Skolkovo Institute of Science and Technology

INTEGRABLE HIERARCHIES OF NONLINEAR DIFFERENTIAL EQUATIONS AND MANY-BODY SYSTEMS

Doctoral Thesis

by

VADIM PROKOFEV

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Supervisor
Professor Anton Zabrodin

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I hereby declare that the work presented in this thesis was carried out by myself at Skolkovo Institute of Science and Technology, Moscow, except where due acknowledgement is made, and has not been submitted for any other degree.

Candidate (Vadim Prokofev)

Supervisor (Prof. Anton Zabrodin)

Abstract

One of the most astonishing quality characterizing integrable systems is their nontrivial interconnections with each other. In particular, there is a connection between integrable spin chains, integrable hierarchies of nonlinear partial equations and classical many-bodies models.

In this thesis we study poles dynamics of singular solutions of integrable hierarchies of KP type and show that it is isomorphic to dynamics of particles in many-body integrable systems on the level of hierarchies. Such connection between two different types of integrable systems has been a long known conjecture. The connection between nonlinear integrable equations and many-body systems was first studied in seminal paper (Airault et al. [1977]). After that in the works such as (Krichever [1978],Krichever [1980], Krichever and Zabrodin [1995]) it was established that for the first nontrivial times dynamics of poles correspond to the motion of particles in systems of Calogero-Moser type with standard Hamiltonians. After that in papers (Shiota [1994], Haine [2007], Zabrodin [2020]) such connection was extended to the level of whole hierarchies, however it was done only for rational or trigonometric solutions which are just a limits of the most general elliptic solutions.

In a series of the articles presented in this thesis authors extend a connection between integrable hierarchies and many-body systems of Calogero type for three different hierarchies such as KP, 2D Toda lattice and matrix KP up to the most general elliptic solutions. The main results of these papers is that authors establish a connection between spectral curves of elliptic many-body systems and Hamiltonians responsible for dynamics of poles in higher times of corresponding hierarchy. Besides that methods developed in these articles could be used to discover pole dynamics for singular solutions of other hierarchies.

List of publications

1. V. Prokofev and A. Zabrodin. Toda lattice hierarchy and trigonometric Ruijsenaars–Schneider hierarchy . *Journal of Physics A: Mathematical and Theoretical* , 2019. doi:10.1088/1751-8121/ab520c

2. V. Prokofev and A. Zabrodin. Matrix Kadomtsev Petviashvili Hierarchy and Spin Generalization of Trigonometric Calogero Moser Hierarchy. *Proceedings of the Steklov Institute of Mathematics* , 2020. doi:10.1134/S0081543820030177

3. V. Prokofev and A. Zabrodin. Elliptic solutions to the KP hierarchy and elliptic Calogero–Moser model . *Journal of Physics A: Mathematical and Theoretical* , 2021b. doi:10.1088/1751-8121/ac0a3

4. V. Prokofev and A. Zabrodin. Elliptic solutions to Toda lattice hierarchy and elliptic Ruijsenaars-Schneider model . *Theoretical and Mathematical Physics* , 2021a. doi:10.1134/S0040577921080080

5. V. Prokofev and A. Zabrodin. Elliptic solutions to matrix KP hierarchy and spin generalization of elliptic Calogero-Moser model . *Journal of Mathematical Physics* , 2021c. doi:10.1063/5.0051713

Contents

Introduction

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In this thesis we study poles dynamics of singular solutions of integrable hierarchies of KP type and show that it is isomorphic to dynamics of particles in many-body integrable systems on the level of hierarchies. Such connection between two different types of integrable systems has been a long known conjecture. The connection between nonlinear integrable equations and many-body systems was first study in seminal paper (Airault et al. [1977]). After that in the works such as (Krichever [1978],Krichever [1980], Krichever and Zabrodin [1995]) it was established that for the first nontrivial times dynamics of poles correspond to the motion of particles in systems of Calogero-Moser type with standard Hamiltonians. After that in papers (Shiota [1994], Haine [2007], Zabrodin [2020]) such connection was extended to the level of whole hierarchies, however it was done only for rational or trigonometric solutions which are just a limits of the most general elliptic solutions.

In a series of the articles presented in this thesis authors extend a connection between integrable hierarchies and many-body systems of Calogero type for three different hierarchies such as KP, 2D Toda lattice and matrix KP up to the most general elliptic solutions. The main results of these paper is that authors establish a connection between spectral curves of elliptic many-body systems and Hamiltonians responsible for dynamics of poles in higher times of corresponding hierarchy. Besides that methods developed in these articles could be used to discover poles dynamics for singular solutions of other hierarchies.

My thesis presents the results of five articles in which I am one of co-authors. In these articles a connection between integrable hierarchies of nonlinear differential equations and integrable many-body systems was studied. These works contain most general results for KP 2d-Toda and matrix KP hierarchies.

Chapter 1

Historical remarks

1.1 Nonlinear differential hierarchies

One of the first discovered integrable equations is a famous Korteveg-de Vris equation (1.1). It was written by (Boussinesq [1877]) and rediscovered in (Korteveg, D.J. and de Vries, G. [1895]) as an attempt to find a mathematical description of solitary waves observed by Russel and described by him in (Russel [1844]).

$$4u_t - 12uu_x - u_{xxx} = 0 \tag{1.1}$$

However, the fact that this equation contains infinitely many conserved quantities $I_i = \int\limits_{-\infty}^{\infty} Q_i(x,t)\mathrm{d}x$ was proven only almost a century after in (Miura et al. [1968]). In this paper authors presented a general formula for Q_{2m+1}'s as a graded polynomials of u, u', u'', etc., where $u' \equiv u_x \equiv \partial u$:

$$\begin{aligned} &Q_{-1}[u] = u, && Q_1[u] = \frac{u^2}{2}, \\ &Q_3[u] = \frac{u^3}{3} - \frac{u_x^2}{12}, && Q_5[u] = \frac{u^4}{4} - \frac{uu_x^2}{4} + \frac{u_{xx}^2}{360}, \\ &\ldots\ldots\ldots\ldots && \ldots\ldots\ldots \end{aligned}$$

The same year in (Lax [1968]) it was discovered that (1.1) can be rewritten through two differential operators as

$$L_t = [A_3, L] = A_3 L - L A_3. \tag{1.2}$$

This form of equations now referred as Lax form.

In (1.2) L and A_3 are:

$$L = \partial_x^2 + u \tag{1.3}$$

$$A_3 = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x = \partial^3 + \frac{3}{4}u\partial_x + \frac{3}{4}\partial_x u \tag{1.4}$$

where in the last formula operator written in a skew-symmetric form for the standard scalar product $(f, g) = \int\limits_{-\infty}^{\infty} f(x)g(x)\mathrm{d}x$.

Equation (1.2) indicates that $L(t) = U(t)L(0)U^{-1}(t)$ where $U(t)$ is an unitary operator. It becomes clear, that $A_3 = U^{\dagger}U_t = -U_t^{\dagger}U$ is skew-symmetric.

Lax also considered a case of higher KdV equations as a generalization of such construction. He introduced general skew-symmetric operators

$$A_{2n+1} = \partial_x^{2n+1} + \sum_{i=1}^{n}(b_i\partial_x^{2i-1} + \partial_x^{2i-1}b_i) \tag{1.5}$$

and put them instead of A_3 into equation (1.2). The fact that $L_{t_{2n+1}} = [A_{2n+1}, L]$ is a function not differential operator imposes n conditions which uniquely determine n coefficients b_i's and equality itself determines a higher order KdV equation.

$$u_{t_{2n+1}} = K_{2n+1}(u). \tag{1.6}$$

Such set of infinite equations is called hierarchy.

Later in (Zakharov and Faddeev [1971]) it was shown that KdV equation have a Hamiltonian form:

$$u_t = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta I_3[u]}{\delta u(x)}. \tag{1.7}$$

Here skew-symmetrical operator $\frac{\mathrm{d}}{\mathrm{d}x}$ is infinite dimensional analogue of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the theory of classical Hamiltonian systems.

Moreover higher order KdV equations can be written also as

$$u_{t_n} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta I_n[u]}{\delta u(x)}. \tag{1.8}$$

It proves that KdV equation can be viewed as an infinite dimensional analogue of classical integrable system from Hamiltonian mechanics.

After these observations it becomes ambiguous to somehow connect KdV equation with some known or unknown finite-dimensional integrable system. In seminal paper (Airault et al. [1977]) connection between class of elliptic solutions of KdV and so-called Calogero-Moser system was shown. Calogero-Moser system (1.17) describes dynamics of non-relativistic particles on complex line with pairwise interaction between every particle with each other (Calogero [1971],Calogero [1975]).

However dynamics of poles was described by special locus and it appears that more natural connection arise between 3-d generalization of KdV hierarchy – Kadomtsev–Petviashvili (or simply KP) hierarchy and Calogero Moser system. KP hierarchy like KdV hierarchy is generalization of nonlinear differential equation called KP equation.

$$3u_{yy} = (4u_t - 12uu_x - u_{xxx})_x \tag{1.9}$$

Kadomtsev-Petviashvili equation originates from (Kadomtsev and Petviashvili [1970]) in which authors derived the equation as a model to study the evolution of long ion-acoustic waves of small amplitude propagating in plasmas under the effect of long transverse perturbations. In the absence of transverse dynamics, this problem is described by the KdV equation. The KP equation was soon widely accepted as a natural extension of the classical KdV equation to two spatial dimensions.

In a paper (Dryuma [1974]) Lax representation of KP equation was found:

$$L_t = [A, L] \tag{1.10}$$

with $L = \partial_y + \partial_x^2 + 2u$ and $A = \partial_x^3 + 3u\partial_x + \int^x u_y \mathrm{d}x$.

However more natural way to describe KP equation was suggested in (Sato

[1983]), where author wrote down the whole hierarchy.

The main idea was to consider a pseudo-differential operator

$$\mathcal{L} = \partial + \sum_{m=1}^{\infty} u_m \partial^{-m} \tag{1.11}$$

where ∂ is ordinary differential operator acting on x with following standard commutation relation with function $\partial f = f' + f\partial$. Multiplying both sides of this equality by ∂^{-1} from left and from right gives $\partial^{-1} f = f\partial^{-1} - \partial^{-1} f' \partial^{-1}$. The multiple application of this rule yields:

$$\partial^{-n} f = \sum_{k\geq 0} (-1)^k \begin{pmatrix} k+n-1 \\ k \end{pmatrix} f^{(k)} \partial^{-n-k} \tag{1.12}$$

which is similar to the rule for usual derivative

$$\partial^n f = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} f^{(k)} \partial^{n-k}. \tag{1.13}$$

Equations of KP hierarchy are equivalent to compatibility condition of a system of Lax equations

$$\partial_{t_n} \mathcal{L} = [\mathcal{A}_n, \mathcal{L}]. \tag{1.14}$$

Where $\mathcal{A}_n$ is the monic differential operator of order n. It is clear, that the only way equation (1.14) make sense if r.h.s is pseudo-differential operator with zero coefficients at positive powers of ∂. The easiest way to impose this condition is to take $\mathcal{A}_n$ as purely differential part of $\mathcal{L}^n$. It can be written using standard notation $\mathcal{A}_n = (\mathcal{L}^n)_+$. Indeed, since $[\mathcal{L}^n, \mathcal{L}] = 0$ $[\mathcal{A}_n, \mathcal{L}] = -[\mathcal{L}^n - \mathcal{A}_n, \mathcal{L}]$ and since $\mathcal{L}^n - \mathcal{A}_n$ has zero differential part it is clear that $[\mathcal{A}_n, \mathcal{L}]$ is also have zero differential part.

Following chain of equalities aims to show, that $\partial_{t_n}\partial_{t_m}\mathcal{L} - \partial_{t_m}\partial_{t_n}\mathcal{L} = 0$.

$$\partial_{t_n}\partial_{t_m}\mathcal{L} - \partial_{t_m}\partial_{t_n}\mathcal{L} = \partial_{t_n}[(\mathcal{L}^m)_+, \mathcal{L}] - \partial_{t_m}[(\mathcal{L}^n)_+, \mathcal{L}] =$$
$$[(\mathcal{L}^n)_+, \mathcal{L}^m]_+\mathcal{L} + (\mathcal{L}^m)_+[(\mathcal{L}^n)_+, \mathcal{L}] - \mathcal{L}[(\mathcal{L}^n)_+, \mathcal{L}^m]_+ - [(\mathcal{L}^n)_+, \mathcal{L}](\mathcal{L}^m)_+ - (n \leftrightarrow m) =$$
$$(\mathcal{L}^n)_+(\mathcal{L}^m)_+\mathcal{L} + [(\mathcal{L}^n)_+, (\mathcal{L}^m)_-]_+\mathcal{L} + \mathcal{L}(\mathcal{L}^m)_+(\mathcal{L}^n)_+ - \mathcal{L}[(\mathcal{L}^n)_+, (\mathcal{L}^m)_-]_+ - (n \leftrightarrow m)$$
$$= [(\mathcal{L}^n, \mathcal{L}^m]_+\mathcal{L} - \mathcal{L}[\mathcal{L}^n, \mathcal{L}^m]_+ = 0.$$

For example in case of $n = 1$ one have $\mathcal{A}_1 = \partial$ which means that $\partial_{t_1} = \partial_x = \partial$ and dependence on x can be restored $u(x, t_1, t_2, ...) = u(t_1 + x, t_2, ...)$.

KP equation is a compatibility condition for system with $n = 2, 3$ and it can be written in zero curvature form:

$$\partial_{t_3}\mathcal{A}_2 - \partial_{t_2}\mathcal{A}_3 + [\mathcal{A}_2, \mathcal{A}_3] = 0 \tag{1.15}$$

here t_2 identifies with y.

Higher KP equations are the same for two arbitrary higher times.

$$\partial_{t_n}\mathcal{A}_m - \partial_{t_m}\mathcal{A}_n + [\mathcal{A}_m, \mathcal{A}_n] = 0. \tag{1.16}$$

In series of works (Krichever [1978], Krichever [1980]) author showed that function $u = c + 2\sum_{j=1}^{n} \wp(x - x_j(y, t))$ is solution of equation (1.9) if and only if dynamics of x_i with respect to y coincide with dynamics of elliptic Calogero-Moser system:

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 - \sum_{i \neq j} \wp(x_i - x_j) \tag{1.17}$$

The dynamics of x_i with respect to $t = t_3$ coincide with Hamiltonian flow of the same system govern by Hamiltonian which is cubic in momenta.

In (1.17) $\wp(x)$ is Weierstrass p-function which can be viewed as averaging of x^{-2} on lattice:

$$\wp(x; \omega_1, \omega_2) = \frac{1}{x^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(x + 2\omega_1 m + 2\omega_2 n)^2} - \frac{1}{(2\omega_1 m + 2\omega_2 n)^2} \right). \tag{1.18}$$

It is well known fact, that Weierstrass p-function degenerates into elementary functions when one or both ω's goes to infinity. In last case it is clear, that $\wp(x)$ becomes just x^{-2}. In case when just ω_1 goes to infinity, we put $\omega_2 = \frac{\pi i}{\gamma}$ and

$$\wp(x; \omega_1, \omega_2) \to \frac{\gamma^2}{\sinh^2(\gamma x)} + \frac{1}{3}\gamma^2. \tag{1.19}$$

These limits called rational and trigonometric (hyperbolic) limits of elliptic functions.

1.2 Many body systems

The other objects of study in this thesis is a classical many body systems integrable according to Liouville i.e. containing maximal number of independent, Poisson-commuting integrals of motion. The first integrable many-body system was discovered by Toda in (Toda [1967a],Toda [1967b]). Having arbitrary number of particles on the line this model consider only interaction between neighbours. With Hamiltonian

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i=1}^{n-1} e^{x_i - x_{i+1}} \tag{1.20}$$

and equations of motion

$$\ddot{x}_1 = e^{x_1 - x_2} \tag{1.21}$$

$$\ddot{x}_i = e^{x_i - x_{i+1}} - e^{x_{i-1} - x_i} \quad \text{for } 1 < i < n \tag{1.22}$$

$$\ddot{x}_n = -e^{x_{n-1} - x_n}. \tag{1.23}$$

After that in (Calogero [1971]) a system with interaction between every particles with each other was found. However author consider only quantum integrability of what will be refereed as Calogero system or rational limit of Calogero-Moser system.

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 - \sum_{i \neq j} \frac{1}{(x_i - x_j)^2} \tag{1.24}$$

Later in (Sutherland [1972]) more general system with potential $\sin^{-2}(x_i - x_j)$

was studied but still for quantum case.

The classical analogues of these systems were proven to be integrable in a works (Calogero and Marchioro [1974], Moser, J. [1975]). In last paper author showed, that equations of motion can be rewritten in Lax form i.e. system (1.24) can be rewritten as:

$$\dot{L} = [M, L] \tag{1.25}$$

where L and M are $n \times n$ matrices with following entries

$$L_{ij} = \delta_{ij} p_i + \frac{(1 - \delta_{ij})}{x_i - x_j} \tag{1.26}$$

$$M_{ij} = -2\delta_{ij} \sum_{k \neq i} \frac{1}{(x_i - x_k)^2} + \frac{2(1 - \delta_{ij})}{(x_i - x_j)^2} \tag{1.27}$$

in rational and

$$L_{ij} = \delta_{ij} p_i + (1 - \delta_{ij}) \coth(\gamma(x_i - x_j)) \tag{1.28}$$

$$M_{ij} = 2\delta_{ij} \sum_{k \neq i} \frac{1}{\sinh^2(\gamma(x_i - x_k))} - \frac{2(1 - \delta_{ij})}{\sinh^2(\gamma(x_i - x_j))} \tag{1.29}$$

in trigonometric (or rather hyperbolic) case.

Lax matrix L becomes an important object in studies of classical integrable systems. Equation (1.25) appears almost in every known integrable system with some important exception such as double elliptic system (Braden et al. [2000]), and system, which can be obtain from BKP hierarchies (Rudneva and Zabrodin [2020]). It was shown that not only $I_m = \mathrm{tr} L^m$ are conserved quantities, which is obvious from equation (1.25), but they also commute with each other, which makes first n of them integrals of motion.

Eventually elliptic generalization (1.17) was obtained in the work (Calogero [1975]). Later in (Krichever [1980]) Lax representation for elliptic case was found but with both L and M matrices depend on additional parameter λ which is not included in equations of motion.

$$L_{ij} = \delta_{ij} p_i + (1 - \delta_{ij}) \Phi(x_i - x_j, \lambda) \tag{1.30}$$

$$M_{ij} = -2\delta_{ij} \sum_{k \neq i} \wp(x_i - x_k) - 2(1 - \delta_{ij})\Phi'(x_i - x_j, \lambda) \tag{1.31}$$

here $\Phi(x, \lambda)$ is Lame function and $\Phi'(x, \lambda) = \partial_x \Phi(x, \lambda)$

$$\Phi(x, \lambda) = \frac{\sigma(x)\sigma(\lambda)}{\sigma(\lambda + x)} e^{-x\zeta(\lambda)} \tag{1.32}$$

$$\sigma(x) = \sigma(x; \omega_1, \omega_2) = x \prod_{s \neq 0} \left(1 - \frac{x}{s}\right) e^{\frac{x}{s} + \frac{x^2}{2s^2}}, \quad s = 2m\omega_1 + 2n\omega_2 \tag{1.33}$$

with integers m, n. $\zeta(x) = \partial_x \log(\sigma(x))$ and $\wp(x) = -\zeta'(x)$.

