

$$\begin{aligned}\mathrm{Tr}\Big(\mathcal{L}^{ij}(z)\mathcal{L}^{ji}(z)\Big) &= \mathrm{Tr}_{123}\Big(R^z_{12}(q_{ij})P_{12}R^z_{13}(q_{ji})P_{13}\mathcal{S}^{ij}_2\mathcal{S}^{ji}_3\Big) = \\ &= \mathrm{Tr}_{123}\Big(R^z_{12}(q_{ij})R^z_{23}(q_{ji})P_{12}P_{13}\mathcal{S}^{ij}_2\mathcal{S}^{ji}_3\Big) \overset{(2.69)}{=} \\ &= \mathrm{Tr}_{123}\Big(\Big([r_{13}(z), r_{32}(q_{ij})] - F^0_{13}(z) + F^0_{32}(q_{ij})\Big)P_{23}\mathcal{S}^{ij}_2\mathcal{S}^{ji}_3\Big).\end{aligned} \tag{2.82}$$

Again, the commutator term vanishes after taking the trace over the first tensor component. Therefore,

$$\begin{aligned}\mathrm{Tr}\Big(\mathcal{L}^{ij}(z)\mathcal{L}^{ji}(z)\Big) &= \mathrm{Tr}_{123}\Big(\Big(-F^0_{13}(z) + F^0_{32}(q_{ij})\Big)P_{23}\mathcal{S}^{ij}_2\mathcal{S}^{ji}_3\Big) = \\ &= \tilde{E}_2(z)\mathrm{Tr}\Big(\mathcal{S}^{ij}\mathcal{S}^{ji}\Big) + N\mathrm{Tr}_{12}\Big(F^0_{21}(q_{ij})P_{12}\mathcal{S}^{ij}_1\mathcal{S}^{ji}_2\Big).\end{aligned} \tag{2.83}$$

Finally, for the potential term from (2.1) we have

$$\mathcal{U}(\mathcal{S}^{ij}, \mathcal{S}^{ji}, q_{ij}) = \mathrm{Tr}_{12}\Big(F^0_{21}(q_{ij})P_{12}\mathcal{S}^{ij}_1\mathcal{S}^{ji}_2\Big) \tag{2.84}$$

and the Hamiltonian (2.1) is of the form:

$$\mathcal{H} = \sum_{i=1}^{M}\frac{p_i^2}{2} + \frac{1}{2}\sum_{i=1}^{M}\mathrm{Tr}_{12}\Big(m_{12}(0)\mathcal{S}^{ii}_1\mathcal{S}^{ii}_2\Big) + \sum_{\substack{i,j\\ i>j}}^{M}\mathrm{Tr}_{12}\Big(F^0_{21}(q_{ij})P_{12}\mathcal{S}^{ij}_1\mathcal{S}^{ji}_2\Big). \tag{2.85}$$

In $M = 1$ case $\mathcal{H}$ reproduce the Hamiltonian of the integrable top, while in the $M = 1$ case we obtain the spin Calogero–Moser Hamiltonian (2.16) up to terms containing S_{ii} — they are constant in this case (2.17).

Poisson brackets. The Poisson structure (before reduction (2.29)) consists of the canonical brackets for positions and momenta

$$\{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0, \quad i = 1, \ldots, M. \tag{2.86}$$

and the linear Poisson–Lie brackets for the $\mathcal{S}$ variables. They are of the form (2.148) but for $\mathrm{Mat}(NM,\mathbb{C})$ case instead of $\mathrm{Mat}(M,\mathbb{C})$ in (2.148). It is convenient to write down these brackets in terms of $\mathrm{Mat}(N,\mathbb{C})$–valued blocks $\mathcal{S}^{ij}$. For $i,j,k,l=1,\dots,M$ and $a,b,c,d=1,\dots,N$:

$$\{\mathcal{S}^{ij}_{ab},\mathcal{S}^{kl}_{cd}\}=\mathcal{S}^{kj}_{cb}\delta^{il}\delta_{ad}-\mathcal{S}^{il}_{ad}\delta^{kj}\delta_{bc} \tag{2.87}$$

or

$$\{\mathcal{S}^{ij}_1,\mathcal{S}^{kl}_2\}=P_{12}\mathcal{S}^{kj}_1\delta^{il}-\mathcal{S}^{il}_1P_{12}\delta^{kj}, \tag{2.88}$$

where P_{12} as before the permutation operator in $\mathrm{Mat}(N,\mathbb{C})^{\otimes 2}$. For the diagonal blocks we have

$$\{\mathcal{S}^{ii}_1,\mathcal{S}^{kk}_2\}=[P_{12},\mathcal{S}^{ii}_1]\delta^{ik}. \tag{2.89}$$

It is verified directly that the Poisson structure (2.86), (2.88) and the Hamiltonian (2.85) provides equations of motion (2.54)–(2.56), i.e. for the l.h.s. of the Lax equation (2.37) we have

$$\dot{\mathcal{L}}(z)=\{\mathcal{H},\mathcal{L}(z)\}. \tag{2.90}$$

2.2.4 Interacting tops

Suppose the matrix $\mathcal{S}$ is of rank one, i.e. (2.25) is fulfilled. Consider the potential

$$\mathrm{Tr}_{12}\Big(F^0_{21}(q_{ij})P_{12}\mathcal{S}^{ij}_1\mathcal{S}^{ji}_2\Big)=\sum_{a,b,c,d=1}^{N}(F^0_{12}(q_{ji})P_{12})_{ab,cd}\mathcal{S}^{ij}_{ba}\mathcal{S}^{ji}_{dc}. \tag{2.91}$$

The right multiplication of an element $T_{12}=\sum\limits_{i,j,k,l=1}^{N}T_{ijkl}E_{ij}\otimes E_{kl}\in\mathrm{Mat}(N,\mathbb{C})^{\otimes 2}$ by permutation operator P_{12} yields $T_{ijkl}\to T_{ilkj}$, i.e.

$$\mathrm{Tr}_{23}\Big(F^0_{32}(q_{ij})P_{23}\mathcal{S}^{ij}_2\mathcal{S}^{ji}_3\Big)=\sum_{a,b,c,d=1}^{N}(F^0_{12}(q_{ji}))_{ad,cb}\mathcal{S}^{ij}_{ba}\mathcal{S}^{ji}_{dc}. \tag{2.92}$$

In the rank one case we have

$$\mathcal{S}^{ij}_{ba}\mathcal{S}^{ji}_{dc}=\xi^i_b\eta^j_a\xi^j_d\eta^i_c=\mathcal{S}^{ii}_{bc}\mathcal{S}^{jj}_{da}. \tag{2.93}$$

Therefore,

$$\mathrm{Tr}_{23}\Big(F^0_{32}(q_{ij})P_{23}\mathcal{S}^{ij}_2\mathcal{S}^{ji}_3\Big)=\mathrm{Tr}_{12}\Big(F^0_{12}(q_{ji})\mathcal{S}^{jj}_1\mathcal{S}^{ii}_2\Big)=\mathrm{Tr}_{12}\Big(F^0_{12}(q_{ij})\mathcal{S}^{ii}_1\mathcal{S}^{jj}_2\Big). \tag{2.94}$$

The Hamiltonian of interacting tops model acquires the form:

$$\mathcal{H}^{\rm tops} = \sum_{i=1}^{M} \frac{p_i^2}{2} + \frac{1}{2}\sum_{i=1}^{M} {\rm Tr}_{12}\Big(m_{12}(0)\mathcal{S}_1^{ii}\mathcal{S}_2^{ii}\Big) + \sum_{\substack{i,j\\ i<j}}^{M} {\rm Tr}_{12}\Big(F_{12}^0(q_{ij})\mathcal{S}_1^{ii}\mathcal{S}_2^{jj}\Big). \tag{2.95}$$

From the Poisson brackets (2.86), (2.89) we get the corresponding equations of motion:

$$\dot{\mathcal{S}}^{ii} = [\mathcal{S}^{ii}, {\rm Tr}_2(m_{12}(0)\mathcal{S}_2^{ii})] + \sum_{k\neq i}^{M}[\mathcal{S}^{ii}, {\rm Tr}_2(F_{12}^0(q_{ik})\mathcal{S}_2^{kk})], \tag{2.96}$$

$$\dot{p}_i = -\sum_{k\neq i}^{M} {\rm Tr}_{12}\Big(\partial_{q_i}F_{12}^0(q_{ik})\mathcal{S}_1^{ii}\mathcal{S}_2^{kk}\Big). \tag{2.97}$$

In this model we are left with M matrix variables $\mathcal{S}^{ii} \in {\rm Mat}(N,\mathbb{C})$ of rank one. It is notable that the spin part of the phase space (2.30) is isomorphic to a product of M minimal coadjoint orbits (2.15):

$$\mathcal{O}_{NM}^{\rm min}//\mathfrak{H}'_{NM} \cong \underbrace{O_N^{\rm min} \times \ldots \times O_N^{\rm min}}_{M {\rm times}}. \tag{2.98}$$

Notice that the orbits $O_N^{\rm min}$ come from the constraints conditions (2.53). Hence it appears that

1. For the model of interacting tops the constraints (2.53) play the role of fixation of the Casimir functions for M copies of $\mathfrak{gl}_N^*$ (of rank one). Consequently, equations of motion (2.96) are not changed after reduction. For the $N = 1$ case (the spin Calogero-Moser model) we get $\dot{S}_{ii} = 0$ since the r.h.s. of (2.96) consists of commutators.

2. The model of interacting tops is formulated in terms of M ${\rm Mat}(N,\mathbb{C})$–valued variables of rank one, describing the minimal coadjoint orbits. The integrability condition is that all Casimir functions ${\rm Tr}(\mathcal{S}^{ii})$ are equal to each other [2].

3. The spin part of the phase space for the model of interacting tops coincides with the phase space of GL_N classical spin chain on M sites with the spins described by minimal coadjoint orbits at each site.

Let us also remark that the top like models with matrix–valued variables were studied in [49, 87] and [8]. In contrast to these papers here we deal with the models, where the matrix variables have their own internal dynamics.

[2]More precisely, we can not confirm that the model is not integrable in the case ${\rm Tr}(\mathcal{S}^{ii}) \neq {\rm Tr}(\mathcal{S}^{jj})$, but the presented Lax pair does not work in this case.

2.3 Classical r-matrix

In this Section we describe the classical r-matrix structure for the Lax matrix (2.51). Since $\mathcal{L} \in \mathrm{Mat}(NM, \mathbb{C})$ then the corresponding classical $\mathfrak{gl}_{NM}$ r-matrix $\mathbf{r} \in \mathrm{Mat}(NM, \mathbb{C})^{\otimes 2}$. Recall that for the Lax matrix we use the matrix basis (2.50), in which $\mathcal{L} \in \mathrm{Mat}(M, \mathbb{C}) \otimes \mathrm{Mat}(N, \mathbb{C})$. Let the $\mathrm{Mat}(M, \mathbb{C})$–valued tensor components be numbered by primed numbers, and the $\mathrm{Mat}(N, \mathbb{C})$–valued components — without primes (as before). Introduce the following r-matrix:

$$\mathbf{r}_{1'2'12}(z, w) = \sum_{i=1}^{M} \overset{1'}{E}_{ii} \otimes \overset{2'}{E}_{ii} \otimes r_{12}(z-w) + \sum_{\substack{i,j \\ i \neq j}}^{M} \overset{1'}{E}_{ij} \otimes \overset{2'}{E}_{ji} \otimes R_{12}^{z-w}(q_{ij}) P_{12}, \tag{2.99}$$

so that $\mathbf{r}_{1'2'12} \in \mathrm{Mat}(M, \mathbb{C})^{\otimes 2} \otimes \mathrm{Mat}(N, \mathbb{C})^{\otimes 2}$. In the case $M = 1$ we come to a non-dynamical r-matrix describing the top model, while in the $N = 1$ we reproduce the dynamical r matrix of the spin Calogero-Moser model (2.150). r-matrices of these type are known in $\mathfrak{gl}_{NM}$ case and can be extended for arbitrary complex semisimple Lie algebras [19, 20, 25, 43]. In the elliptic case (2.99) is known in the quantum case as well [44]. At the same time (2.99) includes the cases, which have not been described yet. For instance, the new cases correspond to the rational $R_{12}^{z}(q)$-matrix from [45, 47]. Similarly to the Lax equations the construction of the r-matrix (2.99) is based on the associative Yang–Baxter equation (2.8) and its degenerations.

Proposition.

Consider an R-matrix satisfying the associative Yang–Baxter equation (2.8), the classical limit (2.9) and the set of properties from the Section 2.2.1. Then for the Lax pair (2.50)–(2.51) the following classical exchange relation holds true:

$$\begin{aligned} \{\mathcal{L}_{1'1}(z), \mathcal{L}_{2'2}(w)\} = [\mathcal{L}_{1'1}(z), \mathbf{r}_{1'2'12}(z, w)] - [\mathcal{L}_{2'2}(w), \mathbf{r}_{2'1'21}(w, z)] - \\ - \sum_{k=1}^{M} \mathrm{Tr}(\mathcal{S}^{kk}) \partial_{q_k} \mathbf{r}_{1'2'12}(z, w), \end{aligned} \tag{2.100}$$

where

$$\mathcal{L}_{1'1}(z) = \sum_{i,j=1}^{M} E_{ij} \otimes 1_M \otimes \mathcal{L}^{ij}(z) \otimes 1_N, \tag{2.101}$$

$$\mathcal{L}_{2'2}(w) = \sum_{k,l=1}^{M} 1_M \otimes E_{kl} \otimes 1_N \otimes \mathcal{L}^{kl}(w). \tag{2.102}$$

The Poisson brackets in the l.h.s. of (2.100) are given by (2.86)–(2.89).

Proof:

The proof is direct. Let us demonstrate how to verify (2.100) for several components of $\overset{1'}{E}_{ij} \otimes \overset{2'}{E}_{kl}$, which are similar to those considered in (2.152)–(2.157) for the spin Calogero–Moser model.

the tensor component $\overset{1'}{E}_{ij} \otimes \overset{2'}{E}_{jk}$ $(i \neq j,\ j \neq k,\ i \neq k)$:

l.h.s. of (2.100):

$$\mathrm{Tr}_{34}\Big(R^z_{13}(q_{ij})P_{13}R^w_{24}(q_{jk})P_{24}\{\mathcal{S}^{ij}_3,\mathcal{S}^{jk}_4\}\Big) = -\mathrm{Tr}_3\Big(\mathcal{S}^{ik}_3 R^w_{23}(q_{jk})P_{23}R^z_{13}(q_{ij})P_{13}\Big). \tag{2.103}$$

r.h.s. of (2.100):

$$\mathrm{Tr}_3\Big(\mathcal{S}^{ik}_3(R^z_{13}(q_{ik})P_{13}R^{z-w}_{12}(q_{kj})P_{12} - R^{z-w}_{12}(q_{ij})P_{12}R^w_{23}(q_{ik})P_{23})\Big). \tag{2.104}$$

Expressions (2.103) and (2.104) coincide due to (2.8) and (2.40).

the tensor component $\overset{1'}{E}_{ii} \otimes \overset{2'}{E}_{ij}$ $(i \neq j)$:

l.h.s. of (2.100):

$$\mathrm{Tr}_4\Big(\{p_i, R^w_{24}(q_{ij})\}P_{24}\mathcal{S}^{ij}_4\Big) + \mathrm{Tr}_{34}\Big(r_{13}(z)R^w_{24}(q_{ij})P_{24}\{\mathcal{S}^{ii}_3,\mathcal{S}^{ij}_4\}\Big) =$$
$$= \mathrm{Tr}_3\Big(\mathcal{S}^{ij}_3\partial_{q_i}R^w_{23}(q_{ij})P_{23}\Big) - \mathrm{Tr}_3\Big(\mathcal{S}^{ij}_3 R^w_{23}(q_{ij})P_{23}r_{13}(z)\Big). \tag{2.105}$$

r.h.s. of (2.100):

$$\mathrm{Tr}_3\Big(\mathcal{S}^{ij}_3(R^w_{13}(q_{ij})P_{13}R^{z-w}_{12}(q_{ji})P_{12} - r_{12}(z-w)R^w_{23}(q_{ij})P_{23})\Big). \tag{2.106}$$

Expressions (2.105) and (2.106) coincide due to (2.67) rewritten through the Fourier symmetry (2.42) as

$$R^{q_{ij}}_{ac}(z)R^{q_{ji}}_{bc}(w) = -R^{q_{ij}}_{ab}(z-w)r_{ac}(z) + r_{bc}(w)R^{q_{ij}}_{ab}(z-w) - \partial_{q_i}R^{q_{ij}}_{ab}(z-w) \tag{2.107}$$

for distinct a, b, c.

the tensor component $\overset{1'}{E}_{ij} \otimes \overset{2'}{E}_{ji}$ $(i \neq j)$:

l.h.s. of (2.100):

$$\mathrm{Tr}_3\Big(\mathcal{S}^{jj}_3 R^z_{13}(q_{ij})P_{13}R^w_{23}(q_{ji})P_{23} - \mathcal{S}^{ii}_3 R^w_{23}(q_{ji})P_{23}R^z_{13}(q_{ij})P_{13}\Big). \tag{2.108}$$

r.h.s. of (2.100):

$$\begin{aligned}\mathrm{Tr}_3\Big(\mathcal{S}_3^{ii}(r_{13}(z)R_{12}^{z-w}(q_{ij})P_{12}-R_{12}^{z-w}(q_{ij})P_{12}r_{23}(w))\Big)+\\ +\mathrm{Tr}_3\Big(\mathcal{S}_3^{jj}(-R_{12}^{z-w}(q_{ij})P_{12}r_{13}(z)+r_{23}(w)R_{12}^{z-w}(q_{ij})P_{12})\Big)-\\ -\mathrm{Tr}_3\Big(\Big(\mathcal{S}_3^{ii}-\mathcal{S}_3^{jj}\Big)\partial_{q_i}R_{12}^{z-w}(q_{ij})P_{12}\Big).\end{aligned} \tag{2.109}$$

The last term comes from the second line of (2.100). Again, expressions (2.108) and (2.109) coincide due to (2.107).

The rest of the components are verified similarly.

2.4 Examples

2.4.1 Elliptic models

Let us begin with the elliptic model [90, 44, 31, 88]. The Lax pair is of the form:

$$\mathcal{L}(z)=\sum_{i,j=1}^{M}E_{ij}\otimes\mathcal{L}^{ij}(z),\qquad \mathcal{L}^{ij}(z)\in\mathrm{Mat}_N\qquad \mathcal{L}(z)\in\mathrm{Mat}_{NM}, \tag{2.110}$$

where

$$\begin{aligned}\mathcal{L}^{ij}(z)=\delta_{ij}\Big(p_i1_N+\mathcal{S}_{(0,0)}^{ii}1_NE_1(z)+\sum_{\alpha\neq0}\mathcal{S}_\alpha^{ii}T_\alpha\varphi_\alpha(z,\omega_\alpha)\Big)+\\ +(1-\delta_{ij})\sum_\alpha\mathcal{S}_\alpha^{ij}T_\alpha\varphi_\alpha(z,\omega_\alpha+\frac{q_{ij}}{N}),\end{aligned} \tag{2.111}$$

where the basis (**??**) in $\mathrm{Mat}(N,\mathbb{C})$ is used. Similarly, the M-matrix is of the form

$$\begin{aligned}\mathcal{M}^{ij}(z)=\delta_{ij}\mathcal{S}_{(0,0)}^{ii}\frac{E_1^2(z)-\wp(z)}{2N}1_N+\frac{1}{N}\delta_{ij}\sum_{\alpha\neq0}\mathcal{S}_\alpha^{ii}T_\alpha f_\alpha(z,\omega_\alpha)+\\ +\frac{1}{N}(1-\delta_{ij})\sum_\alpha\mathcal{S}_\alpha^{ij}T_\alpha f_\alpha(z,\omega_\alpha+\frac{q_{ij}}{N}).\end{aligned} \tag{2.112}$$

These formulae can be obtained from (2.50)–(2.52) and the R-matrix (2.167) together with (A.8).

