

Jury Member Report – Doctor of Philosophy thesis.

Name of Candidate: Vadim Prokofev

PhD Program: Mathematics and Mechanics

Title of Thesis: Integrable hierarchies of nonlinear differential equations and many-body systems

Supervisor: Professor Anton Zabrodin

Name of the Reviewer: Alexander P. Veselov

I confirm the absence of any conflict of interest

Date: 24-02-2022

The purpose of this report is to obtain an independent review from the members of PhD defense Jury before the thesis defense. The members of PhD defense Jury are asked to submit signed copy of the report at least 30 days prior the thesis defense. The Reviewers are asked to bring a copy of the completed report to the thesis defense and to discuss the contents of each report with each other before the thesis defense.

If the reviewers have any queries about the thesis which they wish to raise in advance, please contact the Chair of the Jury.

Reviewer's Report

Reviewers report should contain the following items:

- Brief evaluation of the thesis quality and overall structure of the dissertation.
- The relevance of the topic of dissertation work to its actual content
- The relevance of the methods used in the dissertation
- The scientific significance of the results obtained and their compliance with the international level and current state of the art
- The relevance of the obtained results to applications (if applicable)
- The quality of publications

The summary of issues to be addressed before/during the thesis defense

There is a very deep relation between singular solutions of the nonlinear PDEs and particle dynamics. One of the earliest relevant ideas probably goes back already to Einstein, who proposed to use this link to describe the relativistic particle dynamics. In relation with the soliton theory the modern development was initiated in 1977 seminal paper by Airault, McKean and Moser, linking the pole dynamics of singular solutions of the Korteweg- de Vries equation with the Calogero-Moser (CM) dynamical system. Soon afterwards Krichever made another important step in this direction by extending this link to the KP equation. Since then there were many papers in this direction by the authors, including Shiota, Haine, Zabrodin et al.

The dissertation under review is the latest contribution to this important area of research. In collaboration with his supervisor Zabrodin the author extended a connection between most important integrable hierarchies (including matrix KP hierarchy) and many-body systems of Calogero-Moser type, raising this link to the most general elliptic level. I would not go into details of these results, only confirming that they are indeed new and important.

However I have several issues with the Summary and Introduction to the dissertation. Here are my comments for the author. I will refer to the pages in the Summary.

Pages 3-5. The equation (1.1) is not equivalent to (1.2) in that form.
The names Faddeev and Dryuma are misspelt here and later, including the references.
"...Calogero-Moser system (1.17)..." is not introduced yet.

Page 6. "...Where A_n is (?) monic differential operators (?) of order n "

Page 8. "The other objects of study in my thesis is (?) a (?) classical many body systems integrable according to Liouville i.e. contains maximal number of independent integrals of motion." Not quite, condition of the involutivity is missing.

Page 9. "The classical analogues of these systems were proven to be integrable in a works (Calogero and Marchioro [1974], Moser, J. [1974])." The year of Moser's paper is wrong: it should be 1975.

I do not think that this paper by Calogero and Marchioro actually contains the proof of classical integrability, so I would refer here only to the seminal work of Moser.

"Eventually elliptic generalization (1.17) was obtained in the work (Calogero [1975]). For elliptic case Lax representation remains true but both L and M matrices now depends on additional parameter λ" Not quite true: the Lax representation with spectral parameter for the elliptic CM system was first found later by Krichever [1980].

Page 10-11. The integrals I_m for the elliptic CM system can be found from the Lax representation (without spectral parameter) found by Calogero [1975], so there is no need here to use d'Hoker-Phong results.

"In (Shiota [1994]) it was shown, that in order for function ... be a solution to the whole KP hierarchy (1.14), the dynamics of poles with respect to t_m must be the same as a dynamics of particles in rational Calogero-Moser system w.r.t. Hamiltonian $I_m = \text{tr } L^m$." I believe this was done before in Krichever [1978].

Page 12-18. " Here we also introduce an (?) important objects such as Baker-Akhiezer function and tau function. The content of this section follows Chapters 5 and 6 of (Dickey [2003])"

The author indeed follows Chapter 7 (not 5 and 6, as the author wrote) of Dickey very closely, maybe too closely for the summary of PhD thesis. It does not look good, when all the theorems in the Summary (Theorems 1, 2 and 3) are borrowed from Dickey's work.

Most importantly, it should be mentioned that the notion of Baker-Akhiezer function was introduced long before 2003, in the most generality - by Krichever [1977].

In the context of tau-function, one should mention also the important work by Segal and Wilson (Mathem. IHES, 63, 1-64, 1985), who interpreted Krichever's construction in terms of Sato's Grassmannian approach.

Page 12. "...to proof equivalents of two forms..." should be "...to prove equivalence of two forms..." Similarly, later on pages 14, 21 "to proof" instead of "to prove".

Page 19. "was obtained in Appendices ?? and ?? for trigonometric solutions and in Appendices ?? and ??" Something is wrong with the typesetting.

Page 19-21. "Content of this section follows Chapter 13 of (Dickey [2003])." Same as before: why this material should be presented in the Summary with full details, borrowed from somewhere else. What is the role of Lemmas 3 and 4? Why are they called Lemmas?

Page 23. "The content of this section is based on (Ueno and Takasaki [1984])."

Page 26. "Content of this section is based on (Dickey [1997])." Same comments.

In general, my suggestion would be to rewrite the Summary (and the Introduction in the Thesis) by substantially shortening contents of Chapters 2-4 and to extend Chapter 1 to a proper review of the relevant previous results with all the necessary references.

As far as the scientific content of this work I can confirm that the results of the Thesis constitute a substantial contribution to the theory of integrable systems. The results are new, important and published in high quality academic journals.

Provisional Recommendation

☐ *I recommend that the candidate should defend the thesis by means of a formal thesis defense*

☒ *I recommend that the candidate should defend the thesis by means of a formal thesis defense only after appropriate changes would be introduced in candidate's thesis according to the recommendations of the present report*

☐ *The thesis is not acceptable and I recommend that the candidate be exempt from the formal thesis defense*