λ-dependence of Lax matrix in elliptic case becomes important for investigation of correspondence between many body systems and nonlinear differential hierarchy. In trigonometric and rational limits such dependence can be easily factorized

$$L^{tr(rat)} = L^{ell}(\lambda)\Big|_{ell \to tr(rat)} + (E - I) f^{tr(rat)}(\lambda) \tag{1.34}$$

with $f^{tr}(\lambda) = \gamma(\coth(\gamma\lambda) - 1)$ and $f^{rat}(\lambda) = \dfrac{1}{\lambda}$.

Here E is a matrix consists of only unities and I is an identity matrix.

Since $\mathrm{tr}L^m(\lambda)$ in elliptic case depends on λ it cannot be integral of motion. However in (d'Hoker and Phong [1998]) authors found out following expression for spectral curve:

$$\det(z + \zeta(\lambda) - L(\lambda)) = \frac{\sigma(\lambda - \partial_z)}{\sigma(\lambda)} I(z) \tag{1.35}$$

here $I(z)$ is a polynomial of degree n with some integrals of motion as coefficients.

$$I(k) = \sum_{m=0}^{n} I_n z^{n-m} \tag{1.36}$$

$$I_m = e_m(\boldsymbol{p}) + \sum_{l=1}^{[m/2]} \sum_{\substack{|S_i \cap S_j| = 2\delta_{ij} \\ 1 \leq i,j \leq l}} e_{m-2l}(\boldsymbol{p}_{(\cup_{i=1}^{l} S_i)^c}) \prod_{i=1}^{l} \wp(S_i) \tag{1.37}$$

We are using following notation: $e_r(\boldsymbol{p})$ is elementary symmetric polynomial in variables $\{p_i | 1 \leq i \leq n\}$, $e_r(\boldsymbol{p}_S)$ is elementary symmetric polynomial in variables

$\{p_i|i \in S\}$, S^c is a complementary of set S. $\wp(S)$ where $S = \{i,j\}$ is set of power two is just $\wp(x_i - x_j)$. First few examples:

$$
\begin{aligned}
I_0 &= 1\\
I_1 &= \sum p_i\\
I_2 &= \sum{}' \left(\frac{1}{2!}p_ip_j + \frac{1}{2!}\wp(x_i - x_j)\right)\\
I_3 &= \sum{}' \left(\frac{1}{3!}p_ip_jp_k + \frac{1}{2!}p_i\wp(x_j - x_k)\right)\\
I_4 &= \sum{}' \left(\frac{1}{4!}p_ip_jp_kp_l + \frac{1}{2!\cdot 2!}p_ip_j\wp(x_k - x_l) + \frac{1}{2\cdot(2!)^2}\wp(x_i - x_j)\wp(x_k - x_l)\right)
\end{aligned}
$$

where $\sum{}'$ is sum for all non-repeating indices. Coefficients are chosen the way that every unique term will have coefficient 1.

In (Shiota [1994]) it was shown, that in order for function $u(x, \boldsymbol{t}) = 2\sum\limits_{i=1}^{n}(x - x_i(\boldsymbol{t}))^{-2}$ to be a solution of the whole KP hierarchy (1.14), the dynamics of poles x_i with respect to t_m must be the same as a dynamics of particles in rational Calogero-Moser system w.r.t. Hamiltonian $I_m = \mathrm{tr}L^m$. Later in papers (Haine [2007], Zabrodin [2020]) this result was generalize to trigonometric case in which Hamiltonians responsible to higher times are $H_m = \frac{1}{2(m+1)\gamma}\mathrm{tr}\left((L + \gamma I)^{m+1} - (L - \gamma I)^{m+1}\right)$. Result for elliptic case was obtain in (Prokofev and Zabrodin [2021b]) and in this case $H_m = \operatorname*{res}\limits_{z=0}(z^m\lambda(z))$ where $\lambda(z)$ is defined from equation $\det(z + \zeta(\lambda) - L(\lambda)) = 0$

Chapter 2

Tau function and bilinear equation

In Appendix 4.3 one of the crucial elements of the prove is to consider an integral bilinear form of KP hierarchy. In order to make this thesis more self-contained it can be useful to prove equivalents of two forms: integral form of KP hierarchy and a standard one as an infinite set of Lax equations. This section is devoted to proving that statement. Here we also introduce important objects such as Baker-Akhiezer function and tau function.

The content of this section follows Chapters 5 and 6 of (Dickey [2003])

2.1 Baker-Akhiezer function

We will consider pseudo-differential operator for KP hierarchy:

$$\mathcal{L} = \partial + \sum_{m=0}^{\infty} u_m \partial^{-m}. \tag{2.1}$$

It can be viewed in a dressing form:

$$\mathcal{L} = \mathcal{W} \partial \mathcal{W}^{-1}, \tag{2.2}$$

where $\mathcal{W} = \sum_{i=0}^{\infty} w_i \partial^{-i}$ and $w_0 = 1$. It is clear, that all coefficients u_n can be expressed in terms of w_n.

Equations of hierarchy (1.14) can be extended to $\mathcal{W}$

$$\partial_{t_m}\mathcal{W} = -(\mathcal{L}^m)_-\mathcal{W}. \tag{2.3}$$

Here $\mathcal{A}_+$ is a purely differential part of operator $\mathcal{A}$ and $\mathcal{A}_- = \mathcal{A} - \mathcal{A}_+$.

Action of pseudo-differential operators is not defined on functions, however we will define their action on a function $\xi(t,z) = \sum_{k=1}^{\infty} t_k z^k$ following way: $\partial^m \xi(\boldsymbol{t}, z) = \partial_{t_1}^m \xi(\boldsymbol{t}, z) = z^m$ and $\partial^m \exp \xi(\boldsymbol{t}, z) = z^m \exp \xi(\boldsymbol{t}, z)$ for both positive and negative m.

Define Baker-Akhiezer function:

$$\psi(\boldsymbol{t}, z) = \mathcal{W} e^{\xi(\boldsymbol{t},z)} = e^{\xi(\boldsymbol{t},z)} w(\boldsymbol{t}, z) \tag{2.4}$$

with $w(\boldsymbol{t}, z) = \sum_{i=0}^{\infty} w_i(\boldsymbol{t}) z^{-i}$.

Introducing conjugation: $(f\partial)^\dagger = -\partial \cdot f = -(\partial f) - f\partial$ and let $\mathcal{W}^\dagger$ be a formal adjoint to $\mathcal{W}$ define adjoint Baker-Akhiezer function

$$\psi^*(\boldsymbol{t}, z) = (\mathcal{W}^{-1})^\dagger e^{-\xi(\boldsymbol{t},z)} = e^{-\xi(\boldsymbol{t},z)} w^*(\boldsymbol{t}, z). \tag{2.5}$$

These functions satisfy systems:

$$\begin{cases} \mathcal{L}\psi = z\psi \\ \mathcal{A}_n \psi = \partial_n \psi \end{cases} \qquad \begin{cases} \mathcal{L}\psi^* = z\psi^* \\ \mathcal{A}_n \psi^* = -\partial_n \psi^* \end{cases} \tag{2.6}$$

here and further we put $\partial_n = \partial_{t_n}$.

Equations (1.14) can be viewed as compatibility conditions of these systems.

It is typical for both finite and infinite dimensional integrable systems to be just a compatibility conditions of overdetermined systems such as (2.6). It is often useful to study Baker-Akhiezer function instead of infinite set of $\{u_n\}$ or $\{w_n\}$ since it is just one function and it is a solution of infinitely many linear problems.

For an infinite formal series $P(z) = \sum_{-\infty}^{\infty} p_k z^k$ and an infinite series of pseudo-differential operators $\mathcal{P} = \sum_{-\infty}^{\infty} p_k \partial^k$ define operations.

Definition 1. $\operatorname{res}_z(P(z)) = p_{-1}$

Definition 2. $\operatorname*{res}_{\partial}(\mathcal{P}(z)) = p_{-1}$

These two operations connected with useful Lemma

Lemma 1. $\operatorname*{res}_{z}\left[(\mathcal{P}e^{xz})\cdot(\mathcal{Q}e^{-xz})\right] = \operatorname*{res}_{\partial}(\mathcal{P}\mathcal{Q}^{\dagger})$

It can be proven by simple calculation.

With this lemma it becomes easy to prove following theorem

Theorem 1. *The identity*

$$\operatorname*{res}_{z}[(\partial_1^{i_1}...\partial_m^{i_m}\psi)\psi^*] = 0$$

holds for any $(i_1, ..., i_m)$ *with arbitrary* m *if and only if* ψ *and* ψ^* *of the form* $(1 + \sum\limits_{k>0} a_k z^{-k})e^{\pm\xi}$ *are solutions of (2.6).*

Before we will prove this theorem let us show that there is an another way to rewrite it. Indeed instead of $\operatorname*{res}_{z}[(\partial_1^{i_1}...\partial_m^{i_m}\psi(\boldsymbol{t}))\psi^*(\boldsymbol{t})]$ for any $(i_1, i_2, ..., i_m)$ we can write $\operatorname*{res}_{z}[\psi(\boldsymbol{t'})\psi^*(\boldsymbol{t})]$ for any $\boldsymbol{t}, \boldsymbol{t'}$ where $f(\boldsymbol{t'})$ should be understood as a formal expansion:

$$f(\boldsymbol{t'}) = \sum \frac{1}{i_1!...i_m!}(t'_1 - t_1)^{i_1}....(t'_m - t_m)^{i_m}\partial_1^{i_1}...\partial_m^{i_m} f(\boldsymbol{t}).$$

This identity can be rewritten in integral form.

$$\oint\limits_{\infty} e^{\xi(\boldsymbol{t}-\boldsymbol{t'},z)} w(\boldsymbol{t'}, z) w^*(\boldsymbol{t}, z) \mathrm{d}z = 0. \tag{2.7}$$

The integration contour is a big circle around infinity separating the singularities coming from the exponential factor from those coming from the functions w and w^*

Proof. First we will prove that if ψ and ψ^* are solutions of (2.6), then

$$\operatorname*{res}_{z}[(\partial_1^{i_1}...\partial_m^{i_m}\psi)\psi^*] = 0.$$

Since $\partial_s\psi = \mathcal{A}_s\psi$ we need a proof only for $m = 1$.

$$\operatorname*{res}_z[(\partial^i\psi)\psi^*] = \operatorname*{res}_z[(\partial^i\mathcal{W}e^{xz})(\mathcal{W}^\dagger)^{-1}e^{-xz}] =$$
$$= \operatorname*{res}_\partial[(\partial^i\mathcal{W})\mathcal{W}^{-1}] = \operatorname*{res}_\partial(\partial^i) = 0.$$

It completes the first half of the proof.

To prove the converse statement we will consider $\operatorname*{res}_z[(\partial^i w(\boldsymbol{t}, z)w^*(\boldsymbol{t}, z)] = 0$ with $\psi(z) = e^{\xi(\boldsymbol{t},z)}\sum_{i=0}^{\infty} w_i z^{-i}$ and $\psi^*(z) = e^{-\xi(\boldsymbol{t},z)}\sum_{i=0}^{\infty} w_i^* z^{-i}$. Define $\mathcal{W} = \sum_{i=0}^{\infty} w_i\partial^{-i}$ and $\mathcal{W}^* = \sum_{i=0}^{\infty}(-1)^i w_i^*\partial^{-i}$.

Using assumption one can show, that

$$0 = \operatorname*{res}_z[(\partial^i\psi)\psi^*] = \operatorname*{res}_z[(\partial^i\mathcal{W}e^{\xi})\mathcal{W}^*e^{-\xi}] = \operatorname*{res}_\partial[(\partial^i\mathcal{W})(\mathcal{W}^*)^\dagger] = \operatorname*{res}_\partial[\partial^i\mathcal{W}(\mathcal{W}^*)^\dagger].$$

It is true for every i, so if we define purely negative pseudo-different operator $\mathcal{X} = \mathcal{X}_-$ as $\mathcal{W}(\mathcal{W}^*)^\dagger = 1+\mathcal{X}$, proven equations mean, that $\mathcal{X} = 0$ and $\mathcal{W}^* = (\mathcal{W}^\dagger)^{-1}$.

Define $\mathcal{L} = \mathcal{W}\partial\mathcal{W}^{-1}$ for which we have

$$(\partial_m\mathcal{W} + (\mathcal{L}^m)_-\mathcal{W})e^\xi = (\partial_m\cdot\mathcal{W} - \mathcal{W}\partial_m + (\mathcal{L}^m)_-\mathcal{W})e^\xi =$$
$$= (\partial_m\cdot\mathcal{W} - \mathcal{L}^m\mathcal{W} + (\mathcal{L}^m)_-\mathcal{W})e^\xi = (\partial_m - (\mathcal{L}^m)_+)\mathcal{W}e^\xi.$$

From our assumption we know, that

$$0 = \operatorname*{res}_z[(\partial^i(\partial_m - (\mathcal{L}^m)_+)\psi)\psi^*] = \operatorname*{res}_z[(\partial^i(\partial_m - (\mathcal{L}^m)_+)\mathcal{W}e^\xi)((\mathcal{W}^\dagger)^{-1}e^{-\xi})] =$$
$$\operatorname*{res}_z[(\partial^i(\partial_m\mathcal{W} + (\mathcal{L}^m)_-\mathcal{W})e^\xi)((\mathcal{W}^\dagger)^{-1}e^{-\xi})] = \operatorname*{res}_\partial[(\partial^i(\partial_m\mathcal{W} + (\mathcal{L}^m)_-\mathcal{W})(\mathcal{W})^{-1})]$$

This yields $\partial_m\mathcal{W} + (\mathcal{L}^m)_-\mathcal{W} = 0$ which is nothing but equation of KP hierarchy.

□

2.2 Tau function

In the last section it was shown, that the whole KP hierarchy can be rewritten as an integral equation (2.7). However it is possible to simplify it by factorizing z-dependence. In order to do so we will use following easy to prove lemma:

Lemma 2. *If* $f(z) = \sum_{i=0}^{\infty} a_i z^{-i}$ *is a formal series where* $a_0 = 1$ *then*

$$\operatorname*{res}_z f(z)(1 - z/\zeta)^{-1} = \zeta(f(\zeta) - 1).$$

More general if $g(z, \zeta) = \sum_{i=-\infty}^{\infty} b_i(\zeta) z^{-i}$ *then*

$$\operatorname*{res}_z [(1 - z/\zeta)^{-1}] g(z) = \zeta g_-(\zeta, z)|_{z=\zeta}$$

where $g_-(z, \zeta) = \sum_{i=1}^{\infty} b_i(\zeta) z^{-i}$.

Here $(1 - z/\zeta)^{-1}$ is understood as series in ζ^{-1}.

Let $D(\zeta)$ be an operator acting on series in z^{-1} with coefficients depending on $\boldsymbol{t}$ as

$$D(\zeta) f(\boldsymbol{t}, z) = f(\boldsymbol{t} - [\zeta^{-1}], z). \tag{2.8}$$

Here $[\zeta^{-1}] = (\zeta^{-1}, \zeta^{-2}/2, \zeta^{-3}/3, ...)$.

Lemma 3. *Following identities hold:*

$$w^{-1}(\boldsymbol{t}, z) = D(z) w^*(\boldsymbol{t}, z)$$

and

$$\partial \log w(\boldsymbol{t}, z) = (-D(z) + 1) w_1(\boldsymbol{t}).$$

Proof. Equation $\operatorname*{res}_z[\psi(\boldsymbol{t})\psi^*(\boldsymbol{t'})] = 0$ with $\boldsymbol{t'} = \boldsymbol{t} - [\zeta]^{-1}$ and identity

$$\exp \sum_{k=1}^{\infty} \frac{z^k}{k\zeta^k} = (1 - z/\zeta)^{-1}$$

results in

$$\operatorname*{res}_z[w(\boldsymbol{t})D(\zeta)w^*(\boldsymbol{t})(1-z/\zeta)^{-1}] = 0.$$

First part of the lemma 2 allows one to transform it into

$$\zeta(w(\boldsymbol{t},\zeta)D(\zeta)w^*(\boldsymbol{t},\zeta)-1) = 0.$$

Which immediately gives us first equation.

Similarly

$$0 = \operatorname*{res}_z[\partial\psi(z)D(\zeta)\psi^*(z)] = \operatorname*{res}_z[(\partial w(z)+zw(z))(D(\zeta)w^*(z))(1-z/\zeta)^{-1}].$$

Second part of lemma 2 implies

$$0 = [(\partial w(z)+zw(x))D(\zeta)w^*(z)]_-|_{z=\zeta} = (\partial w(\zeta)+\zeta w(\zeta))D(\zeta)w^*(\zeta) - \\ -\zeta - w_1 + D(\zeta)w_1 = (\partial w(\zeta))w^{-1}(\zeta) - (1-D(\zeta))w_1$$

which results in second equality. □

We have shown that derivative of $w(\boldsymbol{t},z)$ with respect to t_1 can be expressed in terms of one function which is not depending on z and the whole z dependence can be hidden inside shift of an arguments. It is remarkable discovery that $w(\boldsymbol{t},z)$ itself can be expressed that way.

Theorem 2. *There is a function $\tau(\boldsymbol{t})$ such that*

$$\log w(\boldsymbol{t},z) = (D(z)-1)\log\tau(\boldsymbol{t})$$

or, in more detail

$$w(\boldsymbol{t},z) = \frac{\tau(\boldsymbol{t}-[z^{-1}])}{\tau(\boldsymbol{t})}. \tag{2.9}$$

It is clear, that since solution of (2.6) can be multiplied by any function depending on z, τ-function also determined up to $c\exp\sum_{i=1}^{\infty}c_it_i$ with $c, c_1, c_2, \ldots$ arbitrary constants.

Proof. We will consider operator $N(z) = \partial_z - \sum_{j=1}^{\infty} z^{-j-1}\partial_j$ which annihilates all functions of the form $D(z)f(\boldsymbol{t})$ moreover for functions $f = \sum_{i=0}^{\infty} f_i z^{-i-1}$ $N(z)f = 0$ implies $f = 0$.

Applying $N(z)$ to $\log w(\boldsymbol{t}, z) = (D(z) - 1)\log\tau(\boldsymbol{t})$ we obtain series of equalities:

$$a_i = \partial_i \log\tau = \operatorname*{res}_z z^i \left(-\sum_{j=1}^{\infty} z^{-j-1}\partial_j + \partial_z\right)\log w.$$

In order for this system to be compatible and agreed with last equation from lemma 3 we need $\partial a_i = -\partial_i w_1$ and $\partial_j a_i = \partial_i a_j$.

First part can be verified by simple calculation. It results in $\partial(\partial_j a_i - \partial_i a_j) = 0$ however $(\partial_j a_i - \partial_i a_j)$ itself should be differential polynomial of w_i which are independent functions. The only way it derivative can be zero is for him be only a constant term. But simple calculation, for all $w_i = 0$ shows that constant term is absent.