The Lax equations hold on the constraints

$$\mathcal{S}^{ii}_{(0,0)} = \text{const}, \quad \forall i. \tag{2.113}$$

Instead of the standard basis (2.3) here we use the basis (**??**) for each $N \times N$ block. Then the Poisson structure (2.5) takes the form

$$\{\mathcal{S}^{ij}_\alpha, \mathcal{S}^{kl}_\beta\} = \delta_{il}\varkappa_{\alpha,\beta}\mathcal{S}^{kj}_{\alpha+\beta} - \delta_{kj}\varkappa_{\beta,\alpha}\mathcal{S}^{il}_{\alpha+\beta}, \tag{2.114}$$

where $\varkappa_{\alpha,\beta}$ are the constants from (A.18).

The Hamiltonian easily follows from $\frac{1}{2N}\text{Tr}\mathcal{L}^2(z) = \frac{1}{2N}E_2(z)\text{Tr}(\mathcal{S}^2) + \mathcal{H}$ due to (A.19) and (A.8):

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^M p_i^2 - \frac{1}{2}\sum_{i=1}^M\sum_{\alpha\neq 0}\mathcal{S}^{ii}_\alpha\mathcal{S}^{ii}_{-\alpha}E_2(\omega_\alpha) - \frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}^M\sum_\alpha \mathcal{S}^{ij}_\alpha\mathcal{S}^{ji}_{-\alpha}E_2(\omega_\alpha + \frac{q_{ij}}{N}). \tag{2.115}$$

Let us show how this Hamiltonian is reproduced from the general formula (2.85). In order to get the second term in (2.115) one should substitute $m_{12}(0)$ into (2.85) from (2.169) and use relation (A.12). For evaluation of the last sum in (2.85) we need to calculate $F^0_{12}(q)P_{12}$. The answer for $F^0_{12}(q)$ is given in (2.170). Multiply it by $P_{12} = (1/N)\sum_b T_b \otimes T_{-b}$ from the left:

$$F^0_{12}(q)P_{12} = -\frac{1}{N^2}E_2(q)\sum_b T_b \otimes T_{-b} +$$
$$+\frac{1}{N^2}\sum_{a\neq(0,0),b}\varphi_a(q,\omega_a)(E_1(q+\omega_a) - E_1(q) + 2\pi i\partial_\tau\omega_\alpha)\varkappa^2_{a,b}T_{a+b}\otimes T_{-a-b}. \tag{2.116}$$

Let us redefine the summation index $b \to b - a$ in the last sum. Since $\varkappa_{a,b} = \varkappa_{a,b-a}$ we have

$$F^0_{12}(q)P_{12} =$$
$$= \frac{1}{N^2}\sum_b T_b\otimes T_{-b}\Big(-E_2(q) + \sum_{a\neq 0,b}\varphi_a(q,\omega_a)(E_1(q+\omega_a) - E_1(q) + 2\pi i\partial_\tau\omega_\alpha)\varkappa^2_{a,b}\Big) \overset{(2.174)}{=}$$
$$= -\frac{1}{N^2}\sum_b T_b\otimes T_{-b}E_2(\omega_b + \frac{q}{N}). \tag{2.117}$$

Finally,

$$\text{Tr}_{12}\Big(F^0_{21}(q_{ij})P_{12}\mathcal{S}^{ij}_1\mathcal{S}^{ji}_2\Big) = \sum_\alpha \mathcal{S}^{ij}_\alpha\mathcal{S}^{ji}_{-\alpha}E_2(\omega_\alpha + \frac{q_{ij}}{N}). \tag{2.118}$$

In the rank one case the answer for the Hamiltonian is given by (2.95). Plugging (2.170) into (2.95) we get

$$\mathcal{H}^{\text{tops}} = \frac{1}{2}\sum_{i=1}^{M} p_i^2 - \frac{1}{2}\sum_{i=1}^{M}\sum_{\alpha\neq 0} \mathcal{S}_{\alpha}^{ii}\mathcal{S}_{-\alpha}^{ii}E_2(\omega_\alpha) -$$

$$-\frac{N}{2}\sum_{\substack{i,j\\ i\neq j}}^{M}\Big(E_2(q_{ij})\mathcal{S}_0^{ii}\mathcal{S}_0^{jj} - \sum_{\alpha\neq 0}\varphi_\alpha(q_{ij},\omega_\alpha)(E_1(q_{ij}+\omega_\alpha) - E_1(q_{ij}) + 2\pi i\partial_\tau\omega_\alpha)\mathcal{S}_{-\alpha}^{ii}\mathcal{S}_{\alpha}^{jj}\Big). \quad (2.119)$$

Let us show how the latter expression appears from (2.115). In the rank one case using (A.19) (so that $\mathcal{S}_{\alpha}^{ij} = \mathrm{Tr}(\mathcal{S}^{ij}T_{-\alpha})/N$) we get

$$\mathcal{S}_{\alpha}^{ij}\mathcal{S}_{-\alpha}^{ji} = \frac{\mathrm{Tr}(\eta^j T_{-\alpha}\xi^i)\mathrm{Tr}(\eta^i T_\alpha \xi^j)}{N^2} = \frac{\mathrm{Tr}(\eta^j T_{-\alpha}\xi^i\eta^i T_\alpha\xi^j)}{N^2} = \frac{\mathrm{Tr}(\mathcal{S}^{ii}T_\alpha\mathcal{S}^{jj}T_{-\alpha})}{N^2}. \quad (2.120)$$

In this way the Hamiltonian (2.115) acquires the form

$$\mathcal{H}^{\text{tops}} = \frac{1}{2}\sum_{i=1}^{M} p_i^2 - \frac{1}{2}\sum_{i=1}^{M}\sum_{\alpha\neq 0} \mathcal{S}_{\alpha}^{ii}\mathcal{S}_{-\alpha}^{ii}E_2(\omega_\alpha) -$$

$$-\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}^{M}\sum_{\alpha}\frac{\mathrm{Tr}(\mathcal{S}^{ii}T_\alpha\mathcal{S}^{jj}T_{-\alpha})}{N^2}E_2(\omega_\alpha + \frac{q_{ij}}{N}), \quad (2.121)$$

which is the model of interacting tops of (2.27) type. The last terms in (2.121) can be simplified in the following way. Substitute $\mathcal{S}^{ii} = \sum_\gamma \mathcal{S}_\gamma^{ii}T_\gamma$ and $\mathcal{S}^{jj} = \sum_\gamma \mathcal{S}_\mu^{jj}T_\mu$ into (2.121). It follows from (A.18)–(A.19) that

$$\mathrm{Tr}(T_\gamma T_\alpha T_\mu T_{-\alpha}) = N\varkappa_{\alpha,\mu}^2\delta_{\mu+\gamma}. \quad (2.122)$$

Therefore,

$$\sum_{\alpha}\frac{\mathrm{Tr}(\mathcal{S}^{ii}T_\alpha\mathcal{S}^{jj}T_{-\alpha})}{N^2}E_2(\omega_\alpha + \frac{q_{ij}}{N}) = \frac{1}{N}\sum_{\alpha,\mu}\mathcal{S}_{-\mu}^{ii}\mathcal{S}_{\mu}^{jj}E_2(\omega_\alpha + \frac{q_{ij}}{N})\varkappa_{\alpha,\mu}^2. \quad (2.123)$$

Using (2.172)–(2.173) and summing up over α we obtain the last term in (2.119).

2.4.2 Trigonometric models

The general classification of the unitary trigonometric R-matrices satisfying associative Yang–Baxter equation was given in [72, 62]. It includes the 7-vertex deformation [16] of the 6-vertex

R-matrix and its GL_N generalizations such as the non-standard R-matrix [2]. The integrable tops and related structures based on these R-matrices were described in [36].

Here we restrict ourselves to the case $N = 2$. The 7-vertex R-matrix is of the following form:

$$R^{\hbar}_{12}(z) = \begin{pmatrix} \coth(z)+\coth(\hbar) & 0 & 0 & 0 \\ 0 & \sinh^{-1}(\hbar) & \sinh^{-1}(z) & 0 \\ 0 & \sinh^{-1}(z) & \sinh^{-1}(\hbar) & 0 \\ C\sinh(z+\hbar) & 0 & 0 & \coth(z)+\coth(\hbar) \end{pmatrix} \tag{2.124}$$

where C is a constant. In the limit $C \to 0$ the lower left-hand corner vanishes and we get the 6-vertex XXZ R-matrix. For the classical r-matrix and its derivative ($F^0_{12}(z) = \partial_z r_{12}(z)$) we have

$$r_{12}(z) = \begin{pmatrix} \coth(z) & 0 & 0 & 0 \\ 0 & 0 & \sinh^{-1}(z) & 0 \\ 0 & \sinh^{-1}(z) & 0 & 0 \\ C\sinh(z) & 0 & 0 & \coth(z) \end{pmatrix} \tag{2.125}$$

and

$$F^0_{12}(q) = \begin{pmatrix} -\dfrac{1}{\sinh^2(q)} & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{\cosh(q)}{\sinh^2(q)} & 0 \\ 0 & -\dfrac{\cosh(q)}{\sinh^2(q)} & 0 & 0 \\ C\cosh(q) & 0 & 0 & -\dfrac{1}{\sinh^2(q)} \end{pmatrix} \tag{2.126}$$

respectively. The Fourier transformed F^0 matrix is of the form:

$$F^0_{12}(q)P_{12} = \begin{pmatrix} -\dfrac{1}{\sinh^2(q)} & 0 & 0 & 0 \\ 0 & -\dfrac{\cosh(q)}{\sinh^2(q)} & 0 & 0 \\ 0 & 0 & -\dfrac{\cosh(q)}{\sinh^2(q)} & 0 \\ C\cosh(q) & 0 & 0 & -\dfrac{1}{\sinh^2(q)} \end{pmatrix} \tag{2.127}$$

From the latter matrix using (2.33) we obtain

$$\mathcal{U}(\mathcal{S}^{ij},\mathcal{S}^{ji},q_i-q_j)=\mathrm{Tr}_{12}\Big(\partial_{q_i}r_{21}(q_{ij})P_{12}\mathcal{S}_1^{ij}\mathcal{S}_2^{ji}\Big)=$$
$$=-\frac{1}{\sinh^2(q_{ij})}\Big(\mathcal{S}_{11}^{ij}\mathcal{S}_{11}^{ji}+\mathcal{S}_{22}^{ij}\mathcal{S}_{22}^{ji}\Big)-\frac{\cosh(q_{ij})}{\sinh^2(q_{ij})}\Big(\mathcal{S}_{11}^{ij}\mathcal{S}_{22}^{ji}+\mathcal{S}_{22}^{ij}\mathcal{S}_{11}^{ji}\Big)+C\cosh(q_{ij})\mathcal{S}_{12}^{ij}\mathcal{S}_{12}^{ji}. \quad (2.128)$$

Similarly, using (2.34) and (2.126) we get the potential for the model of interacting tops:

$$\mathcal{V}(\mathcal{S}^{ii},\mathcal{S}^{jj},q_i-q_j)=\mathrm{Tr}_{12}\Big(\partial_{q_i}r_{12}(q_{ij})\mathcal{S}_1^{ii}\mathcal{S}_2^{jj}\Big)=$$
$$=-\frac{1}{\sinh^2(q_{ij})}\Big(\mathcal{S}_{11}^{ii}\mathcal{S}_{11}^{jj}+\mathcal{S}_{22}^{ii}\mathcal{S}_{22}^{jj}\Big)-\frac{\cosh(q_{ij})}{\sinh^2(q_{ij})}\Big(\mathcal{S}_{12}^{ii}\mathcal{S}_{21}^{jj}+\mathcal{S}_{21}^{ii}\mathcal{S}_{12}^{jj}\Big)+C\cosh(q_{ij})\mathcal{S}_{12}^{ii}\mathcal{S}_{12}^{jj}. \quad (2.129)$$

The top Hamiltonian $\mathcal{H}^{\mathrm{top}}(\mathcal{S}^{ii})$ entering (2.1) or (2.27) is of the form:

$$\mathcal{H}^{\mathrm{top}}(\mathcal{S}^{ii})=\frac{1}{2}\Big((\mathcal{S}_{11}^{ii})^2+(\mathcal{S}_{22}^{ii})^2\Big)+C(\mathcal{S}_{12}^{ii})^2. \quad (2.130)$$

2.4.3 Rational models

The rational R-matrices satisfying the required properties are represented by the 11-vertex deformation [16] of the 6-vertex XXX (Yang's) R-matrix. Its higher rank analogues were derived in [80] and [45, 47]. As in trigonometric case here we restrict ourselves to the case $N=2$. The 11-vertex R-matrix is of the following form:

$$R_{12}^{\hbar}(z)=\begin{pmatrix} \hbar^{-1}+z^{-1} & 0 & 0 & 0\\ -\hbar-z & \hbar^{-1} & z^{-1} & 0\\ -\hbar-z & z^{-1} & \hbar^{-1} & 0\\ -\hbar^3-2z\hbar^2-2\hbar z^2-z^3 & \hbar+z & \hbar+z & \hbar^{-1}+z^{-1}\end{pmatrix} \quad (2.131)$$

In order to get the XXX R-matrix one may take the limit $\lim_{\epsilon\to0}\epsilon^{-1}R^{\epsilon\hbar}(\epsilon z)$.

The classical r-matrix, the F_{12}^0 matrix and its Fourier dual are of the form:

$$r_{12}(z)=\begin{pmatrix} z^{-1} & 0 & 0 & 0\\ -z & 0 & z^{-1} & 0\\ -z & z^{-1} & 0 & 0\\ -z^3 & z & z & z^{-1}\end{pmatrix} \quad (2.132)$$

$$F^0_{12}(q) = \begin{pmatrix} -q^{-2} & 0 & 0 & 0 \\ -1 & 0 & -q^{-2} & 0 \\ -1 & -q^{-2} & 0 & 0 \\ -3q^2 & 1 & 1 & -q^{-2} \end{pmatrix} \tag{2.133}$$

$$F^0_{12}(q)P_{12} = \begin{pmatrix} -q^{-2} & 0 & 0 & 0 \\ -1 & -q^{-2} & 0 & 0 \\ -1 & 0 & -q^{-2} & 0 \\ -3q^2 & 1 & 1 & -q^{-2} \end{pmatrix} \tag{2.134}$$

From (2.134) using (2.33) we obtain

$$\mathcal{U}(\mathcal{S}^{ij}, \mathcal{S}^{ji}, q_i - q_j) = -\frac{1}{(q_i - q_j)^2}\Big(\mathcal{S}^{ij}_{11}\mathcal{S}^{ji}_{11} + \mathcal{S}^{ij}_{22}\mathcal{S}^{ji}_{22} + \mathcal{S}^{ij}_{11}\mathcal{S}^{ji}_{22} + \mathcal{S}^{ij}_{22}\mathcal{S}^{ji}_{11}\Big) +$$
$$+\mathcal{S}^{ij}_{12}\mathcal{S}^{ji}_{22} + \mathcal{S}^{ij}_{22}\mathcal{S}^{ji}_{12} - \mathcal{S}^{ij}_{12}\mathcal{S}^{ji}_{11} - \mathcal{S}^{ij}_{11}\mathcal{S}^{ji}_{12} - 3(q_i - q_j)^2\mathcal{S}^{ij}_{12}\mathcal{S}^{ji}_{12}. \tag{2.135}$$

Similarly, from (2.133) using (2.34) we obtain

$$\mathcal{V}(\mathcal{S}^{ii}, \mathcal{S}^{jj}, q_i - q_j) = -\frac{1}{(q_i - q_j)^2}\Big(\mathcal{S}^{ii}_{11}\mathcal{S}^{jj}_{11} + \mathcal{S}^{ii}_{22}\mathcal{S}^{jj}_{22} + \mathcal{S}^{ii}_{12}\mathcal{S}^{jj}_{21} + \mathcal{S}^{ii}_{21}\mathcal{S}^{jj}_{12}\Big) +$$
$$+\mathcal{S}^{ii}_{12}\mathcal{S}^{jj}_{22} + \mathcal{S}^{ii}_{22}\mathcal{S}^{jj}_{12} - \mathcal{S}^{ii}_{12}\mathcal{S}^{jj}_{11} - \mathcal{S}^{ii}_{11}\mathcal{S}^{jj}_{12} - 3(q_i - q_j)^2\mathcal{S}^{ii}_{12}\mathcal{S}^{jj}_{12}. \tag{2.136}$$

The top Hamiltonian $\mathcal{H}^{\text{top}}(\mathcal{S}^{ii})$ entering (2.1) or (2.27) is of the form:

$$\mathcal{H}^{\text{top}}(\mathcal{S}^{ii}) = \mathcal{S}^{ii}_{12}(\mathcal{S}^{ii}_{22} - \mathcal{S}^{ii}_{11}). \tag{2.137}$$

2.5 Discussion

A possible application of the obtained family of integrable models is in constructing quantum integrable anisotropic long-range spin chains. The basic idea is that such spin chains appear from the models of interacting tops by the so-called freezing trick likewise the Haldane-Shastry-Inozemtsev spin chains [32, 77, 34, 63] come from the ordinary spin Calogero-Moser-Sutherland models. A direct quantization of the interacting tops is a separate problem, which will be discussed elsewhere. At the same time the quantum Hamiltonian of interacting tops appears in the so-called R-matrix–valued Lax pairs for the (classical) spinless Calogero-Moser model [46, 48, 73, 31, 88]. These are the Lax pairs in a large space $\text{Mat}(M, \mathbb{C}) \otimes \text{Mat}(N, \mathbb{C})^{\otimes M}$:

$$\mathcal{L}^{\text{CM}} = \sum_{a,b=1}^{M} E_{ab} \otimes \mathcal{L}_{ab}, \quad \mathcal{L}_{ab} = \delta_{ab} p_a \, 1_N^{\otimes M} + \nu(1 - \delta_{ab}) R^z_{ab}, \quad R^z_{ab} = R^z_{ab}(q_a - q_b). \tag{2.138}$$

and similarly for the accompany M-matrix

$$\mathcal{M}^{\rm CM}_{ab} = \nu\delta_{ab}d_a + \nu(1-\delta_{ab})F^z_{ab} + \nu\delta_{ab}\ \mathcal{F}^0, \quad F^z_{ab} = \partial_{q_a}R^z_{ab}(q_a-q_b), \tag{2.139}$$

where

$$d_a = -\sum_{c\neq a}^{M} F^0_{ac}, \quad \mathcal{F}^0 = \sum_{\substack{b,c\\ b>c}}^{M} F^0_{bc} = \sum_{\substack{b,c\\ b>c}}^{M} \partial_{q_b} r_{bc}(q_{bc}). \tag{2.140}$$

In the $N=1$ case this Lax pair coincides with the widely known Krichever's result [37] for $\mathfrak{gl}_M$ Calogero–Moser model. The last term $\mathcal{F}^0$ in (2.139) enters $\mathcal{M}$ as a scalar (it is an identity matrix in $\mathrm{Mat}(M,\mathbb{C})$ component) in the auxiliary space $\mathrm{Mat}(M,\mathbb{C})$. Therefore, it can be moved to the l.h.s. of the Lax equation. This yields

$$\{H^{\rm CM}, \mathcal{L}^{\rm CM}\} + [\nu\mathcal{F}^0, \mathcal{L}^{\rm CM}(z)] = [\mathcal{L}^{\rm CM}(z), \bar{\mathcal{M}}^{\rm CM}(z)], \tag{2.141}$$

where $\bar{\mathcal{M}}^{\rm CM} = \mathcal{M}^{\rm CM} - \nu 1_M \otimes \mathcal{F}^0$. On the one hand, (2.141) is just a rewritten classical Lax equation for the spinless Calogero–Moser model. On the other hand, we may treat it as half-quantum Lax equation in a sense that the dynamics is given by the interacting tops Hamiltonian (2.95), where the spin variables are already quantized, while the positions and momenta remain classical. Indeed, the quantization of $\mathcal{S}^{ii}_1$ in fundamental representation of GL_N is given by the permutation operator P_{1j}. Plugging it into the potential of (2.95) we get the $\mathcal{F}^0$ term from (2.139) and (2.141).

Thus the R-matrix valued Lax pairs are multidimensional classical Lax pairs for the spinless Calogero–Moser models and at the same time they are quantum Lax pairs for the models of interacting tops with the spin variables being quantized in the fundamental representation of GL_N, i.e. the $\mathcal{F}^0$ term is the quantization of the potential $\mathcal{V}(\mathcal{S}^{ii}, \mathcal{S}^{jj}, q_i - q_j)$ (2.34).

Let us also mention that there is another class of integrable models with the Hamiltonian of type (2.27). These are the Gaudin type models [59]. The corresponding Lax matrix is of size $M\times M$. It has simple poles at n points on elliptic curve (or its degenerations) with the classical spin variables matrices attached to each point. The number of points is not necessarily equal to M. It is an interesting task to find interrelations between the Gaudin models and the models of interacting tops through the spectral duality [52, 53, 54] based on the rank-size duality transformation.

The classical spinless Calogero–Moser model possesses an equilibrium position, where $p_i = 0$ and $q_i = x_i$ (for example, $x_i = i/M$ [17]). At this point the term $\{H^{\rm CM}, \mathcal{L}^{\rm CM}\}$ vanishes from the l.h.s. of (2.141), and we are left with the quantum Lax equation for some long-range (quantum) spin chain. It is an anisotropic generalization [73] of the Haldane–Shastry–Inozemtsev type chains. An open question is which $\mathcal{F}^0$ provide integrable spin chains? To confirm integrability we need to construct higher Hamiltonians, which commute with each other and with $\mathcal{F}^0(q_i = x_i)$. Taking into account all the above we guess that the model of interacting tops together with the freezing trick (the quantum version of the equilibrium position) can be used to calculate higher spin chain Hamiltonians and to prove their commutativity. For this purpose we need to construct a quantization for the model of interacting tops, which will be the subject of the further investigations.

Another one intriguing question is to construct relativistic generalization of the models discussed above. While the classical models of relativistic interacting tops are expected to be relatively simple (the block L^{ij} in (2.51) should be replaced by $\mathrm{Tr}_2(\mathcal{S}_2^{ij} R^z_{12}(q_{ij} + \eta)P_{12})$) its quantum versions and the related long-range spin chain were not studied yet as well as the corresponding R-matrix–valued Lax pairs.

2.6 Appendix

2.6.1 Spin $\mathfrak{gl}_M$ Calogero–Moser model

The Lax equations

$$\dot{L}^{\mathrm{spin}}(z) = [L^{\mathrm{spin}}(z), M^{\mathrm{spin}}(z)] \tag{2.142}$$

with the Lax pair

$$L^{\mathrm{spin}}_{ij}(z) = \delta_{ij}(p_i + S_{ii}E_1(z)) + (1 - \delta_{ij})S_{ij}\phi(z, q_{ij}), \tag{2.143}$$

$$M^{\mathrm{spin}}_{ij}(z) = (1 - \delta_{ij})S_{ij}f(z, q_{ij}). \tag{2.144}$$

provide (after restriction on the constraints (2.17)) equations of motion

$$\dot{q}_i = p_i, \quad \ddot{q}_i = \sum_{j\neq i}^{M} S_{ij}S_{ji}\wp'(q_i - q_j), \tag{2.145}$$

$$\dot{S}_{ii} = 0, \quad \dot{S}_{ij} = \sum_{k\neq i,j}^{M} S_{ik}S_{kj}(\wp(q_i - q_k) - \wp(q_j - q_k)),\ i \neq j. \tag{2.146}$$

The l.h.s. of the Lax equations (2.142) is generated by the Hamiltonian (2.16)

$$\dot{L}^{\mathrm{spin}}(z) = \{H^{\mathrm{spin}}, L^{\mathrm{spin}}(z)\} \tag{2.147}$$

and the linear Poisson–Lie brackets on $\mathfrak{gl}^*_M$:

$$\{S_{ij}, S_{kl}\} = -S_{il}\delta_{kj} + S_{kj}\delta_{il} \quad \text{or} \quad \{S_1, S_2\} = [S_2, P_{12}]. \tag{2.148}$$

Recall that the Poisson reduction with respect to Cartan action (2.18) is non-trivial. For instance, in the rank 1 case (2.23) such reduction leads to the spinless model (2.24). Explicit expression of the reduced Poisson structure depends on a choice of gauge fixation conditions. The equations of motion (2.145)–(2.146) are not the reduced. They are obtained by a simple restriction. To get the final equations one should perform the Dirac reduction and evaluate the Dirac terms.

The classical r-matrix structure is as follows:

$$\begin{gathered}\{L_1^{\rm spin}(z), L_2^{\rm spin}(w)\} = [L_1^{\rm spin}(z), r_{12}^{\rm spin}(z,w)] - [L_2^{\rm spin}(w), r_{21}^{\rm spin}(w,z)] - \\ - \sum_{\substack{i,j \\ i\neq j}} E_{ij}\otimes E_{ji}(S_{ii}-S_{jj})f(z-w,q_{ij})\end{gathered} \tag{2.149}$$

with

$$r_{12}^{\rm spin}(z,w) = E_1(z-w)\sum_{i=1}^M E_{ii}\otimes E_{ii} + \sum_{\substack{i,j \\ i\neq j}}^M \phi(z-w,q_{ij})E_{ij}\otimes E_{ji}. \tag{2.150}$$

Here the linear Poisson brackets (2.148) are assumed as well. The Dirac reduction is not yet performed. However, we can see that the restriction on the constraints (2.17) kills the last term in (2.149), and we are left with the standard linear classical r-matrix structure. It is enough for Poisson commutativity

$$\{{\rm Tr}(L^k(z)), {\rm Tr}(L^n(w))\} = 0, \quad \forall k,n\in\mathbb{Z}_+, \quad z,w\in\mathbb{C} \tag{2.151}$$

necessary for the Liouville integrability. The proof of (2.149) is direct. It is based on the identities (A.6)–(A.9). Let us write down a few examples of verification of (2.149):

<u>the tensor component $E_{ij}\otimes E_{jk}$ $(i\neq j, j\neq k, k\neq i)$:</u>

l.h.s. of (2.149):

$$\{L_{ij}^{\rm spin}(z), L_{jk}^{\rm spin}(w)\} = \{S_{ij}, S_{jk}\}\phi(z,q_{ij})\phi(w,q_{jk}) = -S_{ik}\phi(z,q_{ij})\phi(w,q_{jk}). \tag{2.152}$$

r.h.s. of (2.149):

$$S_{ik}\phi(z,q_{ik})\phi(z-w,q_{kj}) + S_{ik}\phi(w,q_{ik})\phi(w-z,q_{ji}). \tag{2.153}$$

Expressions (2.152) and (2.153) coincide due to (A.6).

<u>the tensor component $E_{ii}\otimes E_{ij}$ $(i\neq j)$:</u>

l.h.s. of (2.149):

$$\begin{gathered}\{L_{ii}^{\rm spin}(z), L_{ij}^{\rm spin}(w)\} = \{p_i, \phi(w,q_{ij})\}S_{ij} + \{S_{ii}, S_{ij}\}E_1(z)\phi(w,q_{ij}) = \\ = S_{ij}f(w,q_{ij}) - S_{ij}E_1(z)\phi(w,q_{ij}).\end{gathered} \tag{2.154}$$

r.h.s. of (2.149):

$$S_{ij}\phi(z,q_{ij})\phi(z-w,q_{ji})+S_{ij}E_1(w-z)\phi(w,q_{ij}). \tag{2.155}$$

Expressions (2.154) and (2.155) coincide due to (A.9).

the tensor component $E_{ij}\otimes E_{ji}$ $(i\neq j)$:

l.h.s. of (2.149):

$$\begin{gathered}\{L^{\rm spin}_{ij}(z),L^{\rm spin}_{i}(w)\}=\\ =\{S_{ij},S_{ji}\}\phi(z,q_{ij})\phi(w,q_{ji})=(S_{ii}-S_{jj})\phi(z,q_{ij})\phi(-w,q_{ij}).\end{gathered} \tag{2.156}$$

The last term from the r.h.s. of (2.149) contributes in this component. The r.h.s. of (2.149):

$$\begin{gathered}(p_i+S_{ii}E_1(z)-p_j-S_{jj}E_1(z))\phi(z-w,q_{ij})-\\ -(p_j+S_{jj}E_1(w)-p_i-S_{ii}E_1(w))\phi(w-z,q_{ji})-(S_{ii}-S_{jj})f(z-w,q_{ij})=\\ =(S_{ii}-S_{jj})\Big((E_1(z)-E_1(w))\phi(z-w,q_{ij})-f(z-w,q_{ij})\Big).\end{gathered} \tag{2.157}$$

Expressions (2.156) and (2.157) coincide due to (A.9).

2.6.2 Integrable $\mathfrak{gl}_N$ tops

It was shown in [49] (see also [45, 47]) that the Lax equations

$$\dot{L}(z,S)=[L(z,S),M(z,S)] \tag{2.158}$$

are equivalent to equations

$$\dot{S}=[S,J(S)] \tag{2.159}$$

for the Lax pair

$$L(z,S)=\mathrm{Tr}_2(r_{12}(z)S_2),\qquad M(z,S)=\mathrm{Tr}_2(m_{12}(z)S_2),\quad S_2=1_N\otimes S \tag{2.160}$$

and

$$J(S)=\mathrm{Tr}_2(m_{12}(0)S_2). \tag{2.161}$$

constructed by means of the coefficients of the (classical limit) expansion (2.9) for an R-matrix satisfying the associative Yang–Baxter equation (2.8) and the properties (2.38)–(2.41). The answer (2.160) can be written more explicitly. For

$$r_{12}(z)=\sum_{i,j,k,l=1}^{N}r_{ijkl}(z)e_{ij}\otimes e_{kl} \tag{2.162}$$

(2.160) means

$$L(z,S) = \sum_{i,j,k,l=1}^{N} r_{ijkl}(z) S_{lk} e_{ij} \tag{2.163}$$

since $\mathrm{Tr}(e_{kl}S) = S_{lk}$.

Let us briefly describe how these formulae reproduce the elliptic top from [40].

Define the set of functions numerated by $a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$:

$$\varphi_a(z, \omega_a + u) = \exp(2\pi i \frac{a_2}{N} z)\phi(z, \omega_a + u), \quad \omega_a = \frac{a_1 + a_2\tau}{N} \tag{2.164}$$

and introduce notation

$$f_a(z, \omega_a + u) = \exp(2\pi i \frac{a_2}{N} z) f(z, \omega_a + u). \tag{2.165}$$

The Baxter–Belavin R-matrix [6, 7, 70] satisfying all required properties including the Fourier symmetry (2.42) is of the form:

$$R_{12}^{BB}(\hbar, z) = \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} \varphi_a(z, \hbar + \omega_a) T_a \otimes T_{-a} \in \mathrm{Mat}(N, \mathbb{C})^{\otimes 2}. \tag{2.166}$$

here φ_a are the elliptic functions and T_a are the corresponding basis matrices in $\mathrm{Mat}_N(\mathbb{C})$ defined in the Appendix.

This R-matrix satisfies required properties but with different normalizations. For example, the Fourier symmetry has form $R_{12}^{BB}(\hbar, z)P_{12} = R_{12}^{BB}(z/N, N\hbar)$ (see the Fourier transformation formulae in [87]). To fulfill all requirements including the normalization (2.41) we consider

$$R_{12}^{\hbar}(z) = R_{12}^{BB}(\hbar/N, z) = \frac{1}{N} \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} \varphi_a(z, \frac{\hbar}{N} + \omega_a) T_a \otimes T_{-a} \in \mathrm{Mat}(N, \mathbb{C})^{\otimes 2}. \tag{2.167}$$

The corresponding classical r-matrix is as follows

$$r_{12}(z) = \frac{1}{N} E_1(z) 1_N \otimes 1_N + \frac{1}{N} \sum_{a \neq (0,0)} \varphi_a(z, \omega_a) T_a \otimes T_{-a} \in \mathrm{Mat}(N, \mathbb{C})^{\otimes 2}, \tag{2.168}$$

and

$$m_{12}(z) = \frac{E_1^2(z) - \wp(z)}{2N^2} 1_N \otimes 1_N + \frac{1}{N^2} \sum_{a \neq (0,0)} f_a(z, \omega_a) T_a \otimes T_{-a} \in \mathrm{Mat}(N, \mathbb{C})^{\otimes 2}, \tag{2.169}$$

where f_a are the derivatives of φ_a functions (see Appendix).

Then the formulae for the Lax pair (2.160) reproduce the Lax pair of the elliptic top. It is contained in the Lax pair (2.110)–(2.112) as a diagonal $N \times N$ block.

The derivative of the classical r-matrix is obtained through (A.5):

$$F^0_{12}(z) = \partial_z r_{12}(z) = -\frac{1}{N}E_2(z)\ 1_N \otimes 1_N + \tag{2.170}$$
$$+\frac{1}{N}\sum_{a\neq(0,0)} \varphi_a(z,\omega_a)(E_1(z+\omega_a) - E_1(z) + 2\pi i\partial_\tau\omega_\alpha)T_a \otimes T_{-a}$$

The Fourier symmetry $R^\hbar_{12}(z) = R^z_{12}(\hbar)P_{12}$ for the R-matrix (2.167) is based on the following set of identities for the functions (2.164):

$$\frac{1}{N}\sum_\alpha \varkappa^2_{\alpha,\gamma}\varphi_\alpha(N\hbar,\omega_\alpha + \frac{z}{N}) = \varphi_\gamma(z,\omega_\gamma+\hbar), \quad \forall\gamma \in \mathbb{Z}_N \times \mathbb{Z}_N. \tag{2.171}$$

Be degeneration of the latter identities on can deduce (see [87]):

$$\sum_\alpha E_2(\omega_\alpha + q) = N^2 E_2(Nq) \tag{2.172}$$

and for $\gamma \neq 0$

$$\sum_\alpha \varkappa^2_{\alpha,\gamma}E_2(\omega_\alpha+q) = -N^2\varphi_\gamma(Nq,\omega_\gamma)(E_1(Nq+\omega_\gamma) - E_1(Nq) + 2\pi i\partial_\tau\omega_\gamma). \tag{2.173}$$

Conversely,

$$-E_2(q) + \sum_\alpha \varkappa^2_{\alpha,\gamma}\varphi_\alpha(q,\omega_\alpha)(E_1(q+\omega_\alpha) - E_1(q) + 2\pi i\partial_\tau\omega_\alpha) = -E_2(\omega_\gamma + \frac{q}{N}). \tag{2.174}$$

Chapter 3

Generalized relativistic interacting integrable tops

This chapter is based on our paper [75] and it continues and develops the ideas of the previous chapter, introducing the relativistic versions of the generalized interacting tops systems. The spin Calogero–Moser system of particles has the relativistic analogue — the spin Ruijsenaars–Schneider model, and the Euler–Arnold integrable top also has a relativistic counterpart. This chapter is devoted to the classical integrable system generalizing both of these relativistic integrable models.

The main result of this chapter is the description of the Lax structure of a generalized model of relativistic interacting tops, constructed via the quantum R-matrix satisfying the associative Yang–Baxter equation. Similar to the models in the previous chapter, these $GL(NM)$ models generalizes the classical spin Ruijsenaars–Schneider systems (obtained in the particular case $N = 1$) and the relativistic integrable tops on $GL(N)$ Lie group (the particular case $M = 1$).

The Hamiltonian structure is not known even in the case of the spin Ruijsenaars–Schneider model, which has been described only in terms of the equations of motion and the Lax structure corresponding to these equations. So, the generalized relativistic model introduced in this section are also obtained by means of the Lax pair with spectral parameters and the equations of motion, matching on a certain constraint. The proof of this matching uses again only the identities on the quantum R-matrices but not their special form.

3.1 Introduction

This chapter ideas are the continuation of the series of articles [30, 73, 31, 74, 89], where the known integrable systems and related structures are extended through the use of quantum R–matrices (in the fundamental representation of $GL(N)$ Lie group) being interpreted as the matrix generalizations of the Kronecker function in the rational, trigonometric and elliptic cases, defined as

$$\phi(z,q)=\begin{cases}1/z+1/q,\\ \coth(z)+\coth(q),\\ \frac{\vartheta'(0)\vartheta(z+q)}{\vartheta(z)\vartheta(q)},\end{cases}\quad E_1(z)=\begin{cases}1/z,\\ \coth(z),\\ \frac{\vartheta'(z)}{\vartheta(z)},\end{cases}\quad \wp(z)=\begin{cases}1/z^2,\\ 1/\sinh^2(z)+\frac{1}{3},\\ -E_1'(z)+\frac{1}{3}\frac{\vartheta'''(0)}{\vartheta(0)}.\end{cases}\tag{3.1}$$

All the functions are complex–valued. So that the trigonometric and hyperbolic cases are actually the same. In (3.1) for all three cases we also give definitions of the first Eisenstein function $E_1(z)$ and the Weierstrass $\wp$-function, which appear in the expansion of $\phi(z,q)$ near its simple pole (with residue equal to one) at $z=0$:

$$\phi(z,q)=\frac{1}{z}+E_1(q)+\frac{z}{2}(E_1^2(q)-\wp(q))+O(z^2).\tag{3.2}$$

The properties of elliptic functions can be found in the Appendix, for the applications the most important ones is the Fay identity of genus 1:

$$\phi(z_1,q_1)\phi(z_2,q_2)=\phi(z_1-z_2,q_1)\phi(z_2,q_1+q_2)\phi(z_2-z_1,q_2)\phi(z_1,q_1+q_2),\tag{3.3}$$

as well as its degenerations corresponding to equal arguments:

$$\phi(z,q_1)\phi(z,q_2)=\phi(z,q_1+q_2)(E_1(z)+E_1(q_1)+E_1(q_2)-E_1(q_1+q_2+z)),\tag{3.4}$$

$$\phi(z,q)\phi(z,-q)=\wp(z)-\wp(q).\tag{3.5}$$

The Fay identity (3.3) can be considered as a particular scalar case of the associative Yang–Baxter equation [27, 61, 48]:

$$R^z_{12}(q_{12})R^w_{23}(q_{23})=R^w_{13}(q_{13})R^{z-w}_{12}(q_{12})+R^{w-z}_{23}(q_{23})R^z_{13}(q_{13}),\quad q_{ab}=q_a-q_b.\tag{3.6}$$

A normalization of the matrix operator $R^z_{ab}(q_{ab})$ is chosen in a way that for $N=1$ the latter reduces to the scalar function $\phi(z,q)$ (3.1). In this respect equation (3.6) is a noncommutative generalization of (3.3), while the operator R — is a noncommutative generalization of the Kronecker function.

In addition to (3.6), one can impose the properties of the skew-symmetry and unitarity (the latter is a matrix analogue for (3.5)):

$$R^z_{12}(q) = -R^{-z}_{21}(-q), \qquad R^z_{12}(q)R^z_{21}(-q) = 1_N \otimes 1_N(\wp(z) - \wp(q)). \tag{3.7}$$

Then such an R–operator satisfies the quantum Yang–Baxter equation

$$R^\hbar_{12}(z_{12})R^\hbar_{13}(z_{13})R^\hbar_{23}(z_{23}) = R^\hbar_{23}(z_{23})R^\hbar_{13}(z_{13})R^\hbar_{12}(z_{12}). \tag{3.8}$$

Let us mention that even in the scalar case the condition (3.6) or (3.3) is very restrictive. At the same time equation (3.8) is not restrictive at all since in the scalar case the quantum Yang–Baxter equation is identically true. A class of quantum R–matrices satisfying the discussed above conditions includes the elliptic Baxter–Belavin R–matrix as well as its trigonometric and rational degenerations, which are equal to the function $\phi(z, q)$ (3.1) in the scalar case. More detailed description of these R–matrices can be found in [1, 49, 45, 36, 88, 86, 48], where an application of this class of R–matrices to integrable system is given — a construction of integrable tops. The main idea goes back to the Sklyanin's paper [78]. He suggested the Hamiltonian description of the classical Euler top by means of the quadratic Poisson algebras, obtained through the classical limit of RLL-relations. That is, the classical Euler top was described as the classical limit of 1 site spin chain. This approach can be developed to obtain explicit description of the Lax pairs with spectral parameters, constructed via the data of R–matrices satisfying (3.6), (3.7). A detailed derivation of equations of motion together with the Hamiltonian description by means of the R–matrix data is given in papers [1, 49] and [45, 36] for non-relativistic and relativistic cases respectively.