□

Equality (2.9) together with first part of lemma 3 gives us following representation of $w^*(\boldsymbol{t}, z)$:

$$w^*(\boldsymbol{t}, z) = \frac{\tau(\boldsymbol{t} + [z^{-1}])}{\tau(\boldsymbol{t})} \tag{2.10}$$

Eventually we can rewrite (2.7) as integral equation on function $\tau(\boldsymbol{t})$ in the form also known as bilinear relation for tau-function.

$$\oint_{\infty} e^{\xi(\boldsymbol{t}-\boldsymbol{t}', z)}\tau(\boldsymbol{t} - [z^{-1}])\tau(\boldsymbol{t} + [z^{-1}])\mathrm{d}z = 0 \tag{2.11}$$

Chapter 3

Further Generalizations

In this section there will be shown ways to generalize KP hierarchy to 2d Toda hierarchy and matrix KP for both these cases pole dynamics of singular solutions was obtained in Appendices A and B for trigonometric solutions and in Appendices D and E these results are generalized to elliptic case.

3.1 Modified KP

Content of this section follows Chapter 13 of (Dickey [2003]).

We start with $\mathcal{L}$ pseudo-differential operator of KP hierarchy, then add infinitely many functions v_i for $i \in \mathbb{Z}$ and determine

$$\mathcal{L}_i = (\partial + v_{i-1})...(\partial + v_0)\mathcal{L}(\partial + v_0)^{-1}...(\partial + v_{i-1})^{-1} \quad \text{for } i > 0 \tag{3.1}$$

$$\mathcal{L}_{-i} = (\partial + v_{-i})^{-1}...(\partial + v_{-1})^{-1}\mathcal{L}(\partial + v_{-1})...(\partial + v_{-i}) \quad \text{for } i > 0 \tag{3.2}$$

$$\mathcal{L}_0 = \mathcal{L}. \tag{3.3}$$

This way we have evident recursion:

$$\mathcal{L}_{i+1}(\partial + v_i) = (\partial + v_i)\mathcal{L}_i. \tag{3.4}$$

Determine dynamics of v_i with respect to t_k following way.

$$\partial_k v_i = (\mathcal{L}_{i+1}^k)_+(\partial + v_i) - (\partial + v_i)(\mathcal{L}_i^k)_+. \tag{3.5}$$

In this case it is easy to show, that

$$\partial_k \mathcal{L}_i = [(\mathcal{L}_i^k)_+, \mathcal{L}_i]. \tag{3.6}$$

One can introduce dressing operators for each $\mathcal{L}_i$

$$\mathcal{W}_i \mathcal{L}_i \mathcal{W}_i^{-1} = \partial \tag{3.7}$$

where $\mathcal{W}_i = \sum\limits_{\alpha=0}^{\infty} w_{i\alpha}\partial^{-\alpha}$, with $w_{i0} = 1$. It is clear that

$$(\partial + v_i)\mathcal{W}_i = \mathcal{W}_{i+1} \cdot \partial. \tag{3.8}$$

Taking similar approach as in section 2.1 we introduce Baker-Akhiezer functions

$$\psi_i(\boldsymbol{t}, z) = \mathcal{W}_i e^{\xi}$$
$$\mathcal{L}_i \psi_i = z\psi_i, \quad (\partial + v_i)\psi_i = z\psi_{i+1}$$

and adjoint Baker-Akhiezer functions

$$\psi_i^*(\boldsymbol{t}, z) = (\mathcal{W}_i^\dagger)^{-1} e^{-\xi}$$
$$\mathcal{L}_i^\dagger \psi_i^* = z\psi_i^*, \quad (\partial - v_i)\psi_{i+1}^* = -z\psi_i^*.$$

Analogically to Lemma 3 we have

Lemma 4. *For two formal series*

$$\psi_i = \sum_\alpha w_{i\alpha} z^{-\alpha} e^{\xi}, \quad \psi^* = \sum_\alpha w_{i\alpha}^* z^{-\alpha} e^{\xi}$$

with $w_{i0} = w_{i0}^* = 1$ *the following two statements are met simultaneously.*

1) ψ_i and ψ_i^* are Baker Akhiezer functions of mKP hierarchy.

2) $\operatorname{res}_z[z^{i-j}(\partial_1^{k_1}...\partial_m^{k_m}\psi_i)\psi_j^*] = 0$ for $i \geq j$ and any $(k_1, ..., k_m)$.

Proof. First we will show, that if condition 1 is satisfied, then condition 2 is also satisfied.

Like in KP case we need to consider only $(k, 0, 0, ..., 0)$ since $\partial_s\psi_i = (\mathcal{L}_i^s)_+\psi_i$.

$$\operatorname{res}_z[z^{i-j}(\partial^k\psi_i)\psi_j^*] = \operatorname{res}_z[(\partial^k\mathcal{W}_i\partial^{i-j}e^{\xi})((\mathcal{W}_j^\dagger)^{-1}e^{-\xi})] = \operatorname{res}_\partial[(\partial^k\mathcal{W}_i\partial^{i-j})\mathcal{W}_j^{-1}] =$$
$$= \operatorname{res}_\partial[(\partial^k(\partial + v_{i-1})...(\partial + v_j)\mathcal{W}_j\mathcal{W}_j^{-1}] = 0.$$

Now we will prove reverse statement. For $i = j$ we have a case of KP hierarchy and it already has been proven, that if $\operatorname{res}_z[(\partial_1^{k_1}...\partial_m^{k_m}\psi_i)\psi_i^*] = 0$, then ψ_i and ψ_i^* are Baker-Akhiezer functions and adjoint BA functions of KP hierarchies with $\mathcal{L}_i$ operators. We are left to prove, that these operators connected through equation (3.4). In order to do so we will consider the case $i = j + 1$ and $(k, 0, 0, ..., 0)$.

$$0 = \operatorname{res}_z[z(\partial^k\psi_{j+1})\psi_j^*] = \operatorname{res}_z[(\partial^k\mathcal{W}_{j+1}\partial e^{\xi})((\mathcal{W}_j^\dagger)^{-1}e^{-\xi})] = \operatorname{res}_\partial[\partial^k\mathcal{W}_{j+1}\partial\mathcal{W}_j^{-1}].$$

which means, that $\mathcal{W}_{j+1}\partial\mathcal{W}_j^{-1}$ is purely differential first order monic operator and we can put $\mathcal{W}_{j+1}\partial\mathcal{W}_j^{-1} = \partial + v_j$. From that equation (3.4) follows immediately.

□

Since every ψ_i is solution of KP Theorem 2 combined with proven Lemma means, that whole mKP hierarchy is equivalent to series of bilinear equations:

$$\oint_\infty z^{n-m}e^{\xi(\boldsymbol{t}-\boldsymbol{t'},z)}\tau_n(\boldsymbol{t} - [z^{-1}])\tau_m(\boldsymbol{t} + [z]^{-1})\mathrm{d}z = 0 \tag{3.9}$$

for $n \geq m$.

But for our purposes it will be convenient to take a different look at mKP hierarchy as a half of more general 2d Toda hierarchy.

3.2 $\mathfrak{gl}((\infty))$ algebra and 2d Toda hierarchy

The content of this section is based on (Ueno and Takasaki [1984]).

We will consider formal Lie algebra $\mathfrak{gl}((\infty))$

Let Λ^j be a j-th shift matrix $\Lambda^j = (\delta_{\mu+j,\nu})_{\mu,\nu\in\mathbb{Z}}$ and E_{ij} be the (i,j)-matrix unit $E_{ij} = (\delta_{\mu i}\delta_{\nu j})_{\mu,\nu\in\mathbb{Z}}$. Let $\mathfrak{gl}((\infty))$ be a formal Lie algebra consisting of all $\mathbb{Z}\times\mathbb{Z}$ matrices

$$\mathfrak{gl}((\infty)) = \left\{ \sum_{i,j\in\mathbb{Z}} a_{ij}E_{ij} | a_{ij} \in \mathbb{C} \right\}. \tag{3.10}$$

A matrix $A \in \mathfrak{gl}((\infty))$ is written in a form

$$A = \sum_{j\in\mathbb{Z}} \mathrm{diag}[a_j(s)]\Lambda^j \tag{3.11}$$

here $\mathrm{diag}[a_j(s)]$ denotes a diagonal matrix $\mathrm{diag}(..., a_j(-1), a_j(0).a_j(1), ...)$ we can define a positive/negative part of matrix of matrix A : $(A)_+ = \sum_{j\geq 0} \mathrm{diag}[a_j(s)]\Lambda^j$ and $(A)_- = \sum_{j<0} \mathrm{diag}[a_j(s)]\Lambda^j$.

If $a_j(s) = 0$ for all $j > m$ we call A is order less than m. If $a_j(s) = 0$ for all $j < m$ we call A is order grater than m. If matrices A and B both less or larger than some m, then product of AB is well-defined.

There is natural correspondence between matrix A and difference operator

$$\mathcal{A}(x) = \sum_{j\in\mathbb{Z}} a_j(x)e^{j\eta\partial_x} \tag{3.12}$$

where operator $e^{j\partial_s}$ define by it action $e^{j\eta\partial_s}f(x) = f(x + j\eta)$.

Definition 3. *Set two copies of time flows $\boldsymbol{t}_+$ and $\boldsymbol{t}_-$. Let $L, \bar{L}$ and M_n for $n \in \mathbb{Z}/\{0\}$ be elements of $\mathfrak{gl}((\infty))$ where*

$$L = \sum_{j\leq 1} \mathrm{diag}[l_j(s)]\Lambda^j \quad \textit{with } l_1(s) = 1 \textit{ for any } s \tag{3.13}$$

$$\bar{L} = \sum_{-1\leq j} \mathrm{diag}[\bar{l}_j(s)]\Lambda^j \quad \textit{with } \bar{l}_{-1}(s) \neq 0 \textit{ for any } s \tag{3.14}$$

$$M_{n>0} = (L^n)_+ \qquad B_{n<0} = (\bar{L}^{-n})_-. \tag{3.15}$$

The Toda lattice hierarchy is a system of equations

$$\partial_n L = [M_n, L] \quad \partial_n \bar{L} = [M_n, \bar{L}]. \tag{3.16}$$

It can be proven the same way it was proven for KP hierarchy that second derivatives commute. In a case of both m and n positive or negative the proof is the same as in KP case. Following chain of equalities will prove it for times from different flows:

$$\partial_n \partial_{-m} L - \partial_{-m} \partial_n L = \partial_n [\bar{L}^m_-, L] - \partial_{-m}[L^n_+, L] =$$
$$[L^n_+, \bar{L}^m]_- L + \bar{L}^m_- [L^n_+, L] - L[L^n_+, \bar{L}^m]_- - [L^n_+, L]\bar{L}^m_- -$$
$$-[\bar{L}^m_-, L^n]_+ L - L^n_+[\bar{L}^m_-, L] + L[\bar{L}^m_-, L^n]_+ + [\bar{L}^m_-, L]L^n_+$$
$$= [L^n_+, \bar{L}^m]L + (\bar{L}^m_- L^n_+ - L^n_+ \bar{L}^m_-)L - L[L^n_+, \bar{L}^m] + L(L^n_+ \bar{L}^m_- - \bar{L}^m_- L^n_+) = 0.$$

Now we will prove some lemmas which help us to obtain the whole Toda lattice hierarchy in bilinear form:

Lemma 5. *There are exist two matrices W and $\bar{W}$ of form*

$$W = \sum_{j=0}^{\infty} \text{diag}[w_j(s)]\Lambda^{-j} \tag{3.17}$$

$$\bar{W} = \sum_{j=0}^{\infty} \text{diag}[\bar{w}_j(s)]\Lambda^{j} \tag{3.18}$$

with $w_0(s) = 1$ and $\bar{w}_0(s) \neq 0$ for any s such that $L = W\Lambda W^{-1}$, $\bar{L} = \bar{W}\bar{\Lambda}^{-1}\bar{W}^{-1}$ and they satisfy equations

$$\partial_n W = -L^n_- W \quad \partial_{-n} W = \bar{L}^n_- W \quad n > 0 \tag{3.19}$$

$$\partial_n \bar{W} = L^n_+ \bar{W} \quad \partial_{-n} W = -\bar{L}^n_+ \bar{W} \quad n > 0. \tag{3.20}$$

Moreover they both defined up to arbitrariness

$$W \to WF^-(\Lambda) \quad \bar{W} \to \bar{W}F^+(\Lambda) \tag{3.21}$$

where $F^{\pm}(\Lambda) = \sum_{j\geq 0} f_j^{\pm}\Lambda^{\pm j}$.

Proof. First of all simple calculation shows that system (3.19)-(3.20) is compatible.

It is clear that there are exist a some constant matrices W_0 and $\bar{W}_0$ of forms (3.17) and (3.18) respectfully such that

$$L = W_0 \Lambda W_0^{-1} \quad \bar{L} = \bar{W}_0 \bar{\Lambda} \bar{W}_0^{-1}.$$

We can consider Cauchy problem for (3.19) and (3.20) with initial conditions W_0, $\bar{W}_0$.

Straightforward calculations show, that $LW - WL$ and $\bar{L}\bar{W} - \bar{W}\bar{L}$ are both solutions of the same systems with zero initial conditions, so the uniqueness of solution obliges them be a null solutions, which means, that we have constructed W and $\bar{W}$ from lemma. □

Using that lemma it is easy to show, that matrices $\Psi = We^{\xi(\boldsymbol{t}_+,\Lambda)}$ and $\bar{\Psi} = \bar{W}e^{\xi(\boldsymbol{t}_-,\Lambda^{-1})}$ are solutions of following linear problems:

$$L\Psi = \Psi\Lambda, \quad \partial_n \Psi = M_n \Psi \tag{3.22}$$

$$\bar{L}\bar{\Psi} = \bar{\Psi}\Lambda^{-1}, \quad \partial_n \bar{\Psi} = M_n \bar{\Psi}. \tag{3.23}$$

$L\Psi = LWe^{\xi(\boldsymbol{t}_+,\Lambda)} = W\Lambda e^{\xi(\boldsymbol{t}_+.\Lambda)} = \Psi\Lambda$ and for $n > 0$ $\partial_n \Psi = -L_-^n \Psi + \Psi\Lambda^n = (-L_-^n + L^n)\Psi = M_n\Psi$ the rest calculations are similar.

Now we have proven, that

$$M_n = (\partial_n \Psi)\Psi^{-1} = (\partial_n \bar{\Psi})\bar{\Psi}^{-1} \tag{3.24}$$

or even

$$(\partial_{i_k}^{n_k} ... \partial_{i_1}^{n_1} \Psi)\Psi^{-1} = (\partial_{i_k}^{n_k} ... \partial_{i_1}^{n_1} \bar{\Psi})\bar{\Psi}^{-1} \tag{3.25}$$

for any $(i_1, ...i_k) \in (\mathbb{Z}/\{0\})^k$ and $(n_1, ...n_k) \in (\mathbb{N}^*)^k$.

It can be written as

$$\Psi(\boldsymbol{t}_+, \boldsymbol{t}_-)\Psi^{-1}(\boldsymbol{t}'_+, \boldsymbol{t}'_-) = \bar{\Psi}(\boldsymbol{t}_+, \boldsymbol{t}_-)\bar{\Psi}^{-1}(\boldsymbol{t}'_+, \boldsymbol{t}'_-) \tag{3.26}$$

for any $s, s', \boldsymbol{t}_+ . \boldsymbol{t}_-, \boldsymbol{t'}_-, \boldsymbol{t'}_+$,

This equation resembles similar ones for KP and mKP case. It can be proven similarly to KP and mKP cases that equation (3.26) defines the whole hierarchy. Resemblance with mKP and KP hierarchies can be continued further. We have

$$W = \sum_{j=0}^{\infty} \mathrm{diag}[w_j(s)]\Lambda^{-j}, \quad W^{-1} = \sum_{j=0}^{\infty} \Lambda^{-j}\mathrm{diag}[w_j^*(s+1)],$$
$$\bar{W} = \sum_{j=0}^{\infty} \mathrm{diag}[\bar{w}_j(s)]\Lambda^{j}, \quad \bar{W}^{-1} = \sum_{j=0}^{\infty} \Lambda^{j}\mathrm{diag}[\bar{w}_j^*(s+1)].$$

And define

$$\psi(s,z) = \sum_{j=0}^{\infty} w_j(s) z^{s-j} e^{\xi(\boldsymbol{t}_+, z)}, \quad \psi^*(s,z) = \sum_{j=0}^{\infty} w_j^*(s) z^{-j-s} e^{-\xi(\boldsymbol{t}_+, z)},$$
$$\bar{\psi}(s,z) = \sum_{j=0}^{\infty} \bar{w}_j(s) z^{j+s} e^{\xi(\boldsymbol{t}_-, z^{-1})}, \quad \bar{\psi}^*(s,z) = \sum_{j=0}^{\infty} \bar{w}_j^*(s) z^{j-s} e^{-\xi(\boldsymbol{t}_-, z^{-1})}.$$

After that equation (3.26) can be rewritten as

$$\oint_{\infty} \psi(s,z;\boldsymbol{t}_+,\boldsymbol{t}_-)\psi^*(s',z;\boldsymbol{t'}_+,\boldsymbol{t'}_-)\frac{\mathrm{d}z}{2\pi i} = \oint_{0} \bar{\psi}(s,z;\boldsymbol{t}_+,\boldsymbol{t}_-)\bar{\psi}^*(s',z;\boldsymbol{t'}_+,\boldsymbol{t'}_-)\frac{\mathrm{d}z}{2\pi i}. \tag{3.27}$$

When $s \geq s'$ and $\boldsymbol{t}_- = \boldsymbol{t'}_-$ right hand side is equal to zero and we obtain mKP equation in bilinear form. We can introduce tau-functions.

$$\psi(s,z,\boldsymbol{t}_+,\boldsymbol{t}_-) = z^s e^{\xi(\boldsymbol{t}_+,z)} \frac{\tau_s(\boldsymbol{t}_+ - [z^{-1}],\boldsymbol{t}_-)}{\tau_s(\boldsymbol{t}_+,\boldsymbol{t}_-)} \tag{3.28}$$
$$\psi^*(s,z,\boldsymbol{t}_+,\boldsymbol{t}_-) = z^{-s} e^{\xi(\boldsymbol{t}_+,z)} \frac{\tau_s(\boldsymbol{t}_+ + [z^{-1}],\boldsymbol{t}_-)}{\tau_s(\boldsymbol{t}_+,\boldsymbol{t}_-)}. \tag{3.29}$$