3.1.1 Relativistic integrable GL_N-top.

In the general case the phase space of GL_N-top is given by the set of coordinate functions S_{ij}, $i, j = 1, \ldots, N$ on the Lie group GL_N. They are unified into matrix $S = \sum_{ij} S_{ij}E_{ij}$ of size $N \times N$. Equations of motion then take the form of the Euler–Arnold equations

$$\dot{S} = [S, J(S)], \tag{3.9}$$

where $J(S)$ is a linear functional on S. It can be written in the form

$$J(S) = \sum_{i,j,k,l=1}^{N} J_{ijkl}E_{ij}S_{lk} \in \mathrm{Mat}(NM, \mathbb{C}) \tag{3.10}$$

or, using the standard notations $S_1 = S \otimes 1_N$, $S_2 = 1_N \otimes S$,

$$J(S) = \mathrm{Tr}_2(J_{12}S_2), \qquad J_{12} = \sum_{i,j,k,l=1}^{N} J_{ijkl}E_{ij} \otimes E_{kl}, \tag{3.11}$$

where Tr_2 — is the trace over the second space in the tensor product. Below we give the Lax pair of the relativistic integrable top using the above notation (in the general case, equations (3.9) are of course not integrable).

For this purpose consider the classical limit of the R–matrix:

$$R^{\hbar}_{12}(z) = \frac{1}{\hbar} 1_N \otimes 1_N + r_{12}(z) + \frac{\hbar}{2}\big(r_{12}(z)^2 - \wp(z) 1 \otimes 1\big) + O(\hbar^2), \tag{3.12}$$

where $r_{12}(z) = -r_{21}(-z)$ — the classical r–matrix, and the $\hbar$–order term follows from (3.7). By comparing this expression with (3.2), we conclude that while the quantum R–matrix is a matrix analogue of the Kronecker function, the classical r–matrix is a matrix analogue of the first Eisenstein function $E_1(z)$ (3.1).

Consider expansions

$$R^z_{12}(q) = \frac{1}{q} P_{12} + R^{z,(0)}_{12} + O(q), \quad r_{12}(q) = \frac{1}{q} P_{12} + r^{(0)}_{12} + O(q), \quad P_{12} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}, \tag{3.13}$$

where P_{12} — is the matrix permutation operator.

Generally speaking, existence of expansions of types (3.12), (3.13) is an additional non-trivial requirement for the R–matrix. Finally, let us impose one more condition for R–matrix:

$$R^z_{12}(q) = R^q_{12}(z) P_{12}. \tag{3.14}$$

In the scalar case it turns into equality $\phi(z, q) = \phi(q, z)$. Using (3.14) and comparing (3.12), (3.13) we easily get

$$r_{12}(z) = R^{z,(0)}_{12} P_{12}. \tag{3.15}$$

Now we can formulate the statement on the Lax pair of the relativistic top. Namely, for a pair of matrices

$$L(z) = \mathrm{Tr}_2(R^{\eta}_{12}(z) S_2) = \mathrm{Tr}_2(R^z_{12}(\eta) P_{12} S_2), \tag{3.16}$$

$$M(z) = -\mathrm{Tr}_2(r_{12}(z) S_2) = -\mathrm{Tr}_2(R^{z,(0)}_{12} P_{12} S_2) \tag{3.17}$$

the Lax equation

$$\dot{L}(z) = [L(z), M(z)] \tag{3.18}$$

is equivalent to equations of motion of the form (3.9), where

$$J_{12} = R^{\eta,(0)}_{12} - r^{(0)}_{12}. \tag{3.19}$$

3.1.2 Spin generalization of the Ruijsenaars–Schneider model.

In the integrable many-body systems the relativistic generalizations are known as the Ruijsenaars–Schneider models [71]. We are going to deal with their spin extensions [39]. The set of dynamical variables consists of particles positions and velocities, and the spin variables are arranged into the matrix $S \in \mathrm{Mat}(M, \mathbb{C})$. The equations of motion take the following form (for the diagonal and off-diagonal parts of the matrix S):

$$\dot{S}_{ii} = -\sum_{k \neq i}^{M} S_{ik} S_{ki} \Big(E_1(q_{ik} + \eta) + E_1(q_{ik} - \eta) - 2E_1(q_{ik}) \Big), \tag{3.20}$$

$$\dot{S}_{ij} = \sum_{k \neq j}^{M} S_{ik} S_{kj} \Big(E_1(q_{kj} + \eta) - E_1(q_{kj}) \Big) - \sum_{k \neq i}^{M} S_{ik} S_{kj} \Big(E_1(q_{ik} + \eta) - E_1(q_{ik}) \Big) \tag{3.21}$$

and

$$\ddot{q}_i = \dot{S}_{ii}, \tag{3.22}$$

where $i \neq j$ and $q_{ij} = q_i - q_j$. The Lax pair with spectral parameter

$$L_{ij}(z) = S_{ij} \phi(z, q_{ij} + \eta), \quad i, j = 1, \ldots, M \quad \operatorname*{res}_{z=0} L(z) = S \in \mathrm{Mat}(M, \mathbb{C}), \tag{3.23}$$

$$M_{ij}(z) = -\delta_{ij}(E_1(z) + E_1(\eta)) S_{ii} - (1 - \delta_{ij}) S_{ij} \phi(z, q_{ij}) \tag{3.24}$$

satisfies the Lax equation with additional term (here $\mu_i = \dot{q}_i - S_{ii}$)

$$\dot{L}(z) = [L(z), M(z)] + \sum_{i,j=1}^{M} E_{ij} (\mu_i - \mu_j) S_{ij} f(z, q_{ij} + \eta), \qquad f(z, q) = \partial_q \phi(z, q), \tag{3.25}$$

which turns into zero on-shell constraints

$$\mu_i = 0 \quad \text{or} \quad S_{ii} = \dot{q}_i, \quad i = 1, \ldots, M. \tag{3.26}$$

More precisely, equation (3.25) is equivalent to (3.20)–(3.21), and under conditions (3.26) the Lax equations with the additional term (3.25) turn into the ordinary Lax equations (3.18), and (3.22) holds true. Besides the original paper [39], the detailed derivation of (3.25) can be also found in [89]. This derivation is convenient for consideration of a more general system, where the functions entering (3.20)–(3.24) are replaced by their R–matrix analogues. Although we do not use the Hamiltonian description, let us mention that it is known for the rational and trigonometric systems (see [4, 5, 68, 23, 24, 15, 21, 22]).

3.1.3 The main result

The main result of the chapter is the simultaneous generalization of both the relativistic top (3.16)–(3.19) and the spin Ruijsenaars–Schneider model (3.20)–(3.24). Consider $\mathrm{Mat}(NM,\mathbb{C})$–valued Lax pair subdivided into $M\times M$ blocks $\mathcal{L}^{ij}(z)=\mathcal{L}^{ij}(\mathcal{S}^{ij},z)$ of sizes $N\times N$ each:

$$\mathcal{L}(z)=\sum_{i,j=1}^{M}E_{ij}\otimes\mathcal{L}^{ij}(z)\in\mathrm{Mat}(NM,\mathbb{C}),\quad \mathcal{L}^{ij}(z)\in\mathrm{Mat}(N,\mathbb{C}),\tag{3.27}$$

$$\mathcal{L}^{ij}(z)=\mathrm{Tr}_2(R^z_{12}(q_{ij}+\eta)P_{12}\mathcal{S}^{ij}_2),\quad \mathcal{S}^{ij}=\operatorname*{res}_{z=0}\mathcal{L}^{ij}(z)\in\mathrm{Mat}(N,\mathbb{C}),\tag{3.28}$$

$$\mathcal{M}(z)=\sum_{i,j=1}^{M}E_{ij}\otimes\mathcal{M}^{ij}(z)\in\mathrm{Mat}(NM,\mathbb{C}),\quad \mathcal{M}^{ij}(z)\in\mathrm{Mat}(N,\mathbb{C}),\tag{3.29}$$

$$\mathcal{M}^{ij}(z)=-\delta^{ij}\mathrm{Tr}_2\Big(R^{(0),z}_{12}P_{12}\mathcal{S}^{ii}_2\Big)-(1-\delta^{ij})\mathrm{Tr}_2\Big(R^z_{12}(q_{ij})P_{12}\mathcal{S}^{ij}_2\Big).\tag{3.30}$$

The R–matrix entering this definition satisfies the associative Yang–Baxter equation (3.6) together with the properties (3.7), (3.14) and the expansions (3.12)–(3.13). Then the Lax equation with the additional term

$$\dot{\mathcal{L}}(z)=[\mathcal{L}(z),\mathcal{M}(z)]+\sum_{i,j=1}^{M}(\mu^i_0-\mu^j_0)E_{ij}\otimes\mathrm{Tr}_2\Big(F^z_{12}(q_{ij}+\eta)P_{12}\mathcal{S}^{ij}_2\Big),\tag{3.31}$$

where by the analogy with (3.25)

$$F^z_{12}(q)=\partial_q R^z_{12}(q)\tag{3.32}$$

and

$$\mu^i_0=\dot{q}_i-\mathrm{Tr}\Big(\mathcal{S}^{ii}\Big),\quad i=1,\dots,M,\tag{3.33}$$

is equivalent to equations of motion (in (3.35) we assume $i\neq j$)

$$\dot{\mathcal{S}}^{ii}=[\mathcal{S}^{ii},J^\eta(\mathcal{S}^{ii})]+\sum_{k\neq i}^{M}\Big(\mathcal{S}^{ik}J^{\eta,q_{ki}}(S^{ki})-J^{\eta,q_{ik}}(\mathcal{S}^{ik})S^{ki}\Big),\tag{3.34}$$

$$\dot{\mathcal{S}}^{ij}=\mathcal{S}^{ij}J^\eta(\mathcal{S}^{jj})-J^\eta(\mathcal{S}^{ii})\mathcal{S}^{ij}+\sum_{k\neq j}^{M}\mathcal{S}^{ik}J^{\eta,q_{kj}}(\mathcal{S}^{kj})-\sum_{k\neq i}^{M}J^{\eta,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{kj}.\tag{3.35}$$

On-shell constraints

$$\mu^i_0=0\quad\text{or}\quad\dot{q}_i=\mathrm{Tr}\Big(\mathcal{S}^{ii}\Big),\quad i=1,\dots,M\tag{3.36}$$

(the equations 3.31) reduce to the Lax equations, and the following equations holds:

$$\ddot{q}_i = \mathrm{Tr}\Big(\dot{\mathcal{S}}^{ii}\Big) = \sum_{k\neq i}^{M} \mathrm{Tr}\Big(\mathcal{S}^{ik} J^{\eta,q_{ki}}(\mathcal{S}^{ki}) - J^{\eta,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{ki} \tag{3.37}$$

The linear functionals J^η, $J^{\eta,q}$ from the equations of motion are given by

$$J^\eta(\mathcal{S}^{ii}) = \mathrm{Tr}_2\Big((R_{12}^{(0),\eta} - r_{12}^{(0)})\mathcal{S}_2^{ii}\Big), \tag{3.38}$$

$$J^{\eta,q}(\mathcal{S}^{ij}) = \mathrm{Tr}_2\Big((R_{12}^{(0),q+\eta} - R_{12}^{(0),q})\mathcal{S}_2^{ij}\Big). \tag{3.39}$$

In the elliptic case the above given Lax pairs and equations of motion reproduce [1] the results of our previous paper [89], and in the non-relativistic limit the results of [30] are reproduced as well. For $N = 1$ the R–matrix operators under consideration become the scalar functions from (3.1), thus reproducing the spin Ruijsenaars–Schneider model (3.20)–(3.26). For $M = 1$ the Lax matrices consists of a single block. In this way we come to relativistic top (3.16)–(3.19). In the non-relativistic elliptic case the models of the above described type were first obtained in [65, 66], they were later described as Hitchin systems on the bundles with non-trivial characteristic classes [44, 90, 92, 41, 42]. Some explicit examples of the systems can be easily obtained using R–matrices used in [30] in the same normalization as in the present article.

3.2 Derivation of equations of motion

3.2.1 R–matrix identities

To derive equations of motion in the spin Ruijsenaars–Schneider model, one should use the identity (3.4). Let us rewrite it in a different manner

$$\phi(z, q_1)\phi(z, q_2) = \phi(z, q_1 + q_2)(E_1(q_1) + E_1(q_2)) - \partial_z\phi(z, q_1 + q_2), \tag{3.40}$$

where we used that (3.1) provides $\partial_z\phi(z, q) = \phi(z, q)(E_1(z + q) - E_1(z))$. Being written in such a form the identity (3.4) possesses R–matrix generalization:

$$R_{12}^z(x)R_{23}^z(y) = R_{13}^z(x + y)r_{12}(x) + r_{23}(y)R_{13}^z(x + y) - \frac{\partial}{\partial z}R_{13}^z(x + y) \tag{3.41}$$

[1] In [89] the elliptic case was described in a slightly different normalization. It differs from the one we use here by $q_j \to q_j/N$. This leads to additional factor $1/N$ in the equations of motion in [89].

Applications of the latter identity can be found in [88]. Let us write down its corollary (3.41):

$$R^z_{12}(q_{ik})R^z_{23}(q_{kj}+\eta) - R^z_{12}(q_{ik}+\eta)R^z_{23}(q_{kj}) = \tag{3.42}$$

$$= R^z_{13}(q_{ij}+\eta)(r_{12}(q_{ik}) - r_{12}(q_{ik}+\eta)) + (r_{23}(q_{kj}+\eta) - r_{23}(q_{kj}))R^z_{13}(q_{ij}+\eta). \tag{3.43}$$

We will use degenerations of (3.41) as well. Consider expansions of its both parts near $x = 0$

$$\Big(\frac{1}{x}P_{12} + R^{(0),z}_{12} + \ldots\Big)R^z_{23}(y) = \Big(R^z_{13}(y) + xF^z_{13}(y) + \ldots\Big)\Big(\frac{1}{x}P_{12} + r^{(0)}_{12} + \ldots\Big) + \tag{3.44}$$

$$+ r_{23}(y)\Big(R^z_{13}(y) + xF^z_{13}(y) + \ldots\Big) - \frac{\partial}{\partial z}\Big(R^z_{13}(y) + xF^z_{13}(y) + \ldots\Big), \tag{3.45}$$

where $F^z_{ab}(y)$ is defined as in (3.32). In the zeroth order in x we have:

$$R^{(0),z}_{12}R^z_{23}(y) = F^z_{13}(y)P_{12} + R^z_{13}(y)r^{(0)}_{12} + r_{23}(y)R^z_{13}(y) - \frac{\partial}{\partial z}R^z_{13}(y). \tag{3.46}$$

Also, by expanding (3.41) near $y = 0$ we similarly obtain

$$R^z_{12}(x)\Big(\frac{1}{y}P_{23} + R^{(0),z}_{23} + \ldots\Big) = \Big(R^z_{13}(x) + yF^z_{13}(x) + \ldots\Big)r_{12}(x) + \tag{3.47}$$

$$+\Big(\frac{1}{y}P_{23} + r^{(0)}_{23} + \ldots\Big)\Big(R^z_{13}(x) + yF^z_{13}(x) + \ldots\Big) - \frac{\partial}{\partial z}\Big(R^z_{13}(x) + yF^z_{13}(x) + \ldots\Big),$$

$$R^z_{12}(x)R^{(0),z}_{23} = R^z_{13}(x)r_{12}(x) + r^{(0)}_{23}R^z_{13}(x) + P_{23}F^z_{13}(x) - \frac{\partial}{\partial z}R^z_{13}(x). \tag{3.48}$$

From (3.46) and (3.48) we deduce

$$R^{(0),z}_{12}R^z_{23}(q_{ij}+\eta) - R^z_{12}(\eta)R^z_{23}(q_{ij}) = \tag{3.49}$$

$$F^z_{13}(q_{ij}+\eta)P_{12} + R^z_{13}(q_{ij}+\eta)(r^{(0)}_{12} - r_{12}(\eta)) + (r_{23}(q_{ij}+\eta) - r_{23}(q_{ij}))R^z_{13}(q_{ij}+\eta).$$

3.2.2 Lax equation

Let us write down the Lax equation with the additional term (3.31) explicitly in terms of $N \times N$ blocks. For the diagonal blocks this yields

$$\dot{\mathcal{L}}_{ii}(z) = \mathcal{L}^{ii}(z)\mathcal{M}^{ii}(z) - \mathcal{M}^{ii}(z)\mathcal{L}^{ii}(z) + \sum_{k\neq i}\Big(\mathcal{L}^{ik}(z)\mathcal{M}^{ki}(z) - \mathcal{M}^{ik}(z)\mathcal{L}^{ki}(z)\Big). \tag{3.50}$$

Similarly, for the off-diagonal part we have

$$\dot{\mathcal{L}}_{ij}(z)=\mathcal{L}^{ii}(z)\mathcal{M}^{ij}(z)-\mathcal{M}^{ii}(z)\mathcal{L}^{ij}(z)+\mathcal{L}^{ij}(z)\mathcal{M}^{jj}(z)-\mathcal{M}^{ij}(z)\mathcal{L}^{jj}(z)+ \\ +\sum_{k\neq i,j}\Big(\mathcal{L}^{ik}(z)\mathcal{M}^{kj}(z)-\mathcal{M}^{ik}(z)\mathcal{L}^{kj}(z)\Big)+(\mu_0^i-\mu_0^j)\mathrm{Tr}_2\Big(F^z_{12}(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ij}\Big). \tag{3.51}$$

Our aim is to show that (3.50) and (3.51) are equivalent to the equations of motion (3.34) and (3.35) respectively. Notice that

$$\operatorname*{res}_{z=0}\mathcal{L}(z)=S=-\operatorname*{res}_{z=0}\mathcal{M}(z)\in\mathrm{Mat}(NM,\mathbb{C}), \tag{3.52}$$

i.e. the second order pole in z is cancelled out in the commutator $[\mathcal{L}(z),\mathcal{M}(z)]$.

Off-diagonal part.