We can also introduce r_n and r_n^* such that

$$\bar{\psi}(s,z,\boldsymbol{t}_+,\boldsymbol{t}_-) = z^s e^{\xi(\boldsymbol{t}_-,z^{-1})} r_s(z,;\boldsymbol{t}_+,\boldsymbol{t}_-) = z^s e^{\xi(\boldsymbol{t}_-,z^{-1})} \sum_{j\geq 0} r_{s,j} z^j \tag{3.30}$$
$$\bar{\psi}^*(s,z,\boldsymbol{t}_+,\boldsymbol{t}_-) = z^{-s} e^{-\xi(\boldsymbol{t}_-,z^{-1})} r_s^*(z,;\boldsymbol{t}_+,\boldsymbol{t}_-) = z^{-s} e^{-\xi(\boldsymbol{t}_-,z^{-1})} \sum_{j\geq 0} r_{s,j}^* z^j. \tag{3.31}$$

Applying calculations similar to one used in Lemma 3 for different choice of

$s - s', \boldsymbol{t}'_- - \boldsymbol{t}_-, \boldsymbol{t}'_+ - \boldsymbol{t}_+$ we can prove following equalities:

$$
\begin{aligned}
r_s^{-1}(z) &= D_-(z^{-1}) r^*_{s+1}(z), && \text{for } \boldsymbol{t}'_+ = \boldsymbol{t}_+,\ \boldsymbol{t}'_- = \boldsymbol{t}_- + [z],\ s' = s+1 \\
\frac{r_s(z)}{r_{s-1}(z)} &= \frac{D_-(z^{-1}) r_{s,0}}{r_{s-1,0}} && \text{for } \boldsymbol{t}'_+ = \boldsymbol{t}_+,\ \boldsymbol{t}'_- = \boldsymbol{t}_- + [z],\ s' = s+2 \\
\frac{D_+(\zeta)\tau_s D_-(z^{-1})\tau_{s+1}}{\tau_s D_+(\zeta) D_-(z^{-1})\tau_{s+1}} &= \frac{r_s(z)}{D_+(\zeta) r_s(z)} && \text{for } \boldsymbol{t}'_+ = \boldsymbol{t}_+ + [\zeta^{-1}],\ \boldsymbol{t}'_- = \boldsymbol{t}_- + [z],\ s' = s+1.
\end{aligned}
$$

Combine last two equations one can find that

$$
\bar\psi(s, z, \boldsymbol{t}_+, \boldsymbol{t}_-) = z^s e^{\xi(\boldsymbol{t}_-, z^{-1})} \frac{\tau_{s+1}(\boldsymbol{t}_+, \boldsymbol{t}_- - [z])}{\tau(\boldsymbol{t}_+, \boldsymbol{t}_-)} \tag{3.32}
$$

$$
\bar\psi^*(s, z, \boldsymbol{t}_+, \boldsymbol{t}_-) = z^{-s} e^{-\xi(\boldsymbol{t}_-, z^{-1})} \frac{\tau_{s-1}(\boldsymbol{t}_+, \boldsymbol{t}_- + [z])}{\tau(\boldsymbol{t}_+, \boldsymbol{t}_-)}. \tag{3.33}
$$

Eventually it results in integral bilinear equation on tau functions of Toda lattice hierarchy:

$$
\begin{aligned}
&\oint_\infty z^{s'-s} e^{\xi(\boldsymbol{t}_+, z) - \xi(\boldsymbol{t}'_+, z)} \tau_s(\boldsymbol{t}_+ - [z^{-1}], \boldsymbol{t}_-) \tau_{s'}(\boldsymbol{t}'_+ + [z^{-1}], \boldsymbol{t}'_-) \frac{\mathrm{d}z}{2\pi i} = \\
&= \oint_0 z^{s'-s} e^{\xi(\boldsymbol{t}_-, z^{-1}) - \xi(\boldsymbol{t}'_-, z^{-1})} \tau_{s+1}(\boldsymbol{t}_+, \boldsymbol{t}_- - [z]) \tau_{s'-1}(\boldsymbol{t}'_+, \boldsymbol{t}'_- + [z]) \frac{\mathrm{d}z}{2\pi i}.
\end{aligned} \tag{3.34}
$$

Or if one consider n not discrete but continues variable and introduce $x = \eta n$, then it can written as

$$
\begin{aligned}
&\oint_\infty z^{\eta(x'-x)} e^{\xi(\boldsymbol{t}_+, z) - \xi(\boldsymbol{t}'_+, z)} \tau(x, \boldsymbol{t}_+ - [z^{-1}], \boldsymbol{t}_-) \tau(x', \boldsymbol{t}'_+ + [z^{-1}], \boldsymbol{t}'_-) \frac{\mathrm{d}z}{2\pi i} = \\
&= \oint_0 z^{\eta(x'-x)} e^{\xi(\boldsymbol{t}_-, z^{-1}) - \xi(\boldsymbol{t}'_-, z^{-1})} \tau(x+\eta, \boldsymbol{t}_+, \boldsymbol{t}_- - [z]) \tau(x'-\eta, \boldsymbol{t}'_+, \boldsymbol{t}'_- + [z]) \frac{\mathrm{d}z}{2\pi i}.
\end{aligned} \tag{3.35}
$$

3.3 Multi-component KP hierarchy

Content of this section is based on (Dickey [1997]).

Another generalization of KP hierarchy comes when we consider elements u_m of operator $\mathcal{L}$ in (2.1) as $n \times n$ matrices

$$\boldsymbol{L} = \partial + U_1\partial^{-1} + U_2\partial^{-2} + ... \tag{3.36}$$

We can introduce times : $t_{k\alpha}$, where $k > 0$ and $1 \leq \alpha \leq n$.

To define dynamics of these elements we introduce operators $\boldsymbol{R}_\alpha$ such that $\boldsymbol{R}_\alpha\boldsymbol{R}_\beta = \delta_{\alpha\beta}\boldsymbol{R}_\alpha$, $[\boldsymbol{R}_\alpha, \boldsymbol{L}] = 0$ and $\sum\limits_\alpha \boldsymbol{R}_\alpha = 1$ (from here and further summations over Greek indices goes from 1 to n).

Now define operators $\boldsymbol{B}_{k\alpha} = (\boldsymbol{L}^k\boldsymbol{R}_\alpha)_+$ where as in regular KP $()_+$ means taking purely differential part. We introduce dynamics

$$\partial_{k\alpha}\boldsymbol{L} = [\boldsymbol{B}_{k\alpha}, L]. \tag{3.37}$$

It is clear from definition, that $\sum\limits_\alpha \partial_{1\alpha} = \partial$.

Similarly to KP case we may introduce dressing operator $\boldsymbol{W} = I + \sum\limits_{k>0} W_k\partial^{-k}$: $\boldsymbol{L} = \boldsymbol{W}\partial\boldsymbol{W}^{-1}$. It is clear, that defined as $\boldsymbol{R}_\alpha = \boldsymbol{W}E_\alpha\boldsymbol{W}^{-1}$, R_α operators are indeed satisfy all requirements . Here $(E_\alpha)_{ij} = \delta_{i\alpha}\delta_{j\alpha}$.

Introducing matrix Baker-Akhiezer and adjoint Baker-Akhiezer functions:

$$\Psi = \boldsymbol{W}e^{\xi(\boldsymbol{t},z)} = e^{\xi(\boldsymbol{t},z)}W \tag{3.38}$$

$$\Psi^* = (\boldsymbol{W}^\dagger)^{-1}e^{-\xi(\boldsymbol{t},z)} = e^{-\xi(\boldsymbol{t},z)}(W^*)^{-1} \tag{3.39}$$

where $\xi(\boldsymbol{t}, z) = \sum\limits_{k\geq 0}\sum\limits_\alpha z^k E_\alpha t_{k\alpha}$.

They are solutions of corresponding generalization of linear problems:

$$\boldsymbol{L}\Psi = z\Psi \qquad \boldsymbol{L}^\dagger\Psi^* = z\Psi^* \tag{3.40}$$

$$\partial_{n\alpha}\Psi = \boldsymbol{B}_{n\alpha}\Psi \quad \partial_{n\alpha}\Psi^* = -\boldsymbol{B}^\dagger_{n\alpha}\Psi^*. \tag{3.41}$$

It can be proven the same way as in scalar case, analogue of Theorem 1:

Theorem 3. *The identity*

$$\operatorname*{res}_z[(\partial^{i_1}_{k_1\alpha_1}...\partial^{i_m}_{k_m\alpha_m}\Psi)\Psi^*] = 0$$

holds for any $(i_1, ..., i_m) \in (\mathbb{N}^*)^m$, $(k_1, ..., k_m) \in (\mathbb{N}^*)^m$ *and* $(\alpha_1, ..., \alpha_m) \in [1, n]^m$ *if and only if* Ψ *and* Ψ^* *of form* $(I + \sum\limits_{k>0} A_i z^{-i})e^{\pm\xi}$ *are solutions of (3.38).*

Or it can be written in integral form

$$\oint\limits_{\infty} \Psi(z; \boldsymbol{t})\Psi^*(z; \boldsymbol{t'})\mathrm{d}z = 0. \tag{3.42}$$

It is possible to generalize notion of τ-function.

First of all we introduce operators $D_\alpha(z) = \exp\left(-\sum\limits_{k>1} \frac{\partial_{k\alpha}}{z^k k}\right)$ which is act by shifting α's times by $[z^{-1}]$ vector. $D_\alpha(z)f(\boldsymbol{t}) = f(..., t_{k\gamma} - \delta_{\alpha\gamma}(1/k)z^{-k}, ...)$.

As in a proof of existence of τ-function for KP it is useful to consider following identities $D_\alpha(\zeta)e^{-\xi(\boldsymbol{t},z)} = (I - E_\alpha + (1 - \zeta/z)^{-1}E_\alpha)e^{-\xi(\boldsymbol{t},z)}$.

Taking (β, β)th and (α, β)th elements of equation (3.42) with $t'_{k\gamma} = t_{k\gamma} + \delta_{\beta\gamma}\frac{1}{k\zeta^k}$ we obtain equations

$$W_{\beta\beta} D_\alpha W^*_{\beta\beta} = 1 \tag{3.43}$$

$$\zeta\frac{W_{\alpha\beta}(\zeta)}{W_{\beta\beta}(\zeta)} = D_\beta(\zeta)W_{1,\alpha\beta}. \tag{3.44}$$

Taking (β, β)th element of equation (3.42) with $t'_{k\gamma} = t_{k\gamma} + \delta_{\beta\gamma}\left(\frac{1}{k\zeta_1^k} + \frac{1}{k\zeta_2^k}\right)$ results in

$$\frac{D_\beta(\zeta_1)W_{\beta\beta}(\zeta_2)}{W_{\beta\beta}(\zeta_2)} = \frac{D_\beta(\zeta_2)W_{\beta\beta}(\zeta_1)}{W_{\beta\beta}(\zeta_1)}. \tag{3.45}$$

Introducing $f_\beta = \log W_{\beta\beta}$ it can be rewritten as

$$(D_\beta(\zeta_1) - 1)f_\beta(\zeta_2) = (D_\beta(\zeta_2) - 1)f_\beta(\zeta_1) \tag{3.46}$$

Combine equations which come from (α, α)th, (α, β)th, (β, β)th and (β, α)th with $t'_{k\gamma} = t_{k\gamma} + \left(\delta_{\beta\gamma}\frac{1}{k\zeta_1^k} + \delta_{\alpha\gamma}\frac{1}{k\zeta_2^k}\right)$ one can show that

$$(D_\alpha(\zeta_1) - 1)f_\beta(\zeta_2) = (D_\beta(\zeta_2) - 1)f_\alpha(\zeta_1). \tag{3.47}$$

As in KP case we will prove that there is function τ such that $f_\alpha(z) = (D_\alpha(z) - 1)\log\tau$.

Introducing operator $N_\alpha(z) = \sum\limits_{j\geq 0} z^{-j-1}\partial_{j\alpha} + \partial_z$ such that $N_\alpha(z)D_\alpha(z)f(\boldsymbol{t}, z) = 0$ and applying it to (3.47) one obtain

$$D_\beta(\zeta_2)N_\alpha(\zeta_1)f_\alpha(\zeta_1) - N_\alpha f_\alpha(\zeta_1) = -\sum_{j\geq 0}\zeta^{-j-1}\partial_{j\alpha}f_\beta(\zeta_2). \tag{3.48}$$

Then multiply this by ζ_1^i and take $\operatorname*{res}\limits_{\zeta_1}$

$$b_{i\alpha} \equiv \operatorname*{res}_{\zeta_1}\zeta_1^i N_\alpha(\zeta_1)f_\alpha(\zeta_1) = D_\beta(\zeta_2)\operatorname*{res}_{\zeta_1}\zeta_1^i N_\alpha(\zeta_1)f_\alpha(\zeta_1) + \partial_{i\alpha}f_\beta(\zeta_2) \tag{3.49}$$

i.e

$$b_{i\alpha} = D_\beta(\zeta_2)b_{i\alpha} + \partial_{i\alpha}f_\beta(\zeta_2). \tag{3.50}$$

Since (i, α) is arbitrary we can differentiate this equality with respect to $t_{j\gamma}$. change indices and substitute one equation from other to obtain equation

$$(D_\beta(\zeta_2) - 1)(\partial_{j\gamma}b_{i\alpha} - \partial_{i\alpha}b_{j\gamma}). \tag{3.51}$$

Since $(D_\beta(\zeta_2) - 1)$ null only functions which are constant for all times, the same argument as in KP case can be applied here to show, that $\partial_{j\gamma}b_{i\alpha} - \partial_{i\alpha}b_{j\gamma} = 0$, which means, that one can introduce τ such that $b_{i\alpha} = \partial_{i\alpha}\log\tau$. Tau function is defined up to multiplication by $c(z)$, however this ambiguity can be hidden inside definition of Baker-Akhiezer functions, which also can be defined up to multiplication by some matrix which depend only on z.

Using equation 3.44 we define $\tau_{\alpha\beta} = \tau W_{1,\alpha\beta}$ and

$$W_{\alpha\beta}(z; \boldsymbol{t}) = \frac{1}{z}\frac{D_\beta(z)\tau_{\alpha\beta}(\boldsymbol{t})}{\tau(\boldsymbol{t})},\ \alpha \neq \beta. \tag{3.52}$$

If we introduce $t_i = \frac{1}{n}\sum\limits_\alpha t_{i\alpha}$ such that $\partial_n = \sum\limits_\alpha \partial_{n\alpha}$ and consider dependence only on t_n variables, we obtain Matrix KP hierarchy.

Chapter 4

Main Results

4.1 KP hierarchy

This thesis continues a series of works started with (Airault et al. [1977]) where authors have considered a singular solutions of KdV equation and shown, that its poles is governed by dynamics of cubic Hamiltonian of Calogero-Moser system in special locus, where H_2 is equal to zero. Following by famous Krichever results (Krichever [1978]) and (Krichever [1980]) where he has shown that connection between pole solutions of nonlinear partial equations and many body systems becomes more natural for KP equation. Shiota in (Shiota [1994]) have extended that correspondence to the whole hierarchy for rational case. He have shown that poles of rational solutions of KP hierarchy evolve with respect to t_m's KP time like particles of rational Calogero-Moser model governed by Hamiltonian $H_m = \mathrm{tr} L^m$ with Calogero-Moser matrix L (1.26).

Later this result was generalized in (Haine [2007]) and (Zabrodin [2020]) for trigonometric solutions with corresponding Hamiltonians
$H_m = \dfrac{1}{2(m+1)\gamma} \mathrm{tr}\left((L+\gamma I)^{m+1} - (L-\gamma I)^{m+1}\right)$ where L is Lax matrix for trigonometric Calogero-Moser system (1.28).

Appendix 4.3 (Prokofev and Zabrodin [2021b]) contains the most general version of this statement. It considers elliptic solution of whole hierarchy in form of

$$\tau(x, \boldsymbol{t}) = \prod_{i=1}^{N} \sigma(x - x_i(\boldsymbol{t})). \tag{4.1}$$

It is proven, that (4.1) gives solution of (2.11) if and only if evolution of x_i's with respect to t_m is governed by

$$H_m = \operatorname*{res}_{z=\infty} (z^m \lambda(z)) \tag{4.2}$$

where $\lambda(z)$ solves

$$\det(L(\lambda) - (z + \zeta(\lambda)I) = 0 \tag{4.3}$$

with elliptic Lax matrix

$$L_{jk} = -p_j \delta_{jk} - (1 - \delta_{jk})\Phi(x_j - x_k, \lambda). \tag{4.4}$$

It appears that there is only one unique solution of (4.3) when $z \to \infty$.

This article also include nontrivial calculations connecting this solution in the limit when one or both periods of elliptic curve goes to infinity with results of previous works.

4.2 2d Toda hierarchy

Dynamics of poles of elliptic solutions to the 2DTL and mKP hierarchies was studied in (Krichever and Zabrodin [1995]). It was proved that the poles move as particles of the integrable Ruijsenaars–Schneider many-body system (Ruijsenaars and Schneider [1986]) which is a relativistic generalization of the Calogero–Moser system. The extension to the level of hierarchies for rational solutions to the mKP equation was made in (Iliev [2007]): again, the evolution of poles with respect to the higher times t_k of the mKP hierarchy is governed by the higher Hamiltonians $-\mathrm{tr}L^k$ of the Ruijsenaars–Schneider system.

Article (Prokofev and Zabrodin [2019]) generalize that result. It contains direct solutions of bilinear relation for the whole 2d Toda lattice (3.35) with trigonometric tau-function of the form

$$\tau(x, \boldsymbol{t}_+, \boldsymbol{t}_-) = \exp\left(-\sum_{k\geq 1} k t_k t_{-k}\right) \prod_{i=1}^{N} \left(e^{2\gamma x} - e^{2\gamma x_i(\boldsymbol{t}_+, \boldsymbol{t}_-)}\right). \tag{4.5}$$

It is shown, that evolution of the x_i's with respect to the time t_m govern by Hamiltonian

$$H_m = -\frac{\sinh(m\gamma\eta)}{m\gamma\eta} \mathrm{tr}(L)^m \tag{4.6}$$

for both positive and negative m. Here

$$L_{ij} = \frac{\gamma\eta e^{\eta p_i}}{\sinh(\gamma(x_i - x_j - \eta))} \prod_{l\neq i} \frac{\sinh(\gamma(x_i - x_l + \eta))}{\sinh(\gamma(x_i - x_l))} \tag{4.7}$$

is the Lax matrix of trigonometric Ruijsenaars–Schneider system.

Generalization to elliptic case is given in (Prokofev and Zabrodin [2021a]) where we consider solutions of 2d Toda lattice hierarchy of the form

$$\tau(x, \boldsymbol{t}_+, \boldsymbol{t}_-) = \exp\left(-\sum_{k\geq 1} k t_k t_{-k}\right) \prod_{i=1}^{N} \sigma\left(x - x_i(\boldsymbol{t}_+, \boldsymbol{t}_-)\right). \tag{4.8}$$

In order for 4.8 to be solution, evolution of x_i with respect to time t_m should be governed by Hamiltonian

$$H_m = \operatorname*{res}_{z=\infty}\left(z^{m-1}\lambda(z)\right) \tag{4.9}$$

for $m > 0$ and

$$H_m = \operatorname*{res}_{z=0}(z^{m-1}\lambda(z)) \tag{4.10}$$

for $m < 0$.