In the l.h.s. of (3.51) we have

$$\dot{\mathcal{L}}^{ij}(z)_1=\mathrm{Tr}_2(F^z_{12}(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ij}\dot q_{ij})+\mathrm{Tr}_2(R^z_{12}(q_{ij}+\eta)P_{12}\dot{\mathcal{S}}_2^{ij}). \tag{3.53}$$

The index 1 in the l.h.s. means that the Lax equation is in the first tensor component. Consider expression in the r.h.s. of (3.51)

$$(\mathcal{L}^{ik}(z)\mathcal{M}^{kj}(z)-\mathcal{M}^{ik}(z)\mathcal{L}^{kj}(z))_1= \tag{3.54}$$

$$=\mathrm{Tr}_{23}\Big(-R^z_{12}(q_{ik}+\eta)P_{12}\mathcal{S}_2^{ik}R^z_{13}(q_{kj})P_{13}\mathcal{S}_3^{kj}+R^z_{12}(q_{ik})P_{12}\mathcal{S}_2^{ik}R^z_{13}(q_{kj}+\eta)P_{13}\mathcal{S}_3^{kj}\Big)= \tag{3.55}$$

$$=\mathrm{Tr}_{23}\Big(\Big(R^z_{12}(q_{ik})R^z_{23}(q_{kj}+\eta)-R^z_{12}(q_{ik}+\eta)R^z_{23}(q_{kj})\Big)P_{12}\mathcal{S}_2^{ik}P_{13}\mathcal{S}_3^{kj}\Big)= \tag{3.56}$$

$$\stackrel{(3.42)}{=}\mathrm{Tr}_{23}\Big(R^z_{13}(q_{ij}+\eta)\Big(r_{12}(q_{ik})-r_{12}(q_{ik}+\eta)\Big)P_{12}\mathcal{S}_2^{ik}P_{13}\mathcal{S}_3^{kj}\Big)+ \tag{3.57}$$

$$+\mathrm{Tr}_{23}\Big(\Big(r_{23}(q_{kj}+\eta)-r_{23}(q_{kj})\Big)R^z_{13}(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ik}P_{13}\mathcal{S}_3^{kj}\Big). \tag{3.58}$$

Two obtained terms are transformed using (3.14)–(3.15) and the permutation operator property $P_{12}U_{12}=U_{21}P_{12}$ (and $P_{12}U_{23}=U_{13}P_{12}$ respectively). Let us transform the first term from the r.h.s. of (3.54):

$$\mathrm{Tr}_{23}\Big(R^z_{13}(q_{ij}+\eta)\Big(r_{12}(q_{ik})-r_{12}(q_{ik}+\eta)\Big)P_{12}\mathcal{S}_2^{ik}P_{13}\mathcal{S}_3^{kj}\Big)= \\ =-\mathrm{Tr}_3\Big(R^z_{13}(q_{ij}+\eta)P_{13}P_{13}\,\mathrm{Tr}_2\Big((R^{(0),q_{ik}+\eta}_{12}-R^{(0),q_{ik}}_{12})\mathcal{S}_2^{ik}\Big)P_{13}\mathcal{S}_3^{kj}\Big)= \\ =-\mathrm{Tr}_3\Big(R^z_{13}(q_{ij}+\eta)P_{13}\mathrm{Tr}_2\Big((R^{(0),q_{ik}+\eta}_{32}-R^{(0),q_{ik}}_{32})\mathcal{S}_2^{ik}\Big)\mathcal{S}_3^{kj}\Big)= \\ =-\mathrm{Tr}_2\Big(R^z_{12}(q_{ij}+\eta)P_{12}\mathrm{Tr}_3\Big((R^{(0),q_{ik}+\eta}_{23}-R^{(0),q_{ik}}_{23})\mathcal{S}_3^{ik}\Big)\mathcal{S}_2^{kj}\Big). \tag{3.59}$$

Using the defintion (3.38) we obtain

$$\mathrm{Tr}_{23}\Big(R^z_{13}(q_{ij}+\eta)\Big(r_{12}(q_{ik})-r_{12}(q_{ik}+\eta)\Big)P_{12}\mathcal{S}^{ik}_2 P_{13}\mathcal{S}^{kj}_3\Big)=$$
$$=-\mathrm{Tr}_2\Big(R^z_{12}(q_{ij}+\eta)P_{12}J^{\eta,q_{ik}}(\mathcal{S}^{ik})_2\mathcal{S}^{kj}_2\Big). \tag{3.60}$$

The second term from the r.h.s. of (3.54) is transformed in a similar manner:

$$\mathrm{Tr}_{23}\Big(\Big(r_{23}(q_{kj}+\eta)-r_{23}(q_{kj})\Big)R^z_{13}(q_{ij}+\eta)P_{12}\mathcal{S}^{ik}_2 P_{13}\mathcal{S}^{kj}_3\Big)=$$
$$=\mathrm{Tr}_{23}\Big(R^z_{13}(q_{ij}+\eta)P_{12}\mathcal{S}^{ik}_2 P_{13}\mathcal{S}^{kj}_3\Big(r_{23}(q_{kj}+\eta)-r_{23}(q_{kj})\Big)\Big)=$$
$$=\mathrm{Tr}_{23}\Big(R^z_{13}(q_{ij}+\eta)P_{13}P_{23}\mathcal{S}^{ik}_2\mathcal{S}^{kj}_3\Big(R^{(0),q_{kj}+\eta}_{23}-R^{(0),q_{kj}}_{23}\Big)P_{23}\Big)=$$

$$=\mathrm{Tr}_{23}\Big(P_{23}R^z_{13}(q_{ij}+\eta)P_{13}P_{23}\mathcal{S}^{ik}_2\mathcal{S}^{kj}_3\Big(R^{(0),q_{kj}+\eta}_{23}-R^{(0),q_{kj}}_{23}\Big)\Big)=$$
$$=\mathrm{Tr}_{23}\Big(R^z_{12}(q_{ij}+\eta)P_{12}\mathcal{S}^{ik}_2\mathcal{S}^{kj}_3\Big(R^{(0),q_{kj}+\eta}_{23}-R^{(0),q_{kj}}_{23}\Big)\Big)=$$
$$=\mathrm{Tr}_2\Big(R^z_{12}(q_{ij}+\eta)P_{12}\mathcal{S}^{ik}_2\mathrm{Tr}_3\Big(\mathcal{S}^{kj}_3(R^{(0),q_{kj}+\eta}_{23}-R^{(0),q_{kj}}_{23})\Big)\Big)=$$
$$=\mathrm{Tr}_2\Big(R^z_{12}(q_{ij}+\eta)P_{12}\mathcal{S}^{ik}_2 J^{\eta,q_{kj}}(\mathcal{S}^{kj})_2\Big). \tag{3.61}$$

Finally, from (3.60) and (3.61) we get the following answer for the initial expression (3.54):

$$(\mathcal{L}^{ik}(z)\mathcal{M}^{kj}(z)-\mathcal{M}^{ik}(z)\mathcal{L}^{kj}(z))_1=$$
$$=\mathrm{Tr}_2\Big(R^z_{12}(q_{ij}+\eta)P_{12}\Big(\mathcal{S}^{ik}J^{\eta,q_{kj}}(\mathcal{S}^{kj})-J^{\eta,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{kj}\Big)_2\Big). \tag{3.62}$$

Next, consider the following expression from (3.51):

$$(\mathcal{L}^{ii}(z)\mathcal{M}^{ij}(z)-\mathcal{M}^{ii}(z)\mathcal{L}^{ij}(z))_1=$$
$$=\mathrm{Tr}_{23}\Big(-R^z_{12}(\eta)P_{12}\mathcal{S}^{ii}_2R^z_{13}(q_{ij})P_{13}\mathcal{S}^{ij}_3+R^{(0),z}_{12}P_{12}\mathcal{S}^{ii}_2R^z_{13}(q_{ij}+\eta)P_{13}\mathcal{S}^{ij}_3\Big)=$$
$$=\mathrm{Tr}_{23}\Big(\Big(R^{(0),z}_{12}R^z_{23}(q_{ij}+\eta)-R^z_{12}(\eta)R^z_{23}(q_{ij})\Big)P_{12}\mathcal{S}^{ii}_2P_{13}\mathcal{S}^{ij}_3\Big). \tag{3.63}$$

Apply relation (3.49):

$$(\mathcal{L}^{ii}(z)\mathcal{M}^{ij}(z)-\mathcal{M}^{ii}(z)\mathcal{L}^{ij}(z))_1=\mathrm{Tr}_{23}\Big(F^z_{13}(q_{ij}+\eta)P_{12}P_{12}\mathcal{S}^{ii}_2P_{13}\mathcal{S}^{ij}_3\Big)+$$
$$+\mathrm{Tr}_{23}\Big(R^z_{13}(q_{ij}+\eta)\Big(r^{(0)}_{12}-r_{12}(\eta)\Big)P_{12}\mathcal{S}^{ii}_2P_{13}\mathcal{S}^{ij}_3\Big)+$$
$$+\mathrm{Tr}_{23}\Big(\Big(r_{23}(q_{ij}+\eta)-r_{23}(q_{ij})\Big)R^z_{13}(q_{ij}+\eta)P_{12}\mathcal{S}^{ii}_2P_{13}\mathcal{S}^{ij}_3\Big). \tag{3.64}$$

Let us simplify all three terms in the r.h.s. of (3.64). Transform the first term:

$$
\begin{aligned}
\mathrm{Tr}_{23}\Big(F^z_{13}(q_{ij}+\eta)P_{12}P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ij}_3\Big) = \mathrm{Tr}_2\Big(\mathcal{S}^{ii}_2\Big)\mathrm{Tr}_3\Big(F^z_{13}(q_{ij}+\eta)P_{13}\mathcal{S}^{ij}_3\Big) = \\
= \mathrm{Tr}\mathcal{S}^{ii}\cdot\mathrm{Tr}_2\Big(F^z_{12}(q_{ij}+\eta)P_{12}\mathcal{S}^{ij}_2\Big).
\end{aligned}
\tag{3.65}
$$

The third term is already known:

$$
\begin{aligned}
\mathrm{Tr}_{23}\Big(\Big(r_{23}(q_{ij}+\eta)-r_{23}(q_{ij})\Big)R^z_{13}(q_{ij}+\eta)P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ij}_3\Big) = \\
= \mathrm{Tr}_2\Big(R^z_{12}(q_{ij}+\eta)P_{12}\mathcal{S}^{ii}_2 J^{\eta,q_{ij}}(\mathcal{S}^{ij})_2\Big).
\end{aligned}
\tag{3.66}
$$

For the second term from the r.h.s. of (3.64) we obtain

$$
\begin{aligned}
\mathrm{Tr}_{23}\Big(R^z_{13}(q_{ij}+\eta)\Big(r^{(0)}_{12}-r_{12}(\eta)\Big)P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ij}_3\Big) = \\
= \mathrm{Tr}_{23}\Big(R^z_{13}(q_{ij}+\eta)P_{13}\Big(r^{(0)}_{32}-r_{32}(\eta)\Big)P_{32}\mathcal{S}^{ii}_2\mathcal{S}^{ij}_3\Big) = \\
= \mathrm{Tr}_2\Big(R^z_{12}(q_{ij}+\eta)P_{12}\mathrm{Tr}_3\Big\{\Big(r^{(0)}_{23}-r_{23}(\eta)\Big)P_{23}\mathcal{S}^{ii}_3\Big\}\mathcal{S}^{ij}_2\Big) = \\
= -\mathrm{Tr}_2\Big(R^z_{12}(q_{ij}+\eta)P_{12}J^\eta(\mathcal{S}^{ii})_2\mathcal{S}^{ij}_2\Big).
\end{aligned}
\tag{3.67}
$$

Thus, the expression (3.64) takes the form:

$$
\begin{aligned}
(\mathcal{L}^{ii}(z)\mathcal{M}^{ij}(z)-\mathcal{M}^{ii}(z)\mathcal{L}^{ij}(z))_1 = \\
= \mathrm{Tr}_2\Big(R^z_{12}(q_{ij}+\eta)P_{12}\Big(\mathcal{S}^{ii}J^{\eta,q_{ij}}(\mathcal{S}^{ij})-J^\eta(\mathcal{S}^{ii}\Big)\mathcal{S}^{ij})_2\Big)+ \\
+\mathrm{Tr}\mathcal{S}^{ii}\cdot\mathrm{Tr}_2\Big(F^z_{12}(q_{ij}+\eta)P_{12}\mathcal{S}^{ij}_2\Big).
\end{aligned}
\tag{3.68}
$$

One more expression $\mathcal{L}^{ij}(z)\mathcal{M}^{jj}(z)-\mathcal{M}^{ij}(z)\mathcal{L}^{jj}(z)$ from (3.51) is transformed similarly to (3.64). This yields

$$
\begin{aligned}
(\mathcal{L}^{ij}(z)\mathcal{M}^{jj}(z)-\mathcal{M}^{ij}(z)\mathcal{L}^{jj}(z))_1 = \\
= \mathrm{Tr}_2\Big(R^z_{12}(q_{ij}+\eta)P_{12}\Big(\mathcal{S}^{ij}J^\eta(\mathcal{S}^{jj})-J^{\eta,q_{ij}}(\mathcal{S}^{ij})\mathcal{S}^{jj}\Big)_2\Big)- \\
-\mathrm{Tr}\mathcal{S}^{jj}\cdot\mathrm{Tr}_2\Big(F^z_{12}(q_{ij}+\eta)P_{12}\mathcal{S}^{ij}_2\Big).
\end{aligned}
\tag{3.69}
$$

Collecting the terms (3.62), (3.68) and (3.69) gives the following answer for ij–block of the commutator:

$$
\begin{gathered}
([\mathcal{L}(z),\mathcal{M}(z)]^{ij})_1=(\mathrm{Tr}\mathcal{S}^{ii}-\mathrm{Tr}\mathcal{S}^{jj})\cdot\mathrm{Tr}_2(F^z_{12}(q_{ij}+\eta)P_{12}\mathcal{S}^{ij}_2)+\mathrm{Tr}_2(R^z_{12}(q_{ij}+\eta)P_{12}A_2), \quad (3.70)\\
A=\mathcal{S}^{ii}J^{\eta,q_{ij}}(\mathcal{S}^{ij})-J^{\eta}(\mathcal{S}^{ii})\mathcal{S}^{ij}+\mathcal{S}^{ij}J^{\eta}(\mathcal{S}^{jj})-J^{\eta,q_{ij}}(\mathcal{S}^{ij})\mathcal{S}^{jj}+\\
+\sum_{k\neq i,j}\Big(\mathcal{S}^{ik}J^{\eta,q_{kj}}(\mathcal{S}^{kj})-J^{\eta,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{kj}\Big).
\end{gathered}
$$

Also, taking into account the last term (with μ_i^0) in the r.h.s. of (3.51), we get (3.21) in the form

$$
\begin{gathered}
\dot{\mathcal{S}}^{ij}=\mathcal{S}^{ii}J^{\eta,q_{ij}}(\mathcal{S}^{ij})-J^{\eta}(\mathcal{S}^{ii})\mathcal{S}^{ij}+\mathcal{S}^{ij}J^{\eta}(\mathcal{S}^{jj})-J^{\eta,q_{ij}}(\mathcal{S}^{ij})\mathcal{S}^{jj}+\\
+\sum_{k\neq i,j}^{M}\Big(\mathcal{S}^{ik}J^{\eta,q_{kj}}(\mathcal{S}^{kj})-J^{\eta,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{kj}\Big).
\end{gathered}
\tag{3.71}
$$

Let us comment on transition from (3.70) to (3.71). Strictly speaking, we have proved that the Lax representation holds true on the equations of motion but we have not proved the inverse. In order to prove the inverse statement we need to see that all components of the matrix equation (3.71) are contained in (3.70) independently taking also into account that R_{12} is a linear operator, which may mix these components somehow. Put it differently, we need to show that $\mathrm{Tr}_2(R^z_{12}(q_{ij}+\eta)P_{12}C_2)=0$ leads to $C=0$. For this purpose consider the Lax equation near $z=0$. It follows from (3.12)–(3.14) that $R^z_{12}(q_{ij}+\eta)P_{12}$ has a simple pole in $z=0$ and the residue is equal to P_{12}. Then the desired statement follows from $\mathrm{Tr}_2(P_{12}A_2)=A$.

Diagonal part.

Consider now the equation (3.50), which l.h.s. is of the form:

$$
\dot{\mathcal{L}}_{ii}(z)_1=\mathrm{Tr}_2(R^z_{12}(\eta)P_{12}\dot{\mathcal{S}}^{ii}_2).
\tag{3.72}
$$

In the r.h.s. we transform the expression under sum using (3.42):

$$
\begin{aligned}
&(\mathcal{L}^{ik}(z)\mathcal{M}^{ki}(z)-\mathcal{M}^{ik}(z)\mathcal{L}^{ki}(z))_1 = \\
&= \mathrm{Tr}_{23}\Big(-R^z_{12}(q_{ik}+\eta)P_{12}\mathcal{S}^{ik}_2 R^z_{13}(q_{ki})P_{13}\mathcal{S}^{ki}_3 + R^z_{12}(q_{ik})P_{12}\mathcal{S}^{ik}_2 R^z_{13}(q_{ki}+\eta)P_{13}\mathcal{S}^{ki}_3\Big) = \\
&= \mathrm{Tr}_{23}\Big(\Big(R^z_{12}(q_{ik})R_{23}(q_{ki}+\eta) - R^z_{12}(q_{ik}+\eta)R^z_{23}(q_{ki})\Big)P_{12}\mathcal{S}^{ik}_2 P_{13}\mathcal{S}^{ki}_3) = \\
&\overset{(3.42)}{=} \mathrm{Tr}_{23}\Big(R^z_{13}(\eta)\Big(r_{12}(q_{ik})-r_{12}(q_{ik}+\eta)\Big)P_{12}\mathcal{S}^{ik}_2 P_{13}\mathcal{S}^{ki}_3\Big) + \\
&+\mathrm{Tr}_{23}\Big(\Big(r_{23}(q_{ki}+\eta)-r_{23}(q_{ki})\Big)R^z_{13}(\eta)P_{12}\mathcal{S}^{ik}_2 P_{13}\mathcal{S}^{ki}_3\Big) = \\
&= \mathrm{Tr}_2\Big(R^z_{12}(\eta)P_{12}(\mathcal{S}^{ik}J^{\eta,q_{ki}}(\mathcal{S}^{ki}) - J^{\eta,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{ki})_2\Big).
\end{aligned}
\tag{3.73}
$$

The rest of the expression in the r.h.s. of (3.50) is simplified via (3.49):

$$
\begin{aligned}
&(\mathcal{L}^{ii}(z)\mathcal{M}^{ii}(z)-\mathcal{M}^{ii}(z)\mathcal{L}^{ii}(z))_1 = \\
&= \mathrm{Tr}_{23}\Big(-R^z_{12}(\eta)P_{12}\mathcal{S}^{ii}_2 R^{(0),z}_{13}P_{13}\mathcal{S}^{ii}_3 + R^{(0),z}_{12}P_{12}\mathcal{S}^{ii}_2 R^z_{13}(\eta)P_{13}\mathcal{S}^{ii}_3\Big) = \\
&= \mathrm{Tr}_{23}\Big(\Big(R^{(0),z}_{12}R^z_{23}(\eta) - R^z_{12}(\eta)R^{(0),z}_{23}\Big)P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ii}_3\Big) = \\
&\overset{(3.49)}{=} \mathrm{Tr}_{23}\Big(R^z_{13}(\eta)\Big(r^{(0)}_{12}-r_{12}(\eta)\Big)P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ii}_3\Big) + \\
&+\mathrm{Tr}_{23}\Big(\Big(r_{23}(\eta)-r^{(0)}_{23}\Big)R^z_{13}(\eta)P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ii}_3\Big) + \\
&+\mathrm{Tr}_{23}\Big(F^z_{13}(\eta)P_{12}P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ii}_3\Big) - \mathrm{Tr}_{23}\Big(P_{23}F^z_{13}(\eta)P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ii}_3\Big).
\end{aligned}
\tag{3.74}
$$

Notice that the two last terms are equal, so that they are cancelled out:

$$
\begin{aligned}
&\mathrm{Tr}_{23}\Big(F^z_{13}(\eta)P_{12}P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ii}_3\Big) = \mathrm{Tr}_{23}\Big(P_{23}F^z_{13}(\eta)P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ii}_3\Big) = \\
&= \mathrm{Tr}\mathcal{S}^{ii}\cdot \mathrm{Tr}_2\Big(F^z_{12}(\eta)P_{12}\mathcal{S}^{ii}_2\Big).
\end{aligned}
\tag{3.75}
$$

The first and the second terms from (3.74) are of the form

$$
\begin{aligned}
&\mathrm{Tr}_{23}\Big(R^z_{13}(\eta)\Big(r^{(0)}_{12}-r_{12}(\eta)\Big)P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ii}_3\Big) + \\
&+\mathrm{Tr}_{23}\Big(\Big(r_{23}(\eta)-r^{(0)}_{23}\Big)R^z_{13}(\eta)P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ii}_3\Big) = \\
&= \mathrm{Tr}_2\Big(R^z_{12}(\eta)P_{12}\Big(J^\eta(\mathcal{S}^{ii})\mathcal{S}^{ii} - \mathcal{S}^{ii}J^\eta(\mathcal{S}^{ii})\Big)_2\Big).
\end{aligned}
\tag{3.76}
$$

Finally, from (3.73) and (3.76) we get the answer

$$([L(z),M(z)]_{ii})_1=\mathrm{Tr}_2\Big(R^z_{12}(\eta)P_{12}B_2\Big), \tag{3.77}$$

where

$$B=J^\eta(\mathcal{S}^{ii})\mathcal{S}^{ii}-\mathcal{S}^{ii}J^\eta(\mathcal{S}^{ii})+\sum_{k\neq i}\Big(\mathcal{S}^{ik}J^{\eta,q_{ki}}(\mathcal{S}^{ki})-J^{\eta,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{ki}\Big). \tag{3.78}$$

Here one should also use the argument given after (3.71). Thus, the equations of motion are verified for the diagonal blocks.

3.2.3 Interacting tops

As was explained in [89], in the particular case $\mathrm{rk}(\mathcal{S})=1$ the equations of motion can be written in terms of the diagonal blocks only. Let us recall the main idea. The property $\mathrm{rk}(\mathcal{S})=1$ yields

$$\mathcal{S}^{ik}_1P_{12}\mathcal{S}^{ki}_1=\mathcal{S}^{ii}_1\mathcal{S}^{kk}_2. \tag{3.79}$$

Next, for an arbitrary $J(S)=\mathrm{Tr}_2(J_{12}S_2)$ of the form (3.11) and $\breve{J}_{12}=J_{12}P_{12}$ we have

$$\begin{aligned}J(S)=\mathrm{Tr}_2(J_{12}S_2)=\mathrm{Tr}_2(\breve{J}_{12}P_{12}S_2)=\mathrm{Tr}_2(S_2\breve{J}_{12}P_{12})=\\ =\mathrm{Tr}_2(S_2P_{12}\breve{J}_{21})=\mathrm{Tr}_2(P_{12}S_1\breve{J}_{21}).\end{aligned} \tag{3.80}$$

Therefore

$$\mathcal{S}^{ik}J(\mathcal{S}^{ki})=\mathrm{Tr}_2\Big(\mathcal{S}^{ik}_1P_{12}\mathcal{S}^{ki}_1\breve{J}_{21}\Big)=\mathcal{S}^{ii}\mathrm{Tr}_2\Big(\breve{J}_{21}\mathcal{S}^{kk}_2\Big), \tag{3.81}$$

where

$$\breve{J}_{21}=P_{12}\breve{J}_{21}P_{12}=P_{12}J_{12}. \tag{3.82}$$

In the same way

$$J(\mathcal{S}^{ik})\mathcal{S}^{ki}=\mathrm{Tr}_2\Big(\breve{J}_{12}\mathcal{S}^{ik}_1P_{12}\mathcal{S}^{ki}_1\Big)=\mathrm{Tr}_2\Big(\breve{J}_{12}\mathcal{S}^{kk}_2\Big)\mathcal{S}^{ii}. \tag{3.83}$$

Finally, equations (3.34) and (3.37) are written in the form

$$\dot{\mathcal{S}}^{ii}=[\mathcal{S}^{ii},J^\eta(\mathcal{S}^{ii})]+\sum_{k\neq i}^{M}\Big(\mathcal{S}^{ii}\widetilde{J}^{\eta,q_{ki}}(S^{kk})-\breve{J}^{\eta,q_{ik}}(\mathcal{S}^{kk})S^{ii}\Big), \tag{3.84}$$

$$\ddot{q}_i=\mathrm{Tr}\Big(\dot{\mathcal{S}}^{ii}\Big)=\sum_{k\neq i}^{M}\mathrm{Tr}\Big(\mathcal{S}^{ii}\widetilde{J}^{\eta,q_{ki}}(S^{kk})-\breve{J}^{\eta,q_{ik}}(\mathcal{S}^{kk})S^{ii}\Big), \tag{3.85}$$

where

$$\widetilde{J}^{\eta,q_{ki}}(\mathcal{S}^{kk}) = \mathrm{Tr}_2\Big(\breve{J}_{21}^{\eta,q_{ki}}\mathcal{S}_2^{kk}\Big) = \mathrm{Tr}_2\Big(P_{12}J_{12}^{\eta,q_{ki}}\mathcal{S}_2^{kk}\Big), \tag{3.86}$$

$$\breve{J}^{\eta,q_{ik}}(\mathcal{S}^{kk}) = \mathrm{Tr}_2\Big(\breve{J}_{12}^{\eta,q_{ik}}\mathcal{S}_2^{kk}\Big) = \mathrm{Tr}_2\Big(J_{12}^{\eta,q_{ik}}P_{12}\mathcal{S}_2^{kk}\Big). \tag{3.87}$$

Being written in the form (3.84)–(3.85) the equations of motion can be treated as dynamics of M particles bearing additional "spin" type degrees of freedom, i.e. the particles can be identified with the tops, which have also positions and velocities besides their own internal degrees of freedom. The interaction between the tops depends on the distance and on the spin dynamical variables.

Chapter 4

Quantum $GL(NM)$ R-matrix and quantum algebra

This chapter is based on our papers [74, 76] and is devoted to the quantization of the structures described in the previous chapters.

In the first part of the chapter, the generalized quantum GL_{NM} dynamical R-matrix is constructed using the GL_N solution of the associative Yang–Baxter equation. This quantum R-matrix is a quantization of the classical r-matrix for the generalized interacting integrable tops systems introduced in the previous sections. The quantum dynamical Yang–Baxter equation in this case can be also considered as the quantum version of the classical dynamical Yang–Baxter equation for the interacting tops models.

In the $N = 1$ case the obtained answer reproduces the GL_M-valued Felder's dynamical R-matrix, while in the $M = 1$ case it provides the GL_N nondynamical R-matrix of vertex type including the Baxter–Belavin's elliptic one and its degenerations.

In the second part of the chapter, the quadratic quantum algebra has been constructed starting from the dynamical RLL-relation, which corresponds to the elliptic version of R-matrices introduced above.

This quantum R–matrix is related to the $SL(NM)$–bundles over the elliptic curve with nontrivial characteristic class and generalizes simultaneously the elliptic nondynamical Baxter–Belavin and dynamical Felder R–matrices, and the obtained quadratic relations generalize both the Sklyanin algebra relations and the relations in the Felder–Tarasov–Varchenko elliptic quantum group, coinsiding with them in the particular cases $M = 1$ and $N = 1$ respectively.

4.1 Quantum dynamical $GL(NM)$ R-matrix

Yang–Baxter equations. Consider a matrix-valued function $R^{\hbar}_{12}(z) \in \mathrm{Mat}(N,\mathbb{C})^{\otimes 2}$, which solves the associative Yang–Baxter equation [27, 61]:

$$R^{\hbar}_{12}(z_{12})R^{\eta}_{23}(z_{23}) = R^{\eta}_{13}(z_{13})R^{\hbar-\eta}_{12}(z_{12}) + R^{\eta-\hbar}_{23}(z_{23})R^{\hbar}_{13}(z_{13}), \quad z_{ab} = z_a - z_b. \tag{4.1}$$

Let the solution of (4.1) satisfy also the properties of skew-symmetry

$$R^{\hbar}_{12}(z) = -R^{-\hbar}_{21}(-z) = -P_{12}R^{-\hbar}_{12}(-z)P_{12}, \qquad P_{12} = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} \tag{4.2}$$

and unitarity

$$R^{\hbar}_{12}(z)R^{\hbar}_{21}(-z) = (\wp(\hbar) - \wp(z))1_N \otimes 1_N, \tag{4.3}$$

where $\wp(x)$ — is the Weierstrass $\wp$-function. We assume that it is equal to $1/\sinh^2(x)$ or $1/x^2$ for trigonometric (hyperbolic) or rational R-matrices respectively. Notice that solution of (4.1) with the properties (4.2)–(4.3) is a true quantum R-matrix of vertex type, i.e. it satisfies the quantum (non-dynamical) Yang–Baxter equation [1]:

$$R^{\hbar}_{12}(z_{12})R^{\hbar}_{13}(z_{13})R^{\hbar}_{23}(z_{23}) = R^{\hbar}_{23}(z_{23})R^{\hbar}_{13}(z_{13})R^{\hbar}_{12}(z_{12}). \tag{4.4}$$

Equation (4.1) can be viewed as the matrix extension of the genus one Fay trisecant identity:

$$\phi(\hbar, z_{12})\phi(\eta, z_{23}) = \phi(\eta, z_{13})\phi(\hbar-\eta, z_{12}) + \phi(\eta-\hbar, z_{23})\phi(\hbar, z_{13}), \tag{4.5}$$

which coincides with (4.1) in scalar ($N = 1$) case. It plays a crucial role in the theory of classical and quantum integrable systems [37, 78, 12]. Solution of (4.5) satisfying the (scalar versions of) properties (4.2)–(4.3) is the Kronecker function:

$$\phi(\hbar, z) = \frac{\vartheta'(0)\vartheta(\hbar+z)}{\vartheta(\hbar)\vartheta(z)}, \quad \vartheta(x) = \sum_{k\in\mathbb{Z}} \exp\left(\pi i\tau(k+\frac{1}{2})^2 + 2\pi i(x+\frac{1}{2})(k+\frac{1}{2})\right), \tag{4.6}$$

where $\mathrm{Im}(\tau) > 0$. Its trigonometric and rational limits are given by $\coth(\hbar) + \coth(z)$ and $\hbar^{-1} + z^{-1}$ respectively. The properties of this function is given in the Appendix. Similarly, the elliptic solution of (4.1) with properties (4.2)–(4.3) is known [61] to be given by the Baxter–Belavin's R-matrix [6, 7]. The trigonometric solutions were classified in [72, 62]. They include the XXZ R-matrix, its 7-vertex deformation [16] and their GL_N generalizations [2] (see a brief

[1] The latter statement is easily verified. See e.g. [46, 48, 88].

review in [36]). The rational solutions consist of the XXX R-matrix, its 11-vertex deformation [16] and their GL_N generalizations [80, 45] — deformations of the GL_N Yang's R-matrix $R^{\hbar}_{12}(z) = \hbar^{-1}1_N \otimes 1_N + z^{-1}P_{12}$. Summarizing, we deal with the R-matrices considered as matrix generalizations of the Kronecker function (including its trigonometric and rational versions).

To formulate the main result we also need the Felder's dynamical GL_M R-matrix [26]:

$$\begin{gathered} R^{\mathrm{F}}_{12}(\hbar, z_1, z_2 \mid q) = R^{\mathrm{F}}_{12}(\hbar, z_1 - z_2 \mid q) = \\ = \phi(\hbar, z_1 - z_2) \sum_{i=1}^{M} E_{ii} \otimes E_{ii} + \sum_{\substack{i,j \\ i \neq j}}^{M} E_{ij} \otimes E_{ji} \phi(z_1 - z_2, q_{ij}) + \sum_{\substack{i,j \\ i \neq j}}^{M} E_{ii} \otimes E_{jj} \phi(\hbar, -q_{ij}), \end{gathered} \tag{4.7}$$

where $q_1, \ldots, q_M$ – are (free) dynamical parameters,

$$q_{ij} = q_i - q_j \tag{4.8}$$

and the set $\{E_{ij}\}$ is the standard basis in $\mathrm{Mat}(M, \mathbb{C})$.

The R-matrix (4.7) is a solution of the quantum dynamical Yang–Baxter equation:

$$\begin{gathered} R^{\hbar}_{12}(z_1, z_2 \mid q) R^{\hbar}_{13}(z_1, z_3 \mid, q - \hbar^{(2)}) R^{\hbar}_{23}(z_2, z_3 \mid q) = \\ = R^{\hbar}_{23}(z_2, z_3 \mid q - \hbar^{(1)}) R^{\hbar}_{13}(z_1, z_3 \mid q) R^{\hbar}_{12}(z_1, z_2 \mid q - \hbar^{(3)}), \end{gathered} \tag{4.9}$$

where the shifts of the dynamical arguments $\{q_i\}$ are performed as follows:

$$R^{\hbar}_{12}(z_1, z_2 \mid q + \hbar^{(3)}) = P^{\hbar}_3 \; R^{\hbar}_{12}(z_1, z_2 \mid q) \; P^{-\hbar}_3 \quad P^{\hbar}_3 = \sum_{k=1}^{M} 1_M \otimes 1_M \otimes E_{kk} \exp\left(\hbar \frac{\partial}{\partial q_k}\right). \tag{4.10}$$

Quantum dynamical GL_{NM} R-matrix. Consider the following $\mathrm{Mat}(NM, \mathbb{C})$–valued expression:

$$\begin{gathered} \mathbf{R}^{\hbar}_{1'2'12}(z, w) = \sum_{i=1}^{M} \overset{1'}{E}_{ii} \otimes \overset{2'}{E}_{ii} \otimes R^{\hbar}_{12}(z - w) + \sum_{\substack{i,j \\ i \neq j}}^{M} \overset{1'}{E}_{ij} \otimes \overset{2'}{E}_{ji} \otimes R^{q_{ij}}_{12}(z - w) + \\ + \sum_{\substack{i,j \\ i \neq j}}^{M} \overset{1'}{E}_{ii} \otimes \overset{2'}{E}_{jj} \otimes \overset{1}{1}_N \otimes \overset{2}{1}_N \;\; \phi(\hbar, -q_{ij}), \end{gathered} \tag{4.11}$$

where the $\mathrm{Mat}(NM, \mathbb{C})$ indices are represented in a way that the $\mathrm{Mat}(M, \mathbb{C})$–valued tensor components are numbered by the primed numbers, and the $\mathrm{Mat}(N, \mathbb{C})$–valued components are those without primes (as previously). Put it differently, the indices are arranged through $\mathrm{Mat}(NM, \mathbb{C}^{\otimes 2} \cong \mathrm{Mat}(M, \mathbb{C}^{\otimes 2} \otimes \mathrm{Mat}(N, \mathbb{C}^{\otimes 2}$. The order of tensor components is, in fact, not important. It is chosen as in (4.11) just to emphasize its similarity with the Felder's R-matrix (4.7). The latter is reproduced from (4.11) in the $N = 1$ case, when the GL_N R-matrix entering (4.11) turns into the Kronecker function (4.6).

The results of this section are summarized in the following theorem.

Theorem.

Let $R^{\hbar}_{12}(z)$ be some GL_N quantum non-dynamical R-matrix satisfying the associative Yang–Baxter equation (4.1) and the properties (4.2)–(4.3). Then the expression (4.11) is a quantum dynamical R-matrix, i.e. it satisfies the quantum dynamical Yang–Baxter equation:

$$\mathbf{R}^{\hbar}_{1'2'12}(z_1, z_2 \mid q)\mathbf{R}^{\hbar}_{1'3'13}(z_1, z_3 \mid q - \hbar^{(2)})\mathbf{R}^{\hbar}_{2'3'23}(z_2, z_3 \mid q) = \tag{4.12}$$

$$= \mathbf{R}^{\hbar}_{2'3'23}(z_2, z_3 \mid q - \hbar^{(1)})\mathbf{R}^{\hbar}_{1'3'13}(z_1, z_3 \mid q)\mathbf{R}^{\hbar}_{1'2'12}(z_1, z_2 \mid q - \hbar^{(3)}), \tag{4.13}$$

where the shifts of arguments $\{q_i\}$ are performed similarly to (4.10):

$$\mathbf{R}^{\hbar}_{1'2'12}(z_1, z_2 \mid q + \hbar^{(3)}) = \mathbf{P}^{\hbar}_{3'}\ \mathbf{R}^{\hbar}_{1'2'12}(z_1, z_2 \mid q)\ \mathbf{P}^{-\hbar}_{3'}, \tag{4.14}$$

$$\mathbf{P}^{\hbar}_{3'} = \sum_{k=1}^{M} \overset{1'}{1}_M \otimes \overset{2'}{1}_M \otimes \overset{3'}{E}_{kk} \otimes \overset{1}{1}_N \otimes \overset{2}{1}_N \otimes \overset{3}{1}_N \exp\left(\hbar \frac{\partial}{\partial q_k}\right).$$

<u>*Proof:*</u> It is useful to write (4.1) as

$$R^{\hbar}_{ab}(z_{ab})R^{\eta}_{bc}(z_{bc}) = R^{\eta-\hbar}_{bc}(z_{bc})R^{\hbar}_{ac}(z_{ac}) + R^{\eta}_{ac}(z_{ac})R^{\hbar-\eta}_{ab}(z_{ab}), \tag{4.15}$$

where a, b, c are distinct numbers from the set $\{1, 2, 3\}$. Besides (4.15) and the properties (4.2)–(4.3) the proof of (4.12) uses the Yang–Baxter equation (4.4) for the GL_N R-matrix and the following cubic relation:

$$R^{\hbar}_{ab}(z_{ab})R^{\eta}_{ac}(z_{ac})R^{\hbar}_{bc}(z_{bc}) - R^{\eta}_{bc}(z_{bc})R^{\hbar}_{ac}(z_{ac})R^{\eta}_{ab}(z_{ab}) = R^{\hbar+\eta}_{ac}(z_{ac})(\wp(\hbar) - \wp(\eta)), \tag{4.16}$$

which is true under hypothesis of the theorem. If $\hbar = \eta$ it reduces to (4.4). In the general case (4.16) leads (due to skew-symmetry of its r.h.s.) to

$$R^{\eta}_{ab}R^{\hbar}_{ac}R^{\eta}_{bc} + R^{\hbar}_{ab}R^{\eta}_{ac}R^{\hbar}_{bc} = R^{\eta}_{bc}R^{\hbar}_{ac}R^{\eta}_{ab} + R^{\hbar}_{bc}R^{\eta}_{ac}R^{\hbar}_{ab}, \quad R^{\hbar}_{ab} = R^{\hbar}_{ab}(z_a - z_b), \tag{4.17}$$

known as the Yang–Baxter equation with two Planck constants [50]. The verification of (4.12) is a straightforward but cumber some calculation. Consider, for example, the equation arising in the tensor component $\overset{1'}{E}_{ij} \otimes \overset{2'}{E}_{kk} \otimes \overset{3'}{E}_{ji}$ with $i \neq j \neq k \neq i$:

$$R_{12}^{q_{ik}}(z_{12})R_{13}^{q_{kj}}(z_{13})R_{23}^{q_{ik}}(z_{23}) + \phi(\hbar, q_{ik})\phi(\hbar, q_{ki})R_{13}^{q_{ij}}(z_{13}) = \\ = R_{23}^{q_{kj}}(z_{23})R_{13}^{q_{ik}}(z_{13})R_{12}^{q_{kj}}(z_{12}) + \phi(\hbar, q_{kj})\phi(\hbar, q_{jk})R_{13}^{q_{ij}}(z_{13}). \tag{4.18}$$

To prove it one should use (4.16) written in the form

$$R_{12}^{q_{ik}}(z_{12})R_{13}^{q_{kj}}(z_{13})R_{23}^{q_{ik}}(z_{23}) - R_{23}^{q_{kj}}(z_{23})R_{13}^{q_{ik}}(z_{13})R_{12}^{q_{kj}}(z_{12}) = \\ = (\wp(q_{ik}) - \wp(q_{kj}))R_{13}^{q_{ik}+q_{kj}}(z_{13}) = (\wp(q_{ik}) - \wp(q_{kj}))R_{13}^{q_{ij}}(z_{13}) \tag{4.19}$$

and the well-known property of the Kronecker function (scalar version of the unitarity) (see also the Appendix)

$$\phi(\hbar, q_{ik})\phi(\hbar, q_{ki}) = \wp(\hbar) - \wp(q_{ik}), \qquad \phi(\hbar, q_{kj})\phi(\hbar, q_{jk}) = \wp(\hbar) - \wp(q_{kj}). \tag{4.20}$$

The rest of the tensor components are verified similarly.

In the elliptic case, when $R_{12}^{\hbar}(z)$ is the Baxter–Belavin's R-matrix, the result of the theorem is known [44]. Similar results for the classical r-matrices were obtained previously by P. Etingof and O. Schiffmann [19, 20] and later in [41, 42, 43, 92], where the Hitchin type systems were described on the Higgs bundles with non-trivial characteristic classes. Recently, these type models appeared in the context of R-matrix valued Lax pairs and quantum long-range spin chains [31, 73, 30]. In [44] the answer (4.11) was verified explicitly in the elliptic case without use of the associative Yang–Baxter equation. In this respect the approach of this work provides much simpler proof. What is more important, the answer (4.11) is also valid for all trigonometric and rational degenerations of the elliptic R-matrix (satisfying the properties required in the Theorem). In the light of results of [30] the R-matrix (4.11) is the one necessary for quantization of the (generalized) model of interacting tops.