$\lambda(z)$ can be found from the equation

$$\det(L(\lambda) - z^{\eta\zeta(\lambda)}) = 0 \tag{4.11}$$

with elliptic Lax matrix L

$$L_{ij}(\lambda) = e^{p_i}\Phi(x_i - x_j - \eta, \lambda)\prod_{l\neq i}\frac{\sigma(x_i - x_l + \eta)}{\sigma(x_i - x_l)}. \tag{4.12}$$

Equation (4.11) have unique solution near $z = \infty$.

Nontrivial calculations conducted in this paper prove that, degeneration of elliptic curve to its rational or trigonometric limits gives the same results as ones obtained before.

4.3 Matrix KP

The singular (in general, elliptic) solutions to the matrix KP equation were investigated in (Krichever and Zabrodin [1995]). It was shown that the evolution of data of such solutions (positions of poles and some internal degrees of freedom) with respect to the time t_2 is isomorphic to the dynamics of a spin generalization of the Calogero–Moser system (the Gibbons–Hermsen system (Gibbons and Hermsen [1984])). The generalization of this connection to whole hierarchy was studied in (Pashkov and Zabrodin [2018]) for rational solutions. It appears, that dynamics in t_m is governed by Hamiltonian $H_m = \mathrm{tr} L^m$.

Trigonometric version of this result is considered in (Prokofev and Zabrodin [2020]). There are trigonometric solutions of matrix KP hierarchy constructed in this paper. It is proven, that

$$\tau = \prod_{i=1}^{N}(e^{2\gamma x} - e^{2\gamma x_i(\boldsymbol{t})}) \tag{4.13}$$

with

$$W_{1,\alpha\beta} = S_{\alpha\beta} - \sum_i \frac{2\gamma e^{2\gamma x_i(\boldsymbol{t})} a_i^\alpha(\boldsymbol{t}) b_i^\beta(\boldsymbol{t})}{e^{2\gamma x} - e^{2\gamma x_i(\boldsymbol{t})}} \tag{4.14}$$

are solutions to whole matrix KP hierarchy if and only if dynamics of $x_i(\boldsymbol{t})$, $a_i^\alpha(\boldsymbol{t})$, $b_i^\alpha(\boldsymbol{t})$ in t_m is governed by Hamiltonian

$$H_m = \frac{1}{2(m+1)\gamma}\mathrm{tr}\left((L+\gamma I)^{m+1} - (L-\gamma I)^{m+1}\right) \tag{4.15}$$

where

$$L_{jk} = -p_j\delta_{jk} - (1-\delta_{jk})\frac{\gamma\sum\limits_{\alpha} b_j^\alpha a_k^\alpha}{\sinh(\gamma(x_j - x_k))} \tag{4.16}$$

with nonzero Poisson brackets $\{x_i, p_j\} = \delta_{ij}$ and $\{a_i^\alpha, b_j^\beta\} = \delta_{\alpha\beta}\delta_{ij}$.

Appendix 4.3 (Prokofev and Zabrodin [2021c]) contains further generalization to the elliptic level.

$$\tau = \prod_{i=1}^{N} \sigma(x - x_i(\boldsymbol{t})) \tag{4.17}$$

with

$$W_{1,\alpha\beta} = S_{\alpha\beta} - \sum_i a_i^\alpha(\boldsymbol{t}) b_i^\beta(\boldsymbol{t})\zeta(x - x_i(\boldsymbol{t})) \tag{4.18}$$

is solution of matrix KP if dynamics of poles and spins in t_m is governed by

$$H_m = \operatorname*{res}_{z=\infty}(z^m\lambda(z)) \tag{4.19}$$

where $\lambda(z) = \sum\limits_\alpha \lambda_\alpha(z)$ and each $\lambda_\alpha(z)$ is different solution of

$$\det(L(\lambda_\alpha) - (z + \zeta(\lambda_\alpha))I) = 0 \tag{4.20}$$

with elliptic Lax matrix

$$L_{jk} = -p_j\delta_{jk} - (1-\delta_{jk})\Phi(x_j - x_k, \lambda)\sum_\nu b_\nu^j a_\nu^k. \tag{4.21}$$

It appears, that equation (4.20) has n different solutions near $z = \infty$ and each $\lambda_\alpha(z)$ is generating functions of $H_{n\alpha}$-Hamiltonians corresponding $t_{n\alpha}$ flow. So we obtained not only correspondence between Matrix KP and spin Calogero-Moser, but between multi-component KP and spin Calogero-Moser.

Rational and trigonometric limits are also found and they match with results from previous papers.

Bibliography

H. Airault, P. McKean, and J. Moser. Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem. *Communications on Pure and Applied Mathematics*, 1977. URL https://doi.org/10.1002/cpa.3160300106.

J. Boussinesq. Essai sur la theorie des eaux courantes. *' l'Acad. des Sci. Inst. Nat. France*, 1877. URL https://gallica.bnf.fr/ark:/12148/bpt6k56673076.texteImage.

H. W. Braden, A. Marshakov, A. Mironov, and A. Morozov. On Double-Elliptic Integrable Systems. 1. A Duality Argument for the case of SU(2). *Nuclear Physics B*, 2000. URL https://arxiv.org/abs/hep-th/9906240.

F. Calogero. Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials . *Journal of Mathematical Physics*, 1971. URL https://aip.scitation.org/doi/10.1063/1.1665604.

F. Calogero. Exactly solvable one-dimensional many-body problems. *Lettere al Nuovo Cimento*, 1975. URL https://doi.org/10.1007/BF02790495.

F. Calogero and C. Marchioro. Exact solution of a one-dimensional three-body scattering problem with two-body and/or three-body inverse-square potentials. *Journal of Mathematical Physics*, 1974. URL https://aip.scitation.org/doi/10.1063/1.1666827.

E. d'Hoker and D. H. Phong. Order parameters, free fermions, and conservation laws for Calogero–Moser systems. *Asian Journal of Mathematics*, 1998. URL https://arxiv.org/abs/hep-th/9808156.

L A Dickey. *Soliton Equations and Hamiltonian Systems*. WORLD SCIENTIFIC, 2nd edition, 2003. doi:10.1142/5108. URL https://www.worldscientific.com/doi/abs/10.1142/5108.

L.A. Dickey. *Algebraic Aspects of Integrable Systems. Progress in Nonlinear Differential Equations and Their Applications*, volume 26, chapter On τ-Functions of Zakharov—Shabat and Other Matrix Hierarchies of Integrable Equations. 1997. URL https://arxiv.org/pdf/hep-th/9502063.pdf.

V. S. Dryuma. Analytic solution of the two-dimensional Korteweg-de Vries (KdV) equation. *Journal of Experimental and Theoretical Physics Letters*, 1974. URL http://jetpletters.ru/ps/1782/article_27139.shtml.

J. Gibbons and T. Hermsen. A generalisation of the Calogero–Moser system. *Physica D: Nonlinear Phenomena* , 1984. URL https://doi.org/10.1016/0167-2789(84)90015-0.

L. Haine. KP Trigonometric Solitons and an Adelic Flag Manifold. *Symmetry, Integrability and Geometry: Methods and Applications*, 2007. URL https://www.emis.de/journals/SIGMA/2007/015/.

P. Iliev. Rational Ruijsenaars–Schneider hierarchy and bispectral difference operators. *Physica D Nonlinear Phenomena* , 2007. URL https://arxiv.org/abs/math-ph/0609011.

B. B. Kadomtsev and V. I. Petviashvili. On the stability of solitary waves in weakly dispersing media. *Dokl. Akad. Nauk SSSR*, 1970. URL http://mi.mathnet.ru/dan35447.

Korteveg, D.J. and de Vries, G. On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary wave . *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 1895. URL https://www.tandfonline.com/doi/abs/10.1080/14786449508620739.

I. Krichever and A. Zabrodin. Spin generalization of the Ruijsenaars–Schneider model, non- abelian 2D Toda chain and representations of Sklyanin algebra. *Russian Mathematical Surveys* , 1995. URL http://mi.mathnet.ru/eng/umn1121.

I. M. Krichever. Rational solutions of the Kadomtsev — Petviashvili equation and integrable systems of N particles on a line. *Functional Analysis and Its Applications*, 1978. URL https://doi.org/10.1007/BF01077570.

I. M. Krichever. Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles. *Functional Analysis and Its Applications*, 1980. URL https://doi.org/10.1007/BF01078304.

P.D. Lax. Integrals of Nonlinear Equations of Evolution and Solitary Waves . *Communications on Pure and Applied Mathematics*, 1968. URL https://doi.org/10.1002/cpa.3160210503.

R. M. Miura, C.S. Gardner, and M.D. Kruscal. Korteweg-de Vries Equation and Generalizations. II. Existence of Conservation Laws and Constants of Motion. *Journal of Mathematical Physics*, 1968. URL https://doi.org/10.1063/1.1664701.

Moser, J. Three integrable Hamiltonian systems connected with isospectral deformations. *Advances in Mathematics*, 1975. URL https://doi.org/10.1016/0001-8708(75)90151-6.

V. Pashkov and A. Zabrodin. Spin generalization of the Calogero–Moser hierarchy and the matrix KP hierarchy. *Journal of Physics A: Mathematical and Theoretical* , 2018. URL https://iopscience.iop.org/article/10.1088/1751-8121/aabd9c.

V. Prokofev and A. Zabrodin. Toda lattice hierarchy and trigonometric Ruijsenaars–Schneider hierarchy . *Journal of Physics A: Mathematical and Theoretical* , 2019. doi:10.1088/1751-8121/ab520c.

V. Prokofev and A. Zabrodin. Matrix Kadomtsev Petviashvili Hierarchy and Spin Generalization of Trigonometric Calogero Moser Hierarchy. *Proceedings of the Steklov Institute of Mathematics* , 2020. doi:10.1134/S0081543820030177.

V. Prokofev and A. Zabrodin. Elliptic solutions to Toda lattice hierarchy and elliptic Ruijsenaars-Schneider model . *Theoretical and Mathematical Physics* , 2021a. doi:10.1134/S0040577921080080.

V. Prokofev and A. Zabrodin. Elliptic solutions to the KP hierarchy and elliptic Calogero–Moser model . *Journal of Physics A: Mathematical and Theoretical* , 2021b. doi:10.1088/1751-8121/ac0a3.

V. Prokofev and A. Zabrodin. Elliptic solutions to matrix KP hierarchy and spin generalization of elliptic Calogero-Moser model . *Journal of Mathematical Physics* , 2021c. doi:10.1063/5.0051713.

D. Rudneva and A. Zabrodin. Dynamics of poles of elliptic solutions to the BKP equation. *Journal of Physics A: Mathematical and Theoretical*, 2020. URL `https://arxiv.org/abs/1903.00968`.

S. N. M. Ruijsenaars and H. Schneider. A new class of integrable systems and its relation to solitons. *Annals of Physics* , 1986. URL `https://www1.maths.leeds.ac.uk/~siru/papers/p27.pdf`.

J. S. Russel. Report on Waves. *Report of the 14th Meeting of the British Association for the Advancement of Science*, 1844.

Mikio Sato. Soliton equations as dynamical systems on infinite dimensional grassmann manifold. In Hiroshi Fujita, Peter D. Lax, and Gilbert Strang, editors, *Nonlinear Partial Differential Equations in Applied Science; Proceedings of The U.S.-Japan Seminar, Tokyo, 1982*, volume 81 of *North-Holland Mathematics Studies*, pages 259–271. North-Holland, 1983. URL `https://www.sciencedirect.com/science/article/pii/S0304020808720966`.

T. Shiota. Calogero–Moser hierarchy and KP hierarchy. *Journal of Mathematical Physics*, 1994. URL `https://aip.scitation.org/doi/10.1063/1.530713`.

B. Sutherland. Exact Results for a Quantum Many-Body Problem in One Dimension. II. *Physical review A* , 1972. URL `https://journals.aps.org/pra/abstract/10.1103/PhysRevA.5.1372`.

I. M. Toda. Vibration of a Chain with Nonlinear Interaction. *Journal of the Physical Society of Japan* , 1967a. URL `https://journals.jps.jp/doi/10.1143/JPSJ.22.431`.

I. M. Toda. Wave Propagation in Anharmonic Lattices. *Journal of the Physical Society of Japan*, 1967b. URL https://journals.jps.jp/doi/10.1143/JPSJ.23.501.

K. Ueno and K. Takasaki. Toda Lattice Hierarchy . *Advanced Studies in Pure Mathematics*, 1984. URL https://doi.org/10.2969/aspm/00410001.

A. Zabrodin. KP hierarchy and trigonometric Calogero–Moser hierarchy. *Journal of Mathematical Physics*, 2020. URL https://aip.scitation.org/doi/10.1063/1.5120344.

V.E. Zakharov and L.D. Faddeev. Korteweg–de Vries equation: A completely integrable Hamiltonian system . *Functional Analysis and Its Applications*, 1971. URL http://mi.mathnet.ru/eng/faa2612.

Appendix A

V.Prokofev, A Zabrodin "Toda lattice hierarchy and trigonometric Ruijsenaars–Schneider hierarchy"

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Contribution: I conducted all calculations independently. I suggested to use functions $\phi(x)$ and $\bar{\phi}(x)$ for which linear problems (4.9), (4.10) are the same as for Baker-Akhiezer functions of mKP part of hierarchy up to variable changing. Using formula for inverse Cauchy matrix (4.21) I found an explicit formulas for Hamiltonians corresponding to dynamics in negative times.

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Toda lattice hierarchy and trigonometric Ruijsenaars–Schneider hierarchy

V Prokofev[1,2] **and A Zabrodin**[2,3,4]

[1] Moscow Institute of Physics and Technology, Dolgoprudny, Institutsky per., 9, Moscow region, 141700, Russia
[2] Skolkovo Institute of Science and Technology, 143026 Moscow, Russian
[3] National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russian
[4] ITEP, 25 B.Cheremushkinskaya, Moscow 117218, Russian

E-mail: vadim.prokofev@phystech.edu and zabrodin@itep.ru

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Abstract
We consider solutions of the 2D Toda lattice hierarchy, which are trigonometric functions of the 'zeroth' time $t_0 = x$. It is known that their poles move as particles of the trigonometric Ruijsenaars–Schneider model. We extend this correspondence to the level of hierarchies: the dynamics of poles with respect to the mth hierarchical time t_m (respectively, $\bar{t}_m$) of the 2D Toda lattice hierarchy is shown to be governed by the Hamiltonian which is proportional to the mth Hamiltonian $\mathrm{tr}\, L^m$ (respectively, $\mathrm{tr}\, L^{-m}$) of the Ruijsenaars–Schneider model, where L is the Lax matrix.

Keywords: Toda lattice hierarchy, Ruijsenaars–Schneider system, dynamics of poles

1. Introduction

The 2D Toda lattice (2DTL) hierarchy is an infinite set of compatible nonlinear differential-difference equations involving infinitely many time variables $\mathbf{t} = \{t_1, t_2, t_3, \ldots\}$ ('positive' times), $\bar{\mathbf{t}} = \{\bar{t}_1, \bar{t}_2, \bar{t}_3, \ldots\}$ ('negative' times) in which the equations are differential and the 'zeroth' time $t_0 = x$ in which the equations are difference. When the negative times are frozen, the equations involving x and $\mathbf{t}$ variables form the modified Kadomtsev–Petviashvili (mKP) hierarchy which is a subhierarchy of the 2DTL one. Among all solutions to these equations, of special interest are solutions which have a finite number of poles in the variable x in a fundamental domain of the complex plane. In particular, one can consider solutions which are trigonometric or hyperbolic functions of x with poles depending on the times.

The investigation of dynamics of poles of singular solutions to nonlinear integrable equations was initiated in the seminal paper [1], where elliptic and rational solutions to the

Korteweg-de Vries and Boussinesq equations were studied. It was shown that the poles move as particles of the integrable many-body system [2–5] with some restrictions in the phase space. As it was proved in [6, 7], this connection becomes most natural for the more general Kadomtsev–Petviashvili (KP) equation, in which case there are no restrictions in the phase space for the Calogero–Moser dynamics of poles. The method suggested by Krichever [8] for elliptic solutions of the KP equation consists in substituting the solution not in the KP equation itself but in the auxiliary linear problem for it (this implies a suitable pole ansatz for the wave fuction). This method allows one to obtain the equations of motion together with their Lax representation.

The further progress was achieved in Shiota's work [9]. Shiota has shown that the correspondence between rational solutions to the KP equation and the Calogero–Moser system with rational potential can be extended to the level of hierarchies. The evolution of poles with respect to the higher times t_k of the KP hierarchy was shown to be governed by the higher Hamiltonians $H_k = \mathrm{tr}\, L^k$ of the integrable Calogero–Moser system, where L is the Lax matrix. Later this correspondence was generalized to trigonometric solutions of the KP hierarchy (see [10, 11]).

Dynamics of poles of elliptic solutions to the 2DTL and mKP hierarchies was studied in [12]. It was proved that the poles move as particles of the integrable Ruijsenaars–Schneider many-body system [13] which is a relativistic generalization of the Calogero–Moser system. The extension to the level of hierarchies for rational solutions to the mKP equation has been made in [14] (see also [15]): again, the evolution of poles with respect to the higher times t_k of the mKP hierarchy is governed by the higher Hamiltonians $\mathrm{tr}\, L^k$ of the Ruijsenaars–Schneider system.

In this paper we study the correspondence of the 2DTL hierarchy and the Ruijsenaars–Schneider hierarchy for trigonometric solutions of the former. Our method consists in a direct solution of the auxiliary linear problems for the wave function and its adjoint using a suitable pole ansatz. The tau-function of the 2DTL hierarchy for trigonometric solutions has the form

$$\tau(x, \mathbf{t}, \bar{\mathbf{t}}) = \exp\Big(-\sum_{k\geqslant 1} k t_k \bar{t}_k\Big) \prod_{i=1}^{N} \Big(\mathrm{e}^{2\gamma x} - \mathrm{e}^{2\gamma x_i(\mathbf{t}, \bar{\mathbf{t}})}\Big),$$

where γ is a complex parameter. (The zeros x_i of the tau-function are poles of the solution.) When γ is purely imaginary (respectively, real), one deals with trigonometric (respectively, hyperbolic) solutions. The limit $\gamma \to 0$ corresponds to rational solutions. We show that the evolution of the x_i's with respect to the time t_m is governed by the Hamiltonian

$$H_m = -\frac{\sinh(m\gamma\eta)}{m\gamma\eta}\, \mathrm{tr}\, L^m, \tag{1.1}$$

where the parameter η has the meaning of the inverse velocity of light and

$$L_{ij} = \frac{\gamma\eta\, \mathrm{e}^{\eta p_i}}{\sinh(\gamma(x_i - x_j - \eta))} \prod_{l\neq i} \frac{\sinh(\gamma(x_i - x_l + \eta))}{\sinh(\gamma(x_i - x_l))} \tag{1.2}$$

is the Lax matrix of the trigonometric Ruijsenaars–Schneider system. In particular,

$$H_1 = \sum_i \mathrm{e}^{\eta p_i} \prod_{l\neq i} \frac{\sinh(\gamma(x_i - x_l + \eta))}{\sinh(\gamma(x_i - x_l))} \tag{1.3}$$

is the standard first Hamiltonian of the Ruijsenaars–Schneider system. In a similar way, the evolution of the x_i's with respect to the time $\bar{t}_m$ is governed by the Hamiltonian

$$\bar{H}_m = -\frac{\sinh(m\gamma\eta)}{m\gamma\eta}\,\mathrm{tr}\,L^{-m}. \tag{1.4}$$

The paper is organized as follows. In section 2 we remind the reader the main facts about the 2DTL hierarchy. Section 3 is devoted to solutions which are trigonometric functions of $x = t_0$. We derive equations of motion for their poles as functions of the time t_1. In section 4 we consider the dynamics of poles with respect to the higher times and derive the corresponding Hamiltonian equations. In section 5 we derive the self-dual form of equations of motion and show that it encodes all higher equations of motion in the generating form. In section 6 the determinant formula for the tau-function of trigonometric solutions is proved.

2. The 2D Toda latttice hierarchy

Here we very briefly review the 2DTL hierarchy (see [16]). Let us consider the pseudo-difference operators

$$\mathcal{L} = \mathrm{e}^{\eta\partial_x} + \sum_{k\geqslant 0} U_k(x)\mathrm{e}^{-k\eta\partial_x}, \quad \bar{\mathcal{L}} = c(x)\mathrm{e}^{-\eta\partial_x} + \sum_{k\geqslant 0} \bar{U}_k(x)\mathrm{e}^{k\eta\partial_x}, \tag{2.1}$$

where $\mathrm{e}^{\eta\partial_x}$ is the shift operator ($\mathrm{e}^{\pm\eta\partial_x}f(x) = f(x\pm\eta)$) and the coefficient functions U_k, $\bar{U}_k$ are functions of x, $\mathbf{t}$ and $\bar{\mathbf{t}}$. They are the Lax operators of the 2DTL hierarchy. The equations of the hierarchy are differential-difference equations for the functions U_k, $\bar{U}_k$. They are encoded in the Lax equations

$$\partial_{t_m}\mathcal{L} = [\mathcal{A}_m, \mathcal{L}], \quad \partial_{t_m}\bar{\mathcal{L}} = [\mathcal{A}_m, \bar{\mathcal{L}}] \qquad \mathcal{A}_m = (\mathcal{L}^m)_{\geqslant 0}, \tag{2.2}$$

$$\partial_{\bar{t}_m}\mathcal{L} = [\bar{\mathcal{A}}_m, \mathcal{L}], \quad \partial_{\bar{t}_m}\bar{\mathcal{L}} = [\bar{\mathcal{A}}_m, \bar{\mathcal{L}}] \qquad \bar{\mathcal{A}}_m = (\bar{\mathcal{L}}^m)_{<0}, \tag{2.3}$$

where $\left(\sum_{\mathbb{Z}} U_k \mathrm{e}^{k\eta\partial_x}\right)_{\geqslant 0} = \sum_{k\geqslant 0} U_k \mathrm{e}^{k\eta\partial_x}$, $\left(\sum_{k\in\mathbb{Z}} U_k \mathrm{e}^{k\eta\partial_x}\right)_{<0} = \sum_{k<0} U_k \mathrm{e}^{k\eta\partial_x}$ For example, $\mathcal{A}_1 = \mathrm{e}^{\eta\partial_x} + U_0(x)$, $\bar{\mathcal{A}}_1 = c(x)\mathrm{e}^{-\eta\partial_x}$.

An equivalent formulation is through the zero curvature (Zakharov–Shabat) equations

$$\partial_{t_n}\mathcal{A}_m - \partial_{t_m}\mathcal{A}_n + [\mathcal{A}_m, \mathcal{A}_n] = 0, \tag{2.4}$$

$$\partial_{\bar{t}_n}\mathcal{A}_m - \partial_{t_m}\bar{\mathcal{A}}_n + [\mathcal{A}_m, \bar{\mathcal{A}}_n] = 0, \tag{2.5}$$

$$\partial_{\bar{t}_n}\bar{\mathcal{A}}_m - \partial_{\bar{t}_m}\bar{\mathcal{A}}_n + [\bar{\mathcal{A}}_m, \bar{\mathcal{A}}_n] = 0. \tag{2.6}$$

In particular, at $n = 1$, $m = 2$ we obtain from (2.4)

$$\begin{cases} \partial_{t_1}\Big(U_0(x) + U_0(x-\eta)\Big) = U_1(x+\eta) - U_1(x-\eta) \\[2ex] \partial_{t_2}U_0(x) = \partial_{t_1}\Big(U_0^2(x) + U_1(x+\eta) + U_1(x)\Big). \end{cases} \tag{2.7}$$

Excluding U_1 from this system, one gets the mKP equation for $v(x) = U_0(x)$:

$$\partial_{t_2}\Big(v(x+\eta) - v(x)\Big) = \partial_{t_1}^2\Big(v(x+\eta) + v(x)\Big) + \partial_{t_1}\Big(v^2(x+\eta) - v^2(x)\Big). \tag{2.8}$$

From (2.5) at $m = n = 1$ we have

$$\begin{cases} \partial_{t_1} \log c(x) = v(x) - v(x-\eta) \\ \partial_{\bar{t}_1} v(x) = c(x) - c(x+\eta). \end{cases}$$

Excluding $v(x)$, we get the second order differential-difference equation for $c(x)$:

$$\partial_{t_1}\partial_{\bar{t}_1} \log c(x) = 2c(x) - c(x+\eta) - c(x-\eta)$$

which is one of the forms of the 2D Toda equation. After the change of variables $c(x) = \mathrm{e}^{\varphi(x)-\varphi(x-\eta)}$ it acquires the most familiar form

$$\partial_{t_1}\partial_{\bar{t}_1} \varphi(x) = \mathrm{e}^{\varphi(x)-\varphi(x-\eta)} - \mathrm{e}^{\varphi(x+\eta)-\varphi(x)}. \tag{2.9}$$

The zero curvature equations are compatibility conditions for the auxiliary linear problems

$$\partial_{t_m}\psi = \mathcal{A}_m(x)\psi, \quad \partial_{\bar{t}_m}\psi = \bar{\mathcal{A}}_m(x)\psi, \tag{2.10}$$

where the wave function ψ depends on a spectral parameter z: $\psi = \psi(z;\mathbf{t})$. The wave function has the following expansion in powers of z:

$$\psi = z^{x/\eta}\mathrm{e}^{\xi(\mathbf{t},z)}\left(1 + \frac{\xi_1(x,\mathbf{t},\bar{\mathbf{t}})}{z} + \frac{\xi_2(x,\mathbf{t},\bar{\mathbf{t}})}{z^2} + \ldots\right), \tag{2.11}$$

where

$$\xi(\mathbf{t},z) = \sum_{k\geqslant 1} t_k z^k. \tag{2.12}$$

The wave operator is the pseudo-difference operator of the form

$$\mathcal{W}(x) = 1 + \xi_1(x)\mathrm{e}^{-\eta\partial_x} + \xi_2(x)\mathrm{e}^{-2\eta\partial_x} + \ldots \tag{2.13}$$

with the same coefficient functions ξ_k as in (2.11), then the wave function can be written as

$$\psi = \mathcal{W}(x) z^{x/\eta}\mathrm{e}^{\xi(\mathbf{t},z)}. \tag{2.14}$$

The adjoint wave function $\psi^\dagger$ is defined by the formula

$$\psi^\dagger = (\mathcal{W}^\dagger(x-\eta))^{-1} z^{-x/\eta}\mathrm{e}^{-\xi(\mathbf{t},z)} \tag{2.15}$$

(see, e.g. [17]), where the adjoint difference operator is defined according to the rule $(f(x)\circ \mathrm{e}^{n\eta\partial_x})^\dagger = \mathrm{e}^{-n\eta\partial_x}\circ f(x)$. The auxiliary linear problems for the adjoint wave function have the form

$$-\partial_{t_m}\psi^\dagger = \mathcal{A}_m^\dagger(x-\eta)\psi^\dagger. \tag{2.16}$$

In particular, we have:

$$\begin{aligned} \partial_{t_1}\psi(x) &= \psi(x+\eta) + v(x)\psi(x), \\ -\partial_{t_1}\psi^\dagger(x) &= \psi^\dagger(x-\eta) + v(x-\eta)\psi^\dagger(x), \end{aligned} \tag{2.17}$$

$$\partial_{\bar{t}_1}\psi(x) = c(x)\psi(x-\eta). \tag{2.18}$$

A common solution to the 2DTL hierarchy is provided by the tau-function $\tau = \tau(x,\mathbf{t},\bar{\mathbf{t}})$ [18, 19]. The hierarchy is encoded in the bilinear relation

$$\oint_{\infty} z^{\frac{x-x'}{\eta}-1} e^{\xi(\mathbf{t},z)-\xi(\mathbf{t}',z)} \tau\Big(x, \mathbf{t}-[z^{-1}], \bar{\mathbf{t}}\Big)\tau\Big(x'+\eta, \mathbf{t}'+[z^{-1}], \bar{\mathbf{t}}'\Big)\mathrm{d}z$$
$$= \oint_{0} z^{\frac{x-x'}{\eta}-1} e^{\xi(\bar{\mathbf{t}},z^{-1})-\xi(\bar{\mathbf{t}}',z^{-1})} \tau\Big(x+\eta, \mathbf{t}, \bar{\mathbf{t}}-[z]\Big)\tau\Big(x', \mathbf{t}', \bar{\mathbf{t}}'+[z]\Big)\mathrm{d}z \tag{2.19}$$

valid for all x, x', $\mathbf{t}, \mathbf{t}'$, $\bar{\mathbf{t}}, \bar{\mathbf{t}}'$, where

$$\mathbf{t}\pm[z] = \Big\{t_1 \pm z, t_2 \pm \frac{1}{2}z^2, t_3 \pm \frac{1}{3}z^3, \dots\Big\}. \tag{2.20}$$

The integration contour in the left hand side is a big circle around infinity separating the singularities coming from the exponential factor from those coming from the tau-functions. The integration contour in the right hand side is a small circle around zero separating the singularities coming from the exponential factor from those coming from the tau-functions. In particular, setting $x = x'$, $\bar{\mathbf{t}} = \bar{\mathbf{t}}'$, one obtains from (2.19) the bilinear relation for the mKP hierarchy

$$\frac{1}{2\pi \mathrm{i}}\oint_{\infty} z^{-1} e^{\xi(\mathbf{t},z)-\xi(\mathbf{t}',z)} \tau\Big(x, \mathbf{t}-[z^{-1}]\Big)\tau\Big(x+\eta, \mathbf{t}'+[z^{-1}]\Big)\mathrm{d}z = \tau(x+\eta, \mathbf{t})\tau(x, \mathbf{t}'). \tag{2.21}$$

Consequences of the bilinear relations (which are in fact equivalent to the whole hierarchy, see [20]) are the equations

$$\mu\tau(x+\eta, \mathbf{t}+[\lambda^{-1}]-[\mu^{-1}], \bar{\mathbf{t}})\tau(x, \mathbf{t}, \bar{\mathbf{t}}) - \lambda\tau(x+\eta, \mathbf{t}, \bar{\mathbf{t}})\tau(x, \mathbf{t}+[\lambda^{-1}]-[\mu^{-1}], \bar{\mathbf{t}})$$
$$+(\lambda-\mu)\tau(x+\eta, \mathbf{t}+[\lambda^{-1}], \bar{\mathbf{t}})\tau(x, \mathbf{t}-[\mu^{-1}], \bar{\mathbf{t}}) = 0, \tag{2.22}$$

$$\tau(x, \mathbf{t}-[\lambda^{-1}], \bar{\mathbf{t}})\tau(x, \mathbf{t}, \bar{\mathbf{t}}-[\nu]) - \tau(x, \mathbf{t}, \bar{\mathbf{t}})\tau(x, \mathbf{t}-[\lambda^{-1}], \bar{\mathbf{t}}-[\nu])$$
$$= \nu\lambda^{-1}\tau(x+\eta, \mathbf{t}, \bar{\mathbf{t}}-[\nu])\tau(x-\eta, \mathbf{t}-[\lambda^{-1}], \bar{\mathbf{t}}). \tag{2.23}$$

There is also an equation similar to (2.22) with shifts of the negative times. Together with the tau-function τ it is convenient to introduce another tau-function, τ', which differs from τ by a simple factor:

$$\tau'(x, \mathbf{t}, \bar{\mathbf{t}}) = \exp\left(\sum_{k\geqslant 1} k t_k \bar{t}_k\right)\tau(x, \mathbf{t}, \bar{\mathbf{t}}). \tag{2.24}$$

The coefficient functions of the Lax operators can be expressed through the tau-function. In particular,

$$U_0(x) = v(x) = \partial_{t_1}\log\frac{\tau(x+\eta)}{\tau(x)}, \quad c(x) = \frac{\tau(x+\eta)\tau(x-\eta)}{\tau^2(x)}. \tag{2.25}$$

After this substitution the mKP equation (2.8) becomes

$$\partial_{t_2}\log\frac{\tau(x+\eta)}{\tau(x)} = \partial_{t_1}^2\log(\tau(x+\eta)\tau(x)) + \left(\partial_{t_1}\log\frac{\tau(x+\eta)}{\tau(x)}\right)^2, \tag{2.26}$$

which can be also represented in the bilinear form

$$\tau(x)\partial_{t_2}\tau(x+\eta) - \tau(x+\eta)\partial_{t_2}\tau(x) = \tau(x+\eta)\partial_{t_1}^2\tau(x) - 2\partial_{t_1}\tau(x+\eta)\partial_{t_1}\tau(x) + \tau(x)\partial_{t_1}^2\tau(x+\eta).$$

The Toda equation (2.9) becomes

$$\partial_{t_1}\partial_{\bar{t}_1}\log\tau(x) = 1 - \frac{\tau(x+\eta)\tau(x-\eta)}{\tau^2(x)}. \tag{2.27}$$

The wave function and its adjoint are expressed through the tau-function according to the formulas

$$\psi = z^{x/\eta}\mathrm{e}^{\xi(\mathbf{t},z)}\frac{\tau(x,\mathbf{t}-[z^{-1}],\bar{\mathbf{t}})}{\tau(x,\mathbf{t},\bar{\mathbf{t}})}, \tag{2.28}$$

$$\psi^\dagger = z^{-x/\eta}\mathrm{e}^{-\xi(\mathbf{t},z)}\frac{\tau(x,\mathbf{t}+[z^{-1}],\bar{\mathbf{t}})}{\tau(x,\mathbf{t},\bar{\mathbf{t}})}. \tag{2.29}$$

One may also introduce the complimentary wave functions $\bar{\psi}$, $\bar{\psi}^\dagger$ by the formulas

$$\bar{\psi} = z^{x/\eta}\mathrm{e}^{\xi(\bar{\mathbf{t}},z^{-1})}\frac{\tau(x+\eta,\mathbf{t},\bar{\mathbf{t}}-[z])}{\tau(x,\mathbf{t},\bar{\mathbf{t}})}, \tag{2.30}$$

$$\bar{\psi}^\dagger = z^{-x/\eta}\mathrm{e}^{-\xi(\bar{\mathbf{t}},z^{-1})}\frac{\tau(x-\eta,\mathbf{t},\bar{\mathbf{t}}+[z])}{\tau(x,\mathbf{t},\bar{\mathbf{t}})}. \tag{2.31}$$

They satisfy the same auxiliary linear problems as the wave functions ψ, $\psi^\dagger$. It will be more convenient for us to work with the renormalized wave functions

$$\phi(x) = \frac{\tau(x)}{\tau(x+\eta)}\,\bar{\psi}(x) = z^{x/\eta}\mathrm{e}^{\xi(\bar{\mathbf{t}},z^{-1})}\frac{\tau(x+\eta,\mathbf{t},\bar{\mathbf{t}}-[z])}{\tau(x+\eta,\mathbf{t},\bar{\mathbf{t}})}, \tag{2.32}$$

$$\phi^\dagger(x) = \frac{\tau(x)}{\tau(x-\eta)}\,\bar{\psi}^\dagger(x) = z^{-x/\eta}\mathrm{e}^{-\xi(\bar{\mathbf{t}},z^{-1})}\frac{\tau(x-\eta,\mathbf{t},\bar{\mathbf{t}}+[z])}{\tau(x-\eta,\mathbf{t},\bar{\mathbf{t}})}. \tag{2.33}$$

It is easy to check that they satisfy the linear equations

$$\partial_{\bar{t}_1}\phi(x) = \phi(x-\eta) - \bar{v}(x)\phi(x), \quad -\partial_{\bar{t}_1}\phi^\dagger(x) = \phi^\dagger(x+\eta) - \bar{v}(x-\eta)\phi^\dagger(x), \tag{2.34}$$

where $\bar{v}(x) = \partial_{\bar{t}_1}\log\frac{\tau(x+\eta)}{\tau(x}$.

Finally, let us point out useful corollaries of the bilinear relation (2.21). Differentiating it with respect to t_m and putting $\mathbf{t} = \mathbf{t}'$ after that, we obtain:

$$\frac{1}{2\pi\mathrm{i}}\oint_\infty z^{m-1}\tau\Big(x,\mathbf{t}-[z^{-1}]\Big)\tau\Big(x+\eta,\mathbf{t}+[z^{-1}]\Big)\mathrm{d}z$$
$$= \partial_{t_m}\tau(x+\eta,\mathbf{t})\tau(x,\mathbf{t}) - \partial_{t_m}\tau(x,\mathbf{t})\tau(x+\eta,\mathbf{t}) \tag{2.35}$$

or

$$\operatorname*{res}_\infty\Big(z^m\psi(x)\psi^\dagger(x+\eta)\Big) = \partial_{t_m}\log\frac{\tau(x+\eta)}{\tau(x)}. \tag{2.36}$$

In a similar way, differentiating the bilinear relation (2.19) with respect to $\bar{t}_m$ and putting $x = x'$, $\mathbf{t} = \mathbf{t}'$, $\bar{\mathbf{t}} = \bar{\mathbf{t}}'$ after that, we obtain the relation

$$\operatorname*{res}_0\Big(z^{-m}\phi(x)\phi^\dagger(x+\eta)\Big) = -\partial_{\bar{t}_m}\log\frac{\tau(x+\eta)}{\tau(x)}. \tag{2.37}$$

Here res_{∞}, res_{0} are defined according to the convention $\mathrm{res}_{\infty}(z^{-n}) = \delta_{n1}$, $\mathrm{res}_{0}(z^{-n}) = \delta_{n1}$.

3. Trigonometric solutions to the mKP equation

We are going to consider solutions which are trigonometric (i.e. single-periodic) functions of the variable x. For trigonometric solutions the tau-function has the form

$$\tau(x, \mathbf{t}) = \prod_{i=1}^{N} \left(e^{2\gamma x} - e^{2\gamma x_i(\mathbf{t})} \right). \tag{3.1}$$

(In this section we ignore the dependence on the negative times keeping them fixed to zero.) This function has a single period $\pi i/\gamma$ in the complex plane. As in [11], we pass to the exponentiated variables

$$w = e^{2\gamma x}, \quad w_i = e^{2\gamma x_i}. \tag{3.2}$$

In these variables, the tau-function becomes a polynomial of degree N with roots w_i which are supposed to be distinct: $\tau = \prod_i (w - w_i)$. The function $v(x) = \partial_{t_1} \log\Big(\tau(x+\eta)/\tau(x)\Big)$ is

$$v(x) = \sum_i \left(\frac{\dot{w}_i}{w - w_i} - \frac{\dot{w}_i}{qw - w_i} \right), \tag{3.3}$$

where

$$q = e^{2\gamma\eta} \tag{3.4}$$

and here and below dot means the t_1-derivative.