Classical $\mathfrak{gl}_{NM}$ r-matrix. As a by-product of the Theorem we also get the classical dynamical Yang–Baxter equation for the classical r-matrix of the generalized interacting tops [30]. Consider the classical limit of the GL_N R-matrix from the Theorem:

$$R_{12}^{\hbar}(z) = \hbar^{-1} 1_N \otimes 1_N + r_{12}(z) + O(\hbar). \tag{4.21}$$

The coefficient $r_{12}(z)$ is the classical r-matrix, and the quantum Yang–Baxter equation (4.4) reduces in the limit (4.21) to the classical (non-dynamical) Yang–Baxter equation:

$$[r_{12}(z_{12}), r_{13}(z_{13})] + [r_{12}(z_{12}), r_{23}(z_{23})] + [r_{13}(z_{13}), r_{23}(z_{23})] = 0. \tag{4.22}$$

Similarly, the classical dynamical r-matrix appears from (4.7) through (4.21). It satisfies the classical dynamical Yang–Baxter equation:

$$[r_{12}(z_{12}), r_{13}(z_{13})] + [r_{12}(z_{12}), r_{23}(z_{23})] + [r_{13}(z_{13}), r_{23}(z_{23})] + \tag{4.23}$$

$$[\hat{\partial}_1, r_{23}(z_{23})] - [\hat{\partial}_2, r_{13}(z_{13})] + [\hat{\partial}_3, r_{12}(z_{12})] = 0, \tag{4.24}$$

which underlies the Poisson structure of the spin Calogero-Moser model [9, 38]. Here

$$\hat{\partial}_3 = \sum_{k=1}^{M} 1_M \otimes 1_M \otimes E_{kk}\ \partial_{q_k} \qquad P_3^{\hbar} \overset{(4.10)}{=} 1_M{}^{\otimes 3} + \hbar\hat{\partial}_3 + O(\hbar^2). \tag{4.25}$$

In the same way, starting from the quantum R-matrix (4.11) one gets the classical r-matrix

$$\mathbf{r}_{1'2'12}(z) = \sum_{i=1}^{M} \overset{1'}{E}_{ii} \otimes \overset{2'}{E}_{ii} \otimes r_{12}(z) + \sum_{\substack{i,j\\ i\neq j}}^{M} \overset{1'}{E}_{ij} \otimes \overset{2'}{E}_{ji} \otimes R_{12}^{q_{ij}}(z), \tag{4.26}$$

and the classical dynamical Yang–Baxter equation follows from (4.12):

$$\begin{gathered}[\mathbf{r}_{1'2'12}(z_{12}), \mathbf{r}_{1'3'13}(z_{13})] + [\mathbf{r}_{1'2'12}(z_{12}), \mathbf{r}_{2'3'23}(z_{23})] + [\mathbf{r}_{1'3'13}(z_{13}), \mathbf{r}_{2'3'23}(z_{23})] + \\ + [\hat{\partial}_{1'}, \mathbf{r}_{2'3'23}(z_{23})] - [\hat{\partial}_{2'}, \mathbf{r}_{1'3'13}(z_{13})] + [\hat{\partial}_{3'}, \mathbf{r}_{1'2'12}(z_{12})] = 0.\end{gathered} \tag{4.27}$$

with

$$\hat{\partial}_{3'} = \sum_{k=1}^{M} \overset{1'}{1}_M \otimes \overset{2'}{1}_M \otimes \overset{3'}{E}_{kk} \otimes \overset{1}{1}_N \otimes \overset{2}{1}_N \otimes \overset{3}{1}_N\ \partial_{q_k}, \quad \mathbf{P}_{3'}^{\hbar} \overset{(4.14)}{=} 1_{MN}{}^{\otimes 3} + \hbar\hat{\partial}_{3'} + O(\hbar^2). \tag{4.28}$$

4.2 Elliptic quantum algebras

4.2.1 Sklyanin algebra

Consider Baxter–Belavin quantum R–matrix [6, 7]:

$$R_{12}^{\mathrm{BB}}(\hbar, u) = \sum_{\alpha\in\mathbb{Z}_N^2} \varphi_\alpha(u, \hbar + \omega_\alpha) T_\alpha \otimes T_{-\alpha}, \tag{4.29}$$

In this defintion the elliptic functions $\varphi_\alpha(u, x+\omega_\alpha)$ and the $N\times N$ basis matrices T_α, connected with these functions, are used. They are defined in the Appendix. This R–matrix satisfies the quantum Yang–Baxter equation in $\mathrm{Mat}(N,\mathbb{C})^{\otimes 3}$

$$R^{\mathrm{BB}}_{12}(\hbar, z_{12})R^{\mathrm{BB}}_{13}(\hbar, z_{13})R^{\mathrm{BB}}_{23}(\hbar, z_{23}) = R^{\mathrm{BB}}_{23}(\hbar, z_{23})R^{\mathrm{BB}}_{13}(\hbar, z_{13})R^{\mathrm{BB}}_{12}(\hbar, z_{12}), \tag{4.30}$$

An operator $L(z)$ is called L–operator for the Baxter–Belavin R–matrix, if it satisfies the RLL–relation

$$R^{\mathrm{BB}}_{12}(\hbar, z_1 - z_2)L_1(z_1)L_2(z_2) = L_2(z_2)L_1(z_1)R^{\mathrm{BB}}_{12}(\hbar, z_1 - z_2). \tag{4.31}$$

In the work [78] Sklyanin suggested a class of the L–operators for the case $N = 2$. After this, his approach was generalized in the case of an arbitrary N and considering other parameters of bundles over the elliptic curve [60, 67, 91, 11, 35]. The constructed L–operators are connected with the quadratic algebra, called Sklyanin algebra.

Consider an L–operator of the form

$$L(z) = \sum_\alpha \varphi_\alpha(z, \hbar + \omega_\alpha)S_\alpha T_\alpha. \tag{4.32}$$

RLL–relation (4.31) for this L–operator is equivalent to the following quadratic relations on operators S_α, labelled by pairs (α, β), which do not depend on the spectral parameters z_1, z_2:

$$\beta \neq 0: \quad \sum_\gamma \varkappa_{\gamma\alpha}\varkappa_{\beta\gamma}\Big(E_1(\omega_\gamma + \hbar) - E_1(\omega_{\alpha-\beta-\gamma+\hbar}) + E_1(\omega_{\alpha-\gamma} + \hbar) - E_1(\omega_{\beta+\gamma} + \hbar)\Big)S_{\alpha-\gamma}S_{\beta+\gamma} = 0, \tag{4.33}$$

$$\beta = 0: \quad \sum_\gamma \varkappa_{\gamma\alpha}\Big(E_2(\omega_\gamma + \hbar) - E_2(\omega_{\alpha-\gamma} + \hbar)\Big)S_{\alpha-\gamma}S_\gamma = 0, \tag{4.34}$$

where $E_1(z)$ and $E_2(z)$ are Eisenstein functions, also defined in the Appendix. A set of numbers

$$\varkappa_{\alpha\beta} = \exp\left(\frac{\pi i}{N}(\beta_1\alpha_2 - \beta_2\alpha_1)\right) \tag{4.35}$$

defines structure coefficients of the relations (4.33)–(4.34), called the Sklyanin algebra relations. For example, operators $S_\alpha = T_{-\alpha}$ satisfy these relations. In this case the RLL–relation becomes the Yang–Baxter equation (4.30).

One can slightly modify the definition (4.32) and relations (4.33)–(4.34). The L–operator can be divided by a function depending only on z, because this function cancels in the both parts of the RLL–relation. Presenting φ_α–function explicitly:

$$\varphi_\alpha(z, \hbar + \omega_\alpha) = \phi(z, \hbar + \omega_\alpha)e^{\frac{2\pi i}{N}\alpha_2 z} = \frac{\vartheta'(0)\vartheta(z + \hbar + \omega_\alpha)}{\vartheta(z)\vartheta(\hbar + \omega_\alpha)}e^{\frac{2\pi i}{N}\alpha_2 z} \tag{4.36}$$

and dividing the L–operator (4.32) by $\vartheta'(0)/\vartheta(z)$, one obtains

$$L^\hbar(z) = \sum_\alpha \frac{\vartheta(z+\hbar+\omega_\alpha)}{\vartheta(\hbar+\omega_\alpha)} e^{\frac{2\pi i}{N}\alpha_2 z} S_\alpha T_\alpha. \tag{4.37}$$

A multiplier $\vartheta(\hbar+\omega_\alpha)$ does not depend on the spectral parameter, then, one can remove it redefining S_α. In this case the L–operator will have the form

$$L^\hbar(z) = \sum_\alpha \vartheta(z+\hbar+\omega_\alpha) e^{\frac{2\pi i}{N}\alpha_2 z} \widetilde{S}_\alpha T_\alpha, \quad \widetilde{S}_\alpha = \frac{S_\alpha}{\vartheta(\hbar+\omega_\alpha)}. \tag{4.38}$$

The Sklyanin algebra relations will also change

$$\beta \neq 0: \quad \sum_\gamma \varkappa_{\gamma\alpha}\varkappa_{\beta\gamma}\Big(E_1(\omega_\gamma+\hbar) - E_1(\omega_{\alpha-\beta-\gamma}+\hbar) + E_1(\omega_{\alpha-\gamma}+\hbar) - E_1(\omega_{\beta+\gamma}+\hbar)\Big)\times$$
$$\times\vartheta(\hbar+\omega_{\alpha-\gamma})\vartheta(\hbar+\omega_{\beta+\gamma})\widetilde{S}_{\alpha-\gamma}\widetilde{S}_{\beta+\gamma} = 0, \tag{4.39}$$
$$\beta = 0: \quad \sum_\gamma \varkappa_{\gamma\alpha}\Big(E_2(\omega_\gamma+\hbar) - E_2(\omega_{\alpha-\gamma}+\hbar)\Big)\vartheta(\hbar+\omega_{\alpha+\gamma})\vartheta(\hbar+\omega_\gamma)\widetilde{S}_{\alpha-\gamma}\widetilde{S}_\gamma = 0 \tag{4.40}$$

Moreover, one can change $\hbar$ in the L–operators to another parameter, shifting z, because the R–matrix depends only on the difference $z_1 - z_2$. Then one can define an operator

$$L^\eta(z) = L^\hbar(z+\eta-\hbar) = \sum_\alpha \vartheta(z+\eta+\omega_\alpha) e^{\frac{2\pi i}{N}\alpha_2 z} S^\eta_\alpha T_\alpha\,, \quad S^\eta_\alpha = \widetilde{S}_\alpha e^{\frac{2\pi i}{N}\alpha_2(\eta-\hbar)}. \tag{4.41}$$

Relations on S^η_α will be analogues of the relations on $\widetilde{S}_\alpha$ accurate to these exponential multipliers.

4.2.2 Elliptic quantum group

Consider Felder dynamical quantum R–matrix [26, 28]:

$$R^{\mathrm{F}}_{12}(\hbar, u \mid q) = \sum_{i=1}^M \phi(u,\hbar) E_{ii}\otimes E_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^M \phi(u, q_{ij}) E_{ij}\otimes E_{ji} + \sum_{\substack{i,j=1\\ i\neq j}}^M \phi(\hbar, -q_{ij}) E_{ii}\otimes E_{jj}, \tag{4.42}$$

where $q_{ij} = q_i - q_j$, E_{ij} — $M\times M$ matrices with matrix elements $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, and ϕ — elliptic functions, defined in the Appendix. Dynamical here means that the R–matrix depends on the dynamical parameters q_i.

The R–matrix (4.42) satisfies the quantum dynamical Yang–Baxter equation

$$\begin{aligned} & R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q) R^{\mathrm{F}}_{13}(\hbar, z_{13} \mid q - \hbar^{(2)}) R^{\mathrm{F}}_{23}(\hbar, z_{23} \mid q) = \\ & = R^{\mathrm{F}}_{23}(\hbar, z_{23} \mid q - \hbar^{(1)}) R^{\mathrm{F}}_{13}(\hbar, z_{13} \mid q) R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q - \hbar^{(3)}) \end{aligned} \tag{4.43}$$

In this equation the shifts along the Cartan subalgebra $\{E_{ii}\}$ in $\mathfrak{gl}(M)$ is used:

$$R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q - \hbar^{(3)}) = e^{-\hbar\hat{\partial}_3} R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q) e^{\hbar\hat{\partial}_3}\,, \quad \hat{\partial}_3 = \sum_k (E_{kk})_3 \partial_{q_k}. \tag{4.44}$$

Besides the quantum dynamical Yang–Baxter equation, the R–matrix satisfies also zero–weight conditions:

$$[(E_{ii})_1 + (E_{ii})_2, R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q)] = 0, \tag{4.45}$$

$$[\hat{\partial}_1 + \hat{\partial}_2, R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q)] = 0. \tag{4.46}$$

Let $h_i, i = 1, 2, \dots M$ be commuting elements. An operator $L(z \mid q)$ is called dynamical L–operator with Cartan elements h_i for the Felder R–matrix if it satisfies the dynamical RLL–relation

$$R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q) L_1(z_1 \mid q - \hbar^{(2)}) L_2(z_2 \mid q) = L_2(z_2 \mid q - \hbar^{(1)}) L_1(z_1 \mid q) R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q - \hbar \cdot h), \tag{4.47}$$

$$R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q - \hbar \cdot h) = e^{-\hbar \sum_k h_k \frac{\partial}{\partial q_k}} R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q) e^{\hbar \sum_k h_k \frac{\partial}{\partial q_k}}.$$

The dynamical Yang–Baxter equation implies the fact that the Felder dynamical R–matrix is the dynamical L–operator with Cartan elements $h_i = (E_{ii})_3$:

$$L_1(z \mid q) = R^{\mathrm{F}}_{13}(\hbar, z \mid q)\,. \tag{4.48}$$

The RLL–relation (4.47) can be rewritten in the equivalent form if one acts on both sides from the left by the operator $e^{\hbar\hat{\partial}_1} e^{\hbar\hat{\partial}_2}$. Using zero–weight property $[\hat{\partial}_1 + \hat{\partial}_2, R^{\hbar}_{12}(u \mid q)] = 0$, one obtains $[e^{\hbar\hat{\partial}_1} e^{\hbar\hat{\partial}_2}, R^{\hbar}_{12}(u \mid q)] = 0$. Then one can get

$$\begin{aligned} e^{\hbar\hat{\partial}_1} e^{\hbar\hat{\partial}_2} R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q) L_1(z_1 \mid q - \hbar^{(2)}) L_2(z_2 \mid q) &= e^{\hbar\hat{\partial}_1} e^{\hbar\hat{\partial}_2} L_2(z_2 \mid q - \hbar^{(1)}) L_1(z_1 \mid q) R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q - \hbar \cdot h), \\ e^{\hbar\hat{\partial}_1} e^{\hbar\hat{\partial}_2} R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q) e^{-\hbar\hat{\partial}_2} L_1(z_1 \mid q) e^{\hbar\hat{\partial}_2} L_2(z_2 \mid q) &= e^{\hbar\hat{\partial}_2} L_2(z_2 \mid q) e^{\hbar\hat{\partial}_1} L_1(z_1 \mid q) R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q - \hbar \cdot h), \\ R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q) e^{\hbar\hat{\partial}_1} L_1(z_1 \mid q) e^{\hbar\hat{\partial}_2} L_2(z_2 \mid q) &= e^{\hbar\hat{\partial}_2} L_2(z_2 \mid q) e^{\hbar\hat{\partial}_1} L_1(z_1 \mid q) R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q - \hbar \cdot h). \end{aligned}$$

Define operators $\widetilde{L}(u \mid q) = e^{\hbar\hat{\partial}} L(u \mid q)$. Then (4.47) can be rewritten in the form

$$R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q) \widetilde{L}_1(z_1 \mid q) \widetilde{L}_2(z_2 \mid q) = \widetilde{L}_2(z_2 \mid q) \widetilde{L}_1(z_1 \mid q) R^{\mathrm{F}}_{12}(\hbar, z_{12} \mid q - \hbar \cdot h). \tag{4.49}$$

In the work [83] V. Tarasov and A. Varchenko constructed the dynamical L–operators and the quadratic algebra connected with them, which is also known as the small elliptic quantum group. Consider q_k and $q_k - \hbar h_k$ in R–matrices in (4.49) as independent coordinates and label these two new sets of variables $q_k^{\{2\}} = q_k,\ q_k^{\{1\}} = q_k - \hbar h_k$. Then the RLL–relation is presented as

$$R_{12}^{\mathrm{F}}(\hbar, z_{12} \mid q^{\{2\}})\widetilde{L}_1(z_1 \mid q^{\{1\}}, q^{\{2\}})\widetilde{L}_2(z_2 \mid q^{\{1\}}, q^{\{2\}}) = \widetilde{L}_2(z_2 \mid q^{\{1\}}, q^{\{2\}})\widetilde{L}_1(z_1 \mid q^{\{1\}}, q^{\{2\}})R_{12}^{\mathrm{F}}(\hbar, z_{12} \mid q^{\{1\}}). \tag{4.50}$$

Choose an ansatz for L–operator in the form

$$\widetilde{L}(z \mid q) = \sum_{i,j} \vartheta(z + q_i^{\{2\}} - q_j^{\{1\}}) t_{ji} E_{ij}, \tag{4.51}$$

where t_{ij} are operators do not commute with coordinates $q_k^{\{I\}}$, but shift them by $\hbar$ by the following rule

$$\begin{gathered} t_{ij} f(q_1^{\{1\}}, \ldots, q_i^{\{1\}}, \ldots, q_M^{\{1\}}, q_1^{\{2\}}, \ldots, q_j^{\{2\}}, \ldots, q_M^{\{2\}}) = \\ = f(q_1^{\{1\}}, \ldots, q_i^{\{1\}} + \hbar, \ldots, q_M^{\{1\}}, q_1^{\{2\}}, \ldots, q_j^{\{2\}} + \hbar, \ldots, q_M^{\{2\}}) t_{ij}, \end{gathered} \tag{4.52}$$

where f is an arbitrary function of variables $q_k^{\{I\}}$. The dynamical RLL–relation (4.50) for this L–operator is equivalent to the following quadratic relations for the operators t_{ij}:

$$t_{ij} t_{ik} = t_{ik} t_{ij}, \tag{4.53}$$

$$t_{ik} t_{jk} = \frac{\vartheta(q_{ij}^{\{1\}} - \hbar)}{\vartheta(q_{ij}^{\{1\}} - \hbar)} t_{jk} t_{ik}, \quad i \neq j, \tag{4.54}$$

$$\frac{\vartheta(q_{jl}^{\{2\}} - \hbar)}{\vartheta(q_{jl}^{\{2\}})} t_{ij} t_{kl} - \frac{\vartheta(q_{ik}^{\{1\}} - \hbar)}{\vartheta(q_{ik}^{\{1\}})} t_{kl} t_{ij} = -\frac{\vartheta(\hbar)\vartheta(q_{ik}^{\{1\}} + q_{jl}^{\{2\}})}{\vartheta(q_{ik}^{\{1\}})\vartheta(q_{jl}^{\{2\}})} t_{il} t_{kj}, \quad i \neq k,\ j \neq l. \tag{4.55}$$

These quadratic relations define (small) elliptic Felder–Tarasov–Varchenko quantum group.