We begin with the investigation of the t_1-dynamics of the poles. The ansatz for the ψ-function depending on the spectral parameter z suggested by equation (2.28) is

$$\psi = z^{x/\eta} e^{t_1 z} \left(1 + \sum_i \frac{2\gamma c_i}{w - w_i} \right), \tag{3.5}$$

where we have put $t_k = 0$ for $k \geqslant 2$. The coefficients c_i depend on $\mathbf{t}$ and on z. Substituting ψ and v into the first auxiliary linear problem in (2.17) $-\partial_{t_1}\psi(x) + \psi(x+\eta) + v(x)\psi(x) = 0$, we get:

$$-z \sum_i \frac{c_i}{w - w_i} - \sum_i \frac{\dot{c}_i}{w - w_i} - \sum_i \frac{\dot{w}_i c_i}{(w - w_i)^2} + \sum_i \frac{q^{-1} c_i}{w - q^{-1} w_i}$$

$$+ \frac{1}{2\gamma} \sum_i \left(\frac{\dot{w}_i}{w - w_i} - \frac{\dot{w}_i q^{-1}}{w - q^{-1} w_i} \right) + \sum_i \left(\frac{\dot{w}_i}{w - w_i} - \frac{\dot{w}_i q^{-1}}{w - q^{-1} w_i} \right) \sum_k \frac{c_k}{w - w_k} = 0.$$

The left hand side is a rational function of w vanishing at infinity with simple poles at $w = w_i$ and $w = q^{-1} w_i$ (the second order poles cancel identically). We should equate residues at the poles to zero. This gives the following system of linear equations for the coefficients c_i:

$$\begin{cases} zc_i - q\sum_k \dfrac{\dot{w}_i c_k}{w_i - qw_k} = \dfrac{1}{2\gamma}\,\dot{w}_i \\[2ex] \dot{c}_i = c_i\left(\sum_{k\neq i}\dfrac{\dot{w}_k}{w_i - w_k} - \sum_k \dfrac{\dot{w}_k}{qw_i - w_k}\right) + \sum_{k\neq i}\dfrac{\dot{w}_i c_k}{w_i - w_k} - q\sum_k \dfrac{\dot{w}_i c_k}{w_i - qw_k}. \end{cases} \tag{3.6}$$

In a similar way, the adjoint linear problem $\partial_{t_1}\psi^\dagger(x) + \psi^\dagger(x-\eta) + v(x-\eta)\psi^\dagger(x) = 0$ with the ansatz for the $\psi^\dagger$-function

$$\psi^\dagger = z^{-x/\eta}\mathrm{e}^{-t_1 z}\left(1 + \sum_i \frac{2\gamma c_i^*}{w - w_i}\right) \tag{3.7}$$

leads to the linear equations for the coefficients c_i^*:

$$\begin{cases} zc_i^* - \sum_k \dfrac{\dot{w}_i c_k^*}{w_k - qw_i} = -\dfrac{1}{2\gamma}\,\dot{w}_i \\[2ex] \dot{c}_i^* = c_i^*\left(\sum_{k\neq i}\dfrac{\dot{w}_k}{w_i - w_k} + \sum_k \dfrac{q\dot{w}_k}{w_i - qw_k}\right) + \sum_{k\neq i}\dfrac{\dot{w}_i c_k^*}{w_i - w_k} - \sum_k \dfrac{\dot{w}_i c_k^*}{qw_i - w_k}. \end{cases} \tag{3.8}$$

After the gauge transformation $\tilde{c}_i = c_i w_i^{-1/2}$, $\tilde{c}_i^* = c_i^* w_i^{-1/2}$ the above conditions can be written in the matrix form

$$\left(zI - q^{1/2}L\right)\tilde{\mathbf{c}} = \dot{X}W^{1/2}\mathbf{e}, \quad \partial_{t_1}\tilde{\mathbf{c}} = M\tilde{\mathbf{c}}, \tag{3.9}$$

$$\tilde{\mathbf{c}}^*\dot{X}^{-1}\left(zI - q^{-1/2}L\right) = -\mathbf{e}^T W^{1/2}, \quad \partial_{t_1}\tilde{\mathbf{c}}^* = -\tilde{\mathbf{c}}^*\tilde{M}, \tag{3.10}$$

where $\tilde{\mathbf{c}} = (\tilde{c}_1, \ldots, \tilde{c}_N)^T$ is a column vector, $\tilde{\mathbf{c}}^* = (\tilde{c}_1^*, \ldots, \tilde{c}_N^*)$ is a row vector, $\mathbf{e} = (1, 1, \ldots, 1)^T$, I is the identity matrix and the matrices X, W, L, M, $\tilde{M}$ are

$$X = \mathrm{diag}\,(x_1, x_2, \ldots, x_N), \quad W = \mathrm{diag}\,(w_1, w_2, \ldots, w_N), \tag{3.11}$$

$$L_{ij} = 2\gamma q^{1/2}\,\frac{\dot{x}_i w_i^{1/2} w_j^{1/2}}{w_i - qw_j}, \tag{3.12}$$

$$M_{ij} = \gamma\delta_{ij}\left(\sum_{k\neq i}\frac{w_i + w_k}{w_i - w_k}\,\dot{x}_k - \sum_k \frac{qw_i + w_k}{qw_i - w_k}\,\dot{x}_k\right) + 2\gamma\frac{\dot{x}_i w_i^{1/2} w_j^{1/2}}{w_i - w_j}\,(1 - \delta_{ij}) - 2\gamma q\frac{\dot{x}_i w_i^{1/2} w_j^{1/2}}{w_i - qw_j}, \tag{3.13}$$

$$\tilde{M}_{ji} = -\gamma\delta_{ij}\left(\sum_{k\neq i}\frac{w_i + w_k}{w_i - w_k}\,\dot{x}_k - \sum_k \frac{w_i + qw_k}{w_i - qw_k}\,\dot{x}_k\right) + 2\gamma\frac{\dot{x}_i w_i^{1/2} w_j^{1/2}}{w_j - w_i}\,(1 - \delta_{ij}) - 2\gamma\frac{\dot{x}_i w_i^{1/2} w_j^{1/2}}{w_j - qw_i}. \tag{3.14}$$

The following commutation relation can be checked directly:

$$q^{-1/2}WL - q^{1/2}LW = W^{-1/2}\dot{W}EW^{1/2}. \tag{3.15}$$

Here $E = \mathbf{e} \otimes \mathbf{e}^T$ is the rank 1 matrix with matrix elements $E_{ij} = 1$. This commutation relation will be important in what follows.

The system of linear equation (3.9) is overdetermined. Taking the t_1-derivative of the first equation in (3.9) and substituting the second equation, one obtains the compatibility condition of the system in the form

$$\Big(\dot{L} + [L, M]\Big)\tilde{\mathbf{c}} + q^{-1/2}\Big(\ddot{X} + \gamma\dot{X}^2 - M\dot{X}\Big)W^{1/2}\mathbf{e} = 0. \tag{3.16}$$

A straightforward calculation shows that

$$\dot{L} + [L, M] = RL,$$

$$\Big(\ddot{X} + \gamma\dot{X}^2 - M\dot{X}\Big)W^{1/2}\mathbf{e} = R\dot{X}W^{1/2}\mathbf{e},$$

where

$$R = \ddot{X}\dot{X}^{-1} + D^+ + D^- - 2D^0$$

and the diagonal matrices $D^{\pm}$, D_0 are

$$D^{\pm}_{ij} = \delta_{ij}\gamma \sum_{k\neq i} \frac{q^{\pm 1}w_i + w_k}{q^{\pm 1}w_i - w_k}\,\dot{x}_k, \quad D^0_{ij} = \delta_{ij}\gamma \sum_{k\neq i} \frac{w_i + w_k}{w_i - w_k}\,\dot{x}_k.$$

Therefore, the compatibility condition takes the form $R\tilde{\mathbf{c}} = 0$ which means that $R_{ii} = 0$ for all i. This leads to the equations of motion of the trigonometric Ruijsenaars–Schneider model

$$\begin{aligned}
\ddot{x}_i &= -\gamma \sum_{k\neq i} \dot{x}_i\dot{x}_k \Big(\coth(\gamma(x_{ik}+\eta)) + \coth(\gamma(x_{ik}-\eta)) - 2\coth(\gamma x_{ik})\Big) \\
&= \sum_{k\neq i} \dot{x}_i\dot{x}_k \frac{2\gamma \sinh^2(\gamma\eta)\cosh(\gamma x_{ik})}{\sinh(\gamma x_{ik})\sinh(\gamma(x_{ik}+\eta))\sinh(\gamma(x_{ik}-\eta))},
\end{aligned} \tag{3.17}$$

where $x_{ik} = x_i - x_k$. The matrix equation $\dot{L} + [L, M'] = 0$ with

$$L_{ij} = \frac{\gamma\dot{x}_i}{\sinh(\gamma(x_{ij}-\eta))}, \tag{3.18}$$

$$M'_{ij} = \gamma\delta_{ij}\left(\sum_{k\neq i} \dot{x}_k \coth(\gamma x_{ik}) - \sum_{k} \dot{x}_k \coth(\gamma(x_{ik}+\eta))\right) + (1-\delta_{ij})\,\frac{\gamma\dot{x}_i}{\sinh(\gamma x_{ij})} \tag{3.19}$$

provides the Lax representation for them.

Equations (3.17) are Hamiltonian with the Hamiltonian

$$H_1 = \sum_i \mathrm{e}^{\eta p_i} \prod_{k\neq i} \frac{\sinh(\gamma(x_{ik}+\eta))}{\sinh(\gamma x_{ik})} \tag{3.20}$$

and the canonical Poisson brackets between p_i and x_i. The velocity is given by

$$\dot{x}_i = \frac{\partial H_1}{\partial p_i} = \eta e^{\eta p_i} \prod_{k\neq i} \frac{\sinh(\gamma(x_{ik}+\eta))}{\sinh(\gamma x_{ik})}.$$

The Lax representation implies that the higher conserved quantities are $\mathrm{tr}\, L^m$. It is proved in [21] that they are in involution, i.e. the system is integrable.

Let us consider the transformation of the phase space coordinates $(p_i, x_i) \to (u_i, x_i')$, where $x_i' = x_i$ and

$$u_i = \log \dot{x}_i = \eta p_i + \sum_{k\neq i} \log \frac{\sinh(\gamma(x_{ik}+\eta))}{\sinh(\gamma x_{ik})} + \log \eta. \tag{3.21}$$

Then the derivatives transform as follows:

$$\frac{\partial f}{\partial p_i} = \sum_k \frac{\partial u_k}{\partial p_i}\frac{\partial f}{\partial u_k} + \sum_k \frac{\partial x_k'}{\partial p_i}\frac{\partial f}{\partial x_k'} = \eta \frac{\partial f}{\partial u_i}, \tag{3.22}$$

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \sum_k \frac{\partial u_k}{\partial x_i}\frac{\partial f}{\partial u_k} + \sum_k \frac{\partial x_k'}{\partial x_i}\frac{\partial f}{\partial x_k'} \\ &= \frac{\partial f}{\partial x_i'} + \gamma \frac{\partial f}{\partial u_i} \sum_{l\neq i} (\coth(\gamma(x_{il}+\eta)) - \coth(\gamma x_{il})) \\ &\quad + \gamma \sum_{k\neq i} \frac{\partial f}{\partial u_k} (\coth(\gamma(x_{ik}+\eta)) - \coth(\gamma x_{ik})). \end{aligned} \tag{3.23}$$

At this point we finish the discussion of the t_1-dynamics of poles and pass to the higher times in the next section.

4. The higher Hamiltonian equations

4.1. Positive times

In order to study the dynamics of poles in the higher positive times $\mathbf{t}$, we use the relation (2.36), which, after the substitution of the wave functions for the trigonometric solutions, takes the form

$$\frac{1}{2\pi i}\oint_\infty z^{m-1}\left(1+\sum_i \frac{2\gamma c_i}{w-w_i}\right)\left(1+\sum_k \frac{2\gamma c_k^*}{qw-w_k}\right) dz = \sum_i \left(\frac{\partial_{t_m} w_i}{w-w_i} - \frac{\partial_{t_m} w_i}{qw-w_i}\right).$$

The both sides are rational functions of w with simple poles at $w = w_i$ and $w = q^{-1} w_i$ vanishing at infinity. Identifying the residues at the poles at $w = w_i$ in the both sides, we obtain:

$$\partial_{t_m} x_i = -2\gamma \operatorname*{res}_\infty \left(z^m \tilde{c}_i^* \dot{w}_i^{-1} \tilde{c}_i\right). \tag{4.1}$$

From (3.9) and (3.10) we have:

$$\tilde{\mathbf{c}} = \frac{1}{2\gamma}(zI - q^{1/2}L)^{-1}\dot{W}W^{-1/2}\mathbf{e}, \quad \tilde{\mathbf{c}}^* = -\frac{1}{2\gamma}\mathbf{e}^T W^{1/2}(zI - q^{-1/2}L)^{-1}\dot{W}W^{-1}.$$

Substituting this into (4.1), we get:

$$\partial_{t_m} x_i = \frac{1}{2\gamma} \operatorname*{res}_{\infty} \sum_{k,k'} \left[z^m w_k^{1/2} \Big(\frac{1}{zI - q^{-1/2}L} \Big)_{ki} w_i^{-1} \Big(\frac{1}{zI - q^{1/2}L} \Big)_{ik'} \dot{w}_{k'} w_{k'}^{-1/2} \right]$$

$$= \frac{1}{2\gamma} \operatorname*{res}_{\infty} \operatorname{tr} \left(z^m \dot{W} W^{-1/2} E W^{1/2} \frac{1}{zI - q^{-1/2}L} E_i W^{-1} \frac{1}{zI - q^{1/2}L} \right),$$

where E_i is the diagonal matrix with the matrix elements $(E_i)_{jk} = \delta_{ij}\delta_{ik}$. Using the commutation relation (3.15), we have:

$$\partial_{t_m} x_i = \frac{1}{2\gamma} \operatorname*{res}_{\infty} \operatorname{tr} \left(z^m (q^{-1/2}WL - q^{1/2}LW) \frac{1}{zI - q^{-1/2}L} E_i W^{-1} \frac{1}{zI - q^{1/2}L} \right)$$

$$= \frac{1}{2\gamma} \operatorname*{res}_{\infty} \operatorname{tr} \left(z^m \Big(E_i \frac{1}{zI - q^{-1/2}L} - E_i \frac{1}{zI - q^{1/2}L} \Big) \right).$$

Next, we use the easily proved identity

$$E_i L = \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial u_i} = \eta^{-1} \frac{\partial L}{\partial p_i}$$

(see (3.22)) to continue the chain of equalities:

$$\partial_{t_m} x_i = \frac{1}{2\gamma\eta} \operatorname*{res}_{\infty} \operatorname{tr} \left(z^m \left(\frac{\partial L}{\partial p_i} \frac{L^{-1}}{zI - q^{-1/2}L} - \frac{\partial L}{\partial p_i} \frac{L^{-1}}{zI - q^{1/2}L} \right) \right)$$

$$= \frac{1}{2\gamma\eta} (q^{-m/2} - q^{m/2}) \operatorname{tr} \left(\frac{\partial L}{\partial p_i} L^{m-1} \right) = -\frac{\sinh(m\gamma\eta)}{m\gamma\eta} \frac{\partial}{\partial p_i} \operatorname{tr} L^m. \quad (4.2)$$

In this way we have obtained one half of the Hamiltonian equations for the higher flows

$$\partial_{t_m} x_i = \frac{\partial H_m}{\partial p_i}, \quad H_m = -\frac{\sinh(m\gamma\eta)}{m\gamma\eta} \operatorname{tr} L^m, \quad (4.3)$$

where the Lax matrix L is given by (1.2). In particular, the Hamiltonian H_1 coincides with (3.20).