4.3 A quadratic algebra for the $SL(NM)$ R–matrix

Consider a quantum R–matrix, corresponding to the $SL(NM)$–bundle with nontrivial characteristic class over the elliptic curve. This R–matrix was constructed in the papers [44, 92, 74]. It generalizes simultaneously the nondynamical Baxter–Belavin quantum R–matrix and the dynamical Felder quantum R–matrix and can be presented in the form

$$\mathbf{R}^{\hbar}_{ab12}(z_{12} \mid q) = \sum_i (E_{ii})_a (E_{ii})_b R^{\mathrm{BB}}_{12}(\hbar, z_{12}) +$$
$$+ \sum_{\substack{i,j\\ i\neq j}} (E_{ij})_a (E_{ji})_b R^{\mathrm{BB}}_{12}(q_{ij}, z_{12}) + \sum_{\substack{i,j\\ i\neq j}} (E_{ii})_a (E_{jj})_b \otimes 1_N \otimes 1_N \phi(\hbar, -q_{ij}). \tag{4.56}$$

Here spaces labelled by small latin letters are $M \times M$ matrix spaces in the standard basis, and spaces labelled by numbers — $N \times N$ matrix spaces in the basis (A.15). This quantum R–matrix satisfies the dynamical quantum Yang–Baxter equation with shifts only along the Cartan subalgebra corresponding $M \times M$ matrices (i.e. of the form $h_i \otimes 1_N$):

$$\mathbf{R}^{\hbar}_{ab12}(z_{12} \mid q) \mathbf{R}^{\hbar}_{ac13}(z_{13} \mid q - \hbar^{(b)}) \mathbf{R}^{\hbar}_{bc23}(z_{23} \mid q) =$$
$$= \mathbf{R}^{\hbar}_{bc23}(z_{23} \mid q - \hbar^{(a)}) \mathbf{R}^{\hbar}_{ac13}(z_{13} \mid q) \mathbf{R}^{\hbar}_{ab12}(z_{12} \mid q - \hbar^{(c)}). \tag{4.57}$$

An operator $\mathbf{L}_{a1}(z \mid q^{\{1\}}, q^{\{2\}})$ is called an L–operator for this quantum R–matrix, if it satisfies the following RLL–relation:

$$\mathbf{R}^{\hbar}_{ab12}(z_{12} \mid q^{\{2\}}) \mathbf{L}_{a1}(z_1 \mid q^{\{1\}}, q^{\{2\}}) \mathbf{L}_{b2}(z_2 \mid q^{\{1\}}, q^{\{2\}}) =$$
$$= \mathbf{L}_{b2}(z_2 \mid q^{\{1\}}, q^{\{2\}}) \mathbf{L}_{a1}(z_1 \mid q^{\{1\}}, q^{\{2\}}) \mathbf{R}^{\hbar}_{ab12}(z_{12} \mid q^{\{1\}}). \tag{4.58}$$

The main result of this chapter is the description of a quadratic algebra connected with this RLL–relation. Choose an L–operator in the form

$$\mathbf{L}_{a1}(z_1 \mid q^{\{1\}}, q^{\{2\}}) = \sum_{ij} (E_{ij})_a L^{ij}_1(z_1 \mid q^{\{1\}}, q^{\{2\}}), \tag{4.59}$$

$$L^{ij}(z \mid q) = \sum_{\alpha} \vartheta(z + q^{\{2\}}_i - q^{\{1\}}_j + \omega_\alpha) t^{\alpha}_{ji}. \tag{4.60}$$

The operators t^{α}_{ij} shift coordinates q_k by the rule

$$t^{\alpha}_{ij} f(q^{\{1\}}_1, \ldots, q^{\{1\}}_i, \ldots, q^{\{1\}}_M, q^{\{2\}}_1, \ldots, q^{\{2\}}_j, \ldots, q^{\{2\}}_M) =$$
$$= f(q^{\{1\}}_1, \ldots, q^{\{1\}}_i + \hbar, \ldots, q^{\{1\}}_M, q^{\{2\}}_1, \ldots, q^{\{2\}}_j + \hbar, \ldots, q^{\{2\}}_M) t^{\alpha}_{ij}. \tag{4.61}$$

Then the RLL–relation is equivalent to the following set of quadratic relations for the generators t^{α}_{ij}:

1. For the same pairs of indices i, j the elements $\{t^{\alpha}_{ji} \mid \alpha \in \mathbb{Z}^2_N\}$ satisfy the Sklyanin algebra relations with parameter $\eta = q^{\{2\}}_i - q^{\{1\}}_j$.

2. For the same second index and distinct first indices $i, j, k:\ j \neq k$

$$\sum_\gamma \varkappa_{\gamma\alpha}\varkappa_{\beta\gamma}\phi(\hbar+\omega_\gamma, q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha})t_{ji}^{\alpha-\gamma}t_{ki}^{\beta+\gamma} = \phi(\hbar, -q_{jk}^{\{1\}})t_{ki}^{\beta}t_{ji}^{\alpha}. \tag{4.62}$$

3. For the same first index and distinct second indices $i, j, k:\ j \neq k$

$$\sum_\gamma \varkappa_{\gamma\alpha}\varkappa_{\beta\gamma}\phi(\hbar+\omega_{\alpha-\beta-\gamma}, -q_{jk}^{\{2\}}-\omega_\gamma)t_{ik}^{\alpha-\gamma}t_{ij}^{\beta+\gamma} = \phi(\hbar, -q_{jk}^{\{2\}})t_{ij}^{\alpha}t_{ik}^{\beta}. \tag{4.63}$$

4. For distinct first and second indices $i, j, k, l:\ i \neq k, j \neq l$

$$\sum_\gamma \varkappa_{\gamma\alpha}\varkappa_{\beta\gamma}\phi(q_{ik}^{\{2\}}+\omega_\gamma, q_{jl}^{\{1\}}+\omega_{\beta+\gamma-\alpha})t_{jk}^{\alpha-\gamma}t_{li}^{\beta+\gamma} = \phi(\hbar, -q_{jl}^{\{1\}})t_{lk}^{\beta}t_{ji}^{\alpha} - \phi(\hbar, -q_{ik}^{\{2\}})t_{ji}^{\alpha}t_{lk}^{\beta}. \tag{4.64}$$

One can notice, that in the case $M = 1$ there are only $\{t_{11}^\alpha \mid \alpha \in \mathbb{Z}_N^2\}$ generators, satisfying the Sklyanin algebra relations, and in the case $N = 1$ there are only elliptic quantum groups generators $\{t_{ij}^0 \mid i, j \in 1, 2, \ldots, M\}$. Therefore, the constructed quadratic algebra generalizes these two quantum algebras simultaneously.

The proof of this equivalence is straightforward, it can be checked via elliptic functions identities given in the Appendix. An example of this check in the particular tensor component of the RLL–relation is presented below.

4.3.1 An example calculation to check the RLL–relation

Consider for example the $(E_{ij})_a(E_{ik})_b$–component of the RLL–relation, for $j \neq k$:

$$R_{12}^{\mathrm{BB}}(\hbar, z_{12})L_1^{ij}(z_1)L_2^{ik}(z_2) = L_2^{ik}(z_2)L_1^{ij}(z_1)\phi(\hbar, -q_{jk}^{\{1\}}) + L_2^{ij}(z_2)L_1^{ik}(z_1)R_{12}^{\mathrm{BB}}(q_{kj}^{\{1\}}, z_{12}). \tag{4.65}$$

This relation is in $N \times N$–matrices. Expanding it in the basis T_α, one gets the following in the components $(T_\alpha)_1(T_\beta)_2$ (after cancelling all exponential multipliers):

$$\vartheta(z_2+q_i^{\{2\}}-q_k^{\{1\}}+\omega_\beta)t_{ki}^{\beta}\cdot\vartheta(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\omega_\alpha)t_{ji}^{\alpha}\cdot\phi(\hbar, -q_{jk}^{\{1\}}) =$$
$$= \sum_\gamma \varkappa_{\gamma\alpha}\varkappa_{\beta\gamma}\Big(\phi(z_{12}, \hbar+\omega_\gamma)\vartheta(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma})t_{ji}^{\alpha-\gamma}\vartheta(z_2+q_i^{\{2\}}-q_k^{\{1\}}+\omega_{\beta+\gamma})t_{ki}^{\beta+\gamma} -$$
$$-\vartheta(z_2+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma})t_{ji}^{\alpha-\gamma}\vartheta(z_1+q_i^{\{2\}}-q_k^{\{1\}}+\omega_{\beta+\gamma})t_{ki}^{\beta+\gamma}\phi(z_{12}, q_{kj}^{\{1\}}+\omega_{\alpha-\beta-\gamma})\Big)$$

Moving all t_{ab} to the right, one obtains

$$\vartheta(z_2+q_i^{\{2\}}-q_k^{\{1\}}+\omega_\beta)\vartheta(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\hbar+\omega_\alpha)\phi(\hbar,-q_{jk}^{\{1\}})t_{ki}^\beta t_{ji}^\alpha=$$
$$=\sum_\gamma \varkappa_{\gamma\alpha}\varkappa_{\beta\gamma}\Big(\phi(z_{12},\hbar+\omega_\gamma)\vartheta(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma})\vartheta(z_2+q_i^{\{2\}}-q_k^{\{1\}}+\hbar+\omega_{\beta+\gamma})-$$
$$-\vartheta(z_2+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma})\vartheta(z_1+q_i^{\{2\}}-q_k^{\{1\}}+\hbar+\omega_{\beta+\gamma})\phi(z_{12},q_{kj}^{\{1\}}+\omega_{\alpha-\beta-\gamma})\Big)t_{ji}^{\alpha-\gamma}t_{ki}^{\beta+\gamma}$$

Divide two parts by $\vartheta(z_2+q_i^{\{2\}}-q_k^{\{1\}}+\omega_\beta)\vartheta(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\hbar+\omega_\alpha)$ and consider an expression in the brackets in the right hand side. One can simplify it:

$$\phi(z_{12},\hbar+\omega_\gamma)\frac{\vartheta(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma})\vartheta(z_2+q_i^{\{2\}}-q_k^{\{1\}}+\hbar+\omega_{\beta+\gamma})}{\vartheta(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\hbar+\omega_\alpha)\vartheta(z_2+q_i^{\{2\}}-q_k^{\{1\}}+\omega_\beta)}-$$
$$-\phi(z_{12},q_{kj}^{\{1\}}+\omega_{\alpha-\beta-\gamma})\frac{\vartheta(z_1+q_i^{\{2\}}-q_k^{\{1\}}+\hbar+\omega_{\beta+\gamma})\vartheta(z_2+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma})}{\vartheta(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\hbar+\omega_\alpha)\vartheta(z_2+q_i^{\{2\}}-q_k^{\{1\}}+\omega_\beta)}=$$
$$=\phi(z_{12},\hbar+\omega_\gamma)\frac{\phi(z_2+q_i^{\{2\}}-q_k^{\{1\}}+\omega_\beta,\hbar+\omega_\gamma)}{\phi(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},\hbar+\omega_\gamma)}-$$
$$-\phi(z_{12},q_{kj}^{\{1\}}+\omega_{\alpha-\beta-\gamma})\frac{\phi(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\hbar+\omega_\alpha,q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha})}{\phi(z_2+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha})}=$$
$$=\frac{\phi(z_{12},\hbar+\omega_\gamma)\phi(z_2+q_i^{\{2\}}-q_k^{\{1\}}+\omega_\beta,\hbar+\omega_\gamma)\phi(z_2+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha})}{\phi(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},\hbar+\omega_\gamma)\phi(z_2+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha})}-$$
$$-\frac{\phi(z_{12},q_{kj}^{\{1\}}+\omega_{\alpha-\beta-\gamma})\phi(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\hbar+\omega_\alpha,q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha})\phi(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},\hbar+\omega_\gamma)}{\phi(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},\hbar+\omega_\gamma)\phi(z_2+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha})}$$

Applying Fay identity to ϕ, and using the fact that $\phi(x,-x)=0$, one can derive:

$$\phi(z_2+q_i^{\{2\}}-q_k^{\{1\}}+\omega_\beta,\hbar+\omega_\gamma)\phi(z_2+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha})=$$
$$=\phi(q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha},\hbar+\omega_\gamma)\phi(z_2+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},q_{jk}^{\{1\}}+\hbar+\omega_{\beta+2\gamma-\alpha})$$
$$\phi(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\hbar+\omega_\alpha,q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha})\phi(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},\hbar+\omega_\gamma)=$$
$$=\phi(\hbar+\omega_\gamma,q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha})\phi(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},q_{jk}^{\{1\}}+\hbar+\omega_{\beta+2\gamma-\alpha})$$

One can take out the common factor in the numerator $\phi(\hbar+\omega_\gamma, q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha})$, the rest parts in the numerator are equal to the denominator (via Fay identity):

$$\phi(z_{12},\hbar+\omega_\gamma)\phi(z_2+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},q_{jk}^{\{1\}}+\hbar+\omega_{\beta+2\gamma-\alpha})-$$
$$-\phi(z_{12},q_{kj}^{\{1\}}+\omega_{\alpha-\beta-\gamma})\phi(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},q_{jk}^{\{1\}}+\hbar+\omega_{\beta+2\gamma-\alpha})=$$
$$=\phi(z_1+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},\hbar+\omega_\gamma)\phi(z_2+q_i^{\{2\}}-q_j^{\{1\}}+\omega_{\alpha-\gamma},q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha})$$

Using this simplification, one obtains the required relation without spectral parameters:

$$\sum_\gamma \varkappa_{\gamma\alpha}\varkappa_{\beta\gamma}\phi(\hbar+\omega_\gamma,q_{jk}^{\{1\}}+\omega_{\beta+\gamma-\alpha})t_{ji}^{\alpha-\gamma}t_{ki}^{\beta+\gamma}=\phi(\hbar,-q_{jk}^{\{1\}})t_{ki}^{\beta}t_{ji}^{\alpha}.$$

All other relations can be checked in the same way considering the other components of the RLL–relation.

4.3.2 Discussion

In the chapter the quadratic algebra generalized the elliptic quantum group and Sklyanin algebra is concructed. On the onehand, it is a classification result, which complements and generalizes the known structures of the quadratic algebras in the bundles over the elliptic curve. On the other hand, the obtained results can be applied to the description of the concrete mechanical systems. It was shown in [73, 31, 30], that the considered $SL(NM)$ quantum R–matrix is connected with quantum long–range spin chains and R–matrix–valued Lax pairs. Moreover, this particular R–matrix in the nonrelativistic classical limit describes the system of interacting tops. The relativistic analogue of this system was also obtained recently using the natural ansatz for the Lax pair [89, 75]. So, the result of this work can be also considered as the description of the operator algebra of the quantum relativistic interacting tops.

Conclusion

In this thesis we studied the applications of the quantum R-matrix identities, first of all, the associative Yang–Baxter equation, to the theory of the integrable systems, classical and quantum.

- The systems of generalized interacting integrable tops have been constructed for any quantum R-matrix, which solves the associative Yang–Baxter equation together with skew-symmetry and unitarity conditions. These systems can be considered as the extension of both the spin Calogero–Moser systems of particles and the Euler–Arnold integrable tops.
- The relativistic analogue of the system of generalized interacting integrable tops has been constructed. This system generalizes both the spin Ruijsenaars–Schneider model and the relativistic version of the Euler–Arnold top.
- The classical r-matrix structure for the generalized interacting integrable tops has been quantized, quantum R-matrix and quantum RLL-algebra have been obtained. In the elliptic case, the quadratic algebra corresponding to this RLL-algebra is the simultaneous generalization of the Sklyanin algebra and the small elliptic quantum group.

Appendix: Elliptic functions and their properties

The following set of functions is used in this work [84, 58]. The first one is the Kronecker function:

$$\phi(\eta, z) = \begin{cases} \frac{1}{\eta} + \frac{1}{z}, & \text{rational case,} \\ \coth(\eta) + \coth(z), & \text{trigonometric case,} \\ \frac{\vartheta'(0)\vartheta(z+\eta)}{\vartheta(\eta)\vartheta(z)}, & \text{elliptic case.} \end{cases} \tag{A.1}$$

Its elliptic version is given in terms of the odd theta-function

$$\vartheta(z) = \sum_{k\in\mathbb{Z}} \exp\left(\pi i\tau(k+\frac{1}{2})^2 + 2\pi i(z+\frac{1}{2})(k+\frac{1}{2})\right) \tag{A.2}$$

on elliptic curve with moduli $\tau : (\mathrm{Im}(\tau) > 0)$. The next are the first Eisenstein (odd) function and the Weierstrass (even) $\wp$-function:

$$E_1(z) = \begin{cases} \frac{1}{z}, \\ \coth(z), \\ \frac{\vartheta'(z)}{\vartheta(z)}, \end{cases} \qquad \wp(z) = \begin{cases} \frac{1}{z^2}, \\ \frac{1}{\sinh^2(z)}, \\ -\partial_z E_1(z) + \frac{1}{3}\frac{\vartheta'''(0)}{\vartheta'(0)}. \end{cases} \tag{A.3}$$

We also need the derivatives

$$E_2(z) = -\partial_z E_1(z) \tag{A.4}$$

and

$$f(z,q) = \partial_q \phi(z,q) = \phi(z,q)(E_1(z+q) - E_1(q)). \tag{A.5}$$

The one (A.4) is the second Eisenstein function.

The main relation is the Fay trisecant identity:

$$\phi(z,q)\phi(w,u) = \phi(z-w,q)\phi(w,q+u) + \phi(w-z,u)\phi(z,q+u). \tag{A.6}$$

The following degenerations of (A.6) are necessary for the Lax equations and r-matrix structures:

$$\phi(z,x)f(z,y)-\phi(z,y)f(z,x)=\phi(z,x+y)(\wp(x)-\wp(y)), \tag{A.7}$$

$$\phi(\eta,z)\phi(\eta,-z)=\wp(\eta)-\wp(z)=E_2(\eta)-E_2(z). \tag{A.8}$$

Also

$$\begin{aligned}\phi(z,q)\phi(w,q)&=\phi(z+w,q)(E_1(z)+E_1(w)+E_1(q)-E_1(z+w+q))==\\&=\phi(z+w,q)(E_1(z)+E_1(w))-f(z+w,q).\end{aligned} \tag{A.9}$$

The local behavior of the Kronecker function and the first Eisenstein function near its simple pole at $z=0$ is as follows:

$$\phi(z,u)=\frac{1}{z}+E_1(u)+\frac{z}{2}\,(E_1^2(u)-\wp(u))+O(z^2), \tag{A.10}$$

$$E_1(z)=\frac{1}{z}+\frac{z}{3}\frac{\vartheta'''(0)}{\vartheta'(0)}+O(z^3). \tag{A.11}$$

From (A.10) and (A.5) it follows that

$$f(0,u)=-E_2(u). \tag{A.12}$$

In definitions of R–matrices the shifted Kronecker elliptic functions are used

$$\varphi_\alpha(u,x+\omega_\alpha)=\phi(u,x+\omega_\alpha)e^{\frac{2\pi i}{N}\alpha_2 u},\quad \omega_\alpha=\frac{\alpha_1+\alpha_2\tau}{N}, \tag{A.13}$$

$$\phi(u,x)=\frac{\vartheta'(0)\vartheta(u+x)}{\vartheta(u)\vartheta(x)}, \tag{A.14}$$

In the definition of the Baxter–Belavin quantum R–matrix the basis matrices T_α are used. They can be defined as:

$$T_\alpha=T_{(\alpha_1,\alpha_2)}=\exp\left(\frac{\pi i\alpha_1\alpha_2}{N}\right)Q^{\alpha_1}\Lambda^{\alpha_2}, \tag{A.15}$$

$$Q_{jk}=\delta_{jk}\exp\left(\frac{2\pi i k}{N}\right),\qquad \Lambda_{jk}=\begin{cases}1, & \text{if } j+1=k \mod N,\\ 0, & \text{else.}\end{cases} \tag{A.16}$$

Since

$$\exp\left(\frac{2\pi i}{N}\alpha_1\alpha_2\right)Q^{\alpha_1}\Lambda^{\alpha_2}=\Lambda^{\alpha_2}Q^{\alpha_1} \tag{A.17}$$

one has the multiplication rule

$$T_\alpha T_\beta = \varkappa_{\alpha,\beta} T_{\alpha+\beta}, \quad \varkappa_{\alpha,\beta} = \exp\left(\frac{\pi i}{N}(\beta_1\alpha_2 - \beta_2\alpha_1)\right), \tag{A.18}$$

where $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$. The non-degenerate pairing is given by the matrix trace:

$$\mathrm{Tr}(T_\alpha T_\beta) = N\delta_{\alpha+\beta}, \quad T_0 = 1_N. \tag{A.19}$$

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