The derivation of the second half of the Hamiltonian equations is more involved. The idea of the derivation is the same as in [14]. First of all, we note that (4.3) can be written in the form

$$\partial_{t_m} x_i = -m\eta\kappa_m \operatorname{tr}(E_i L^m), \quad \kappa_m = \frac{\sinh(m\gamma\eta)}{m\gamma\eta}.$$

Differentiating this equality with respect to t_1 and using the Lax equation, we get:

$$\partial_{t_m} \dot{x}_i = -m\eta\kappa_m \operatorname{tr}(E_i [M', L^m]) = -m\eta\kappa_m \operatorname{tr}(L^m [E_i, M']).$$

Now we apply ∂_{t_m} to equation (3.21):

$$\partial_{t_m} p_i = \eta^{-1} \partial_{t_m} \log \dot{x}_i - \eta^{-1} \sum_j \sum_{l \neq i} \frac{\partial}{\partial x_j} \log \frac{\sinh(\gamma(x_{il} + \eta))}{\sinh(\gamma x_{il})} \partial_{t_m} x_j$$

$$= -m\kappa_m \dot{x}_i^{-1}\mathrm{tr}\,(L^m[E_i, M']) + m\kappa_m \sum_j \sum_{l\neq i} \frac{\partial}{\partial x_j} \log \frac{\sinh(\gamma(x_{il}+\eta))}{\sinh(\gamma x_{il})}\,\mathrm{tr}\,(E_j L^m)$$

$$= -m\kappa_m\,\mathrm{tr}\left(A^{(i)}L^{m-1}\right),$$

where the matrix $A^{(i)}$ is

$$A^{(i)} = \dot{x}_i^{-1}(LE_iM' - M'E_iL) - \sum_j \sum_{l\neq i} \frac{\partial}{\partial x_j} \log \frac{\sinh(\gamma(x_{il}+\eta))}{\sinh(\gamma x_{il})}\, E_jL. \tag{4.4}$$

Note that the diagonal part of the matrix M' (3.19) does not contribute, so instead of the matrix M' here we can substitute its off-diagonal part

$$A_{ij} = 2\gamma(1-\delta_{ij})\,\frac{\dot{x}_i w_i^{1/2} w_j^{1/2}}{w_i - w_j}.$$

Let us calculate the matrix elements:

$$(LE_iA)_{jk} = \gamma \dot{x}_i L_{jk}\left(\frac{w_i+w_k}{w_i-w_k} - \frac{qw_i+w_k}{qw_i-w_k}\right)(1-\delta_{ik}),$$

$$(AE_iL)_{jk} = -\gamma \dot{x}_i L_{jk}\left(\frac{w_i+w_j}{w_i-w_j} - \frac{w_i+qw_k}{w_i-qw_k}\right)(1-\delta_{ij}),$$

$$\sum_l \sum_{r\neq i} \frac{\partial}{\partial x_l} \log \frac{\sinh(\gamma(x_{ir}+\eta))}{\sinh(\gamma x_{ir})}\,(E_lL)_{jk}$$

$$= \gamma\delta_{ij}L_{jk}\sum_{r\neq i}\left(\frac{qw_i+w_r}{qw_i-w_r} - \frac{w_i+w_r}{w_i-w_r}\right) - \gamma(1-\delta_{ij})L_{jk}\left(\frac{qw_i+w_j}{qw_i-w_j} - \frac{w_i+w_j}{w_i-w_j}\right).$$

Combining everything together, we obtain the matrix elements of the matrix $A^{(i)}$:

$$\begin{aligned} A^{(i)}_{jk} \;=\; & \gamma L_{jk}\left(\frac{w_i+w_k}{w_i-w_k}(1-\delta_{ik}) - \frac{w_i+qw_k}{w_i-qw_k}(1-\delta_{ij}) + \frac{qw_i+w_j}{qw_i-w_j}(\delta_{ik}-\delta_{ij})\right. \\ & \left. -\delta_{ij}\sum_{r\neq i}\left(\frac{qw_i+w_r}{qw_i-w_r} - \frac{w_i+w_r}{w_i-w_r}\right)\right). \end{aligned} \tag{4.5}$$

Our next goal is to prove that

$$A^{(i)} = -\frac{\partial L}{\partial x_i} - [C^{(i)}, L], \tag{4.6}$$

where the matrix $C^{(i)}$ is given by

$$C^{(i)} = \gamma \sum_l \frac{qw_l+w_i}{qw_l-w_i}\,E_l - \gamma\sum_{l\neq i}\frac{w_l+w_i}{w_l-w_i}\,E_l. \tag{4.7}$$

From this one concludes that

$$\partial_{t_m} p_i = -m\kappa_m \,\mathrm{tr}\left(A^{(i)} L^{m-1}\right) = m\kappa_m \,\mathrm{tr}\left(\frac{\partial L}{\partial x_i} L^{m-1}\right) = \kappa_m \frac{\partial}{\partial x_i} \,\mathrm{tr}\, L^m.$$

This yields the second half of the Hamiltonian equations for the higher flows:

$$\partial_{t_m} p_i = -\frac{\partial H_m}{\partial x_i}. \tag{4.8}$$

In order to prove the identity (4.6), we calculate matrix elements of the right hand side and compare them with (4.5). Indeed, we have:

$$\frac{\partial L_{jk}}{\partial x_i} = \gamma L_{jk} \left(\frac{w_j + q w_k}{w_j - q w_k} (\delta_{ik} - \delta_{ij}) + \frac{w_i + q w_j}{w_i - q w_j} (1 - \delta_{ij}) - \frac{w_i + w_j}{w_i - w_j} (1 - \delta_{ij}) + \delta_{ij} \sum_{r \neq i} \left(\frac{q w_i + w_r}{q w_i - w_r} - \frac{w_i + w_r}{w_i - w_r} \right) \right),$$

$$[C^{(i)}, L]_{jk} = \gamma L_{jk} \left(\frac{q w_j + w_i}{q w_j - w_i} - \frac{q w_k + w_i}{q w_k - w_i} - \frac{w_j + w_i}{w_j - w_i}(1 - \delta_{ij}) + \frac{w_k + w_i}{w_k - w_i}(1 - \delta_{ik}) \right),$$

and one can check that $A^{(i)} + \partial L/\partial x_i + [C^{(i)}, L] = 0$.

4.2. Negative times

In order to investigate the dynamics of zeros of the tau-function in the negative times, we first consider the $\bar{t}_1$-evolution. We will work with the complimentary wave functions ϕ, $\phi^\dagger$ given by (2.32) and (2.33) for which we use the ansatz

$$\phi(x) = z^{x/\eta} e^{\bar{t}_1 z^{-1}} \left(1 + \sum_i \frac{2\gamma b_i}{q w - w_i} \right), \tag{4.9}$$

$$\phi^\dagger(x) = z^{-x/\eta} e^{-\bar{t}_1 z^{-1}} \left(1 + \sum_i \frac{2\gamma b_i^*}{q^{-1} w - w_i} \right), \tag{4.10}$$

where b_i, b_i^* are some unknown coefficients depending on the times and on z but not on x. Substituting them into the linear equation (2.34), we can write down the conditions of cancellation of the poles in the way similar to that of section 3. In fact the equations for b_i are the same as for c_i^* up to changing z to $-z^{-1}$, w to qw and ∂_{t_1} to $\partial_{\bar{t}_1}$. The equations for b_i^* and c_i are connected in a similar way. Passing to $\tilde{b}_i = w_i^{-1/2} b_i$, $\tilde{b}_i^* = w_i^{-1/2} b_i^*$, we have, after some algebra, in the notation of section 3:

$$\tilde{\mathbf{b}}^T (\partial_{\bar{t}_1} X)^{-1} (z^{-1} I + q^{-1/2} \bar{L}) = \mathbf{e}^T W^{1/2}, \tag{4.11}$$

$$(z^{-1} I + q^{1/2} \bar{L}) \tilde{\mathbf{b}}^* = -\partial_{\bar{t}_1} X \, W^{1/2} \mathbf{e}, \tag{4.12}$$

where $\tilde{\mathbf{b}}^T = (\tilde{b}_1, \ldots, \tilde{b}_N)$, $\tilde{\mathbf{b}}^* = (\tilde{b}_1^*, \ldots, \tilde{b}_N^*)$ and the matrix $\bar{L}$ reads

$$\bar{L}_{ij} = 2\gamma q^{1/2} \frac{\partial_{\bar{t}_1} x_i \, w_i^{1/2} w_j^{1/2}}{w_i - q w_j}. \tag{4.13}$$

This matrix satisfies the commutation relation (3.15) with $\partial_{\bar{t}_1} W$ instead of $\dot{W} = \partial_{t_1} W$:

$$q^{-1/2} W\bar{L} - q^{1/2}\bar{L}W = W^{-1/2}\partial_{\bar{t}_1} W\, EW^{1/2}. \tag{4.14}$$

Using the relation (2.37), we find, similarly to (4.1):

$$\partial_{\bar{t}_m} x_i = -2\gamma \operatorname*{res}_0 \Big(z^{-m-2} \tilde{b}_i^* (\partial_{\bar{t}_1} w_i)^{-1} \tilde{b}_i \Big). \tag{4.15}$$

Substituting here the solutions of linear systems (4.11), (4.12) and using (4.14), one can repeat the calculation from section 4.1 with the result

$$\partial_{\bar{t}_m} x_i = (-1)^m \frac{\sinh(m\gamma\eta)}{\gamma} \operatorname{tr}(E_i \bar{L}^m). \tag{4.16}$$

Our next goal is to derive a relation between the Lax matrices L and $\bar{L}$. For this, we need a relation between the velocities $\dot{x}_i = \partial_{t_1} x_i$ and $\partial_{\bar{t}_1} x_i$. The latter can be derived from the Toda equation (2.27). Substituting the tau-function in the form $\tau = \prod_i (w - w_i)$, we get

$$-\sum_i \frac{\partial_{t_1}\partial_{\bar{t}_1} w_i}{w - w_i} - \sum_i \frac{\partial_{t_1} w_i\, \partial_{\bar{t}_1} w_i}{(w - w_i)^2} = 1 - \prod_k \frac{(qw - w_k)(q^{-1}w - w_k)}{(w - w_k)^2}.$$

Identifying the second order poles in the both sides, we obtain the relation

$$\partial_{t_1} w_i\, \partial_{\bar{t}_1} w_i = \frac{\prod_k (qw_i - w_k)(q^{-1}w_i - w_k)}{\prod_{l \neq i} (w_i - w_l)^2} \tag{4.17}$$

or

$$\partial_{t_1} X\, \partial_{\bar{t}_1} X = \frac{1}{4\gamma^2}\, W^{-2} U_+ U_-, \tag{4.18}$$

where the diagonal matrices $U_\pm$ are

$$(U_\pm)_{ij} = \delta_{ij} \frac{\prod_k (w_i - q^{\pm 1} w_k)}{\prod_{l \neq i} (w_i - w_l)}. \tag{4.19}$$

Next, we need the formula for the inverse of the Cauchy matrix

$$C_{ij} = \frac{1}{w_i - qw_j}. \tag{4.20}$$

We have:

$$C_{ij}^{-1} = \frac{1}{qw_i - w_j} \frac{\prod_k (qw_i - w_k)(w_j - qw_k)}{q^{N-1} \prod_{l \neq j} (w_j - w_l) \prod_{l' \neq i} (w_i - w_{l'})} \tag{4.21}$$

or

$$C^{-1} = -qU_- C^T U_+. \tag{4.22}$$

Now we write $L = 2\gamma q^{1/2} \partial_{t_1} X\, W^{1/2} C W^{1/2}$ and find, using (4.22) and (4.18):

$$\begin{aligned} L^{-1} &= \frac{q^{-1/2}}{2\gamma} W^{-1/2} C^{-1} W^{-1/2} (\partial_{t_1} X)^{-1} \\ &= -2\gamma q^{1/2} W^{-1/2} U_- C^T W^{-1/2} \partial_{\bar{t}_1} X\, W^2 U_-^{-1} \end{aligned}$$

$$= -2\gamma q^{1/2} W^{-1} U_- \left(\partial_{\bar{t}_1} X\, W^{1/2} C W^{1/2}\right)^T (W^{-1}U_-)^{-1}$$

$$= -W^{-1}U_-\, \bar{L}^T (W^{-1}U_-)^{-1}.$$

We see that the matrix $-\bar{L}^T$ is connected with L^{-1} by a similarity transformation with a diagonal matrix. Using the fact that $E_i\bar{L} = -\eta^{-1}\partial\bar{L}/\partial p_i$, we can therefore rewrite equation (4.16) as

$$\partial_{\bar{t}_m} x_i = -\frac{\sinh(m\gamma\eta)}{m\gamma\eta}\frac{\partial}{\partial p_i}\operatorname{tr} L^{-m} = \frac{\partial \bar{H}_m}{\partial p_i} \tag{4.23}$$

which is one half of the Hamiltonian equations for the negative time flows.

The derivation of the second half is straightforward. We note that

$$\partial_{\bar{t}_m} x_i = m\eta\kappa_m \operatorname{tr}(E_i L^{-m}),$$

where we use the notation of section 4.1. In the complete analogy with the calculation in the previous subsection, we have

$$\partial_{\bar{t}_m} p_i = m\kappa_m \operatorname{tr}(A^{(i)} L^{-m-1})$$

with the same matrix $A^{(i)}$ (4.4). By virtue of (4.6) we obtain:

$$\partial_{\bar{t}_m} p_i = -m\kappa_m \operatorname{tr}\left(\frac{\partial L}{\partial x_i} L^{-m-1}\right) = m\kappa_m \operatorname{tr}\left(\frac{\partial L^{-1}}{\partial x_i}(L^{-1})^{m-1}\right) = \kappa_m \frac{\partial}{\partial x_i}\operatorname{tr} L^{-m},$$

which are the Hamiltonian equations

$$\partial_{\bar{t}_m} p_i = -\frac{\partial \bar{H}_m}{\partial x_i} \tag{4.24}$$

with $\bar{H}_m$ given by (1.4). In particular,

$$\bar{H}_1 = \frac{\sinh^2(\gamma\eta)}{\gamma^2\eta^2}\sum_i e^{-\eta p_i}\prod_{k\neq i}\frac{\sinh(\gamma(x_{ik}-\eta))}{\sinh(\gamma x_{ik})}. \tag{4.25}$$

5. The generating form of equations of motion in higher times

In the above analysis we parametrized the wave function by residues at its poles. Another possible parametrization is by zeros and poles. It leads to the so-called self-dual form of equations of motion. In this section we derive these equations and show that they provide a generating form of equations of motion for the Ruijsenaars–Schneider model in the higher times.

In this section we keep the negative times $\bar{\mathbf{t}}$ fixed and consider only the dependence on $\mathbf{t}$. In accordance with (2.28) we have $\psi(\mu, x, \mathbf{t}) = \mu^{x/\eta} e^{\xi(\mathbf{t},\mu)}\hat{\tau}(x,\mathbf{t})/\tau(x,\mathbf{t})$ (here $\hat{\tau}(x,\mathbf{t}) = \tau(x, \mathbf{t} - [\mu^{-1}])$), then the auxiliary linear problem (2.17) acquires the form

$$\partial_{t_1}\log\frac{\hat{\tau}(x)}{\tau(x+\eta)} = \mu\frac{\hat{\tau}(x+\eta)\tau(x)}{\tau(x+\eta)\hat{\tau}(x)} - \mu. \tag{5.1}$$

For trigonometric solutions τ is of the form (3.1) and for $\hat{\tau}$ we write

$$\hat{\tau} = \prod_i \left(e^{2\gamma x} - e^{2\gamma y_i}\right) = \prod_i (w - v_i), \quad v_i = e^{2\gamma y_i}, \tag{5.2}$$

parametrizing it by its zeros y_i. Substituting this into (5.1), we have:

$$\sum_i \frac{\dot{w}_i}{qw - w_i} - \sum_i \frac{\dot{v}_i}{w - v_i} = \mu \prod_k \frac{(w - w_k)(qw - v_k)}{(w - v_k)(qw - w_k)} - \mu.$$

Identifying residues at the simple poles at $w = q^{-1}w_i$ and $w = v_i$, we get the system of equations

$$\begin{cases} 2\gamma \dot{x}_i = \mu(1-q) \prod_{j \neq i} \dfrac{w_i - qw_j}{w_i - w_j} \prod_k \dfrac{w_i - v_k}{w_i - qv_k} \\[2ex] 2\gamma \dot{y}_i = \mu(1-q) \prod_{j \neq i} \dfrac{qv_i - v_j}{v_i - v_j} \prod_k \dfrac{v_i - w_k}{qv_i - w_k} \end{cases} \tag{5.3}$$

or

$$\begin{cases} \gamma \dot{x}_i = -\mu \sinh(\gamma\eta) \prod_{j \neq i} \dfrac{\sinh(\gamma(x_i - x_j - \eta))}{\sinh(\gamma(x_i - x_j))} \prod_k \dfrac{\sinh(\gamma(x_i - y_k))}{\sinh(\gamma(x_i - y_k - \eta))} \\[2ex] \gamma \dot{y}_i = -\mu \sinh(\gamma\eta) \prod_{j \neq i} \dfrac{\sinh(\gamma(y_i - y_j + \eta))}{\sinh(\gamma(y_i - y_j))} \prod_k \dfrac{\sinh(\gamma(y_i - x_k))}{\sinh(\gamma(y_i - x_k + \eta))}. \end{cases} \tag{5.4}$$

This is the Ruijsenaars–Schneider analog of the Bäcklund transformation for the Calogero–Moser system [22, 23]. These equations appeared in [24] in the context of the integrable time discretization of the Ruijsenaars–Schneider model (see also [25, 26]). One can show that the equations of motion of the Ruijsenaars–Schneider model for x_i's follow from (5.4) and y_i's obey the same equations (for the proof see [25]).

At the same time these equations contain all the higher equations of motion in an encoded form. To see this, we introduce the differential operator

$$D(\mu) = \sum_{k \geqslant 1} \frac{\mu^{-k}}{k} \partial_{t_k}, \tag{5.5}$$

then $\hat{\tau} = e^{-D(\mu)}\tau$ and $y_i = e^{-D(\mu)}x_i$. Performing an overall time shift in the second equation in (5.4), we can rewrite them in the form

$$\begin{cases} \gamma \dot{x}_i = -\mu \sinh(\gamma\eta) \prod_{j \neq i} \dfrac{\sinh(\gamma(x_i - x_j - \eta))}{\sinh(\gamma(x_i - x_j))} \prod_k \dfrac{\sinh(\gamma(x_i - e^{-D(\mu)}x_k))}{\sinh(\gamma(x_i - e^{-D(\mu)}x_k - \eta))} \\[2ex] \gamma \dot{x}_i = -\mu \sinh(\gamma\eta) \prod_{j \neq i} \dfrac{\sinh(\gamma(x_i - x_j + \eta))}{\sinh(\gamma(x_i - x_j))} \prod_k \dfrac{\sinh(\gamma(x_i - e^{D(\mu)}x_k))}{\sinh(\gamma(x_i - e^{D(\mu)}x_k + \eta))}. \end{cases} \tag{5.6}$$

Dividing one equation by the other, we obtain the equations

$$\prod_{k=1}^{N} \frac{\sinh(\gamma(x_i - e^{D(\mu)}x_k))}{\sinh(\gamma(x_i - e^{D(\mu)}x_k + \eta))} \frac{\sinh(\gamma(x_i - x_k + \eta))}{\sinh(\gamma(x_i - x_k - \eta))} \frac{\sinh(\gamma(x_i - e^{-D(\mu)}x_k - \eta))}{\sinh(\gamma(x_i - e^{-D(\mu)}x_k))} = -1 \tag{5.7}$$

which are, on one hand, equations of motion for the Ruijsenaars-Schheider system in discrete time (see [27]) and, on the other, provide the generating form of the higher equations of motion in continuous hierarchical times. Indeed, expanding (5.7) in (inverse) powers of μ, one gets the set of the higher equations of motion. In particular, equations (3.17) are obtained by expansion of (5.7) up to μ^{-1}.

6. The tau-function for trigonometric solutions

In this section we prove the determinant formula for the tau-function of trigonometric solutions

$$\tau'(x,\mathbf{t},\bar{\mathbf{t}}) = \det_{N\times N}\left(wI - \exp\Big(\sum_{k\geqslant 1}(q^{-k/2}-q^{k/2})(t_k L_0^k - \bar{t}_k L_0^{-k})\Big)W_0\right), \quad (6.1)$$

where $L_0 = L(0)$, $W_0 = W(0)$. We recall that the tau-function τ' is connected with τ by formula (2.24).

The matrix $\exp\Big(\sum_{k\geqslant 1}(q^{-k/2}-q^{k/2})(t_k L_0^k - \bar{t}_k L_0^{-k})\Big)W_0$ can be diagonalized with the help of a diagonalizing matrix V:

$$V\exp\Big(\sum_{k\geqslant 1}(q^{-k/2}-q^{k/2})(t_k L_0^k - \bar{t}_k L_0^{-k})\Big)W_0 V^{-1} = W.$$

There is a freedom in the definition of V: it can be multiplied by a diagonal matrix from the left. We fix this freedom by the condition

$$\mathbf{e}^T W_0^{1/2} = \mathbf{e}^T W^{1/2} V. \quad (6.2)$$

The matrices W_0, L_0 satisfy the commutation relation (3.15) which we write here in the form

$$q^{-1/2}W_0^{1/2}L_0W_0^{-1/2} - q^{1/2}W_0^{-1/2}L_0W_0^{1/2} = W_0^{-1}\dot{W}_0\mathbf{e}\otimes\mathbf{e}^T. \quad (6.3)$$

Let us prove, following [24], that the matrices W and $L = VL_0V^{-1}$ satisfy the same commutation relation. We have:

$$\begin{aligned}
q^{-1/2}W^{1/2}LW^{-1/2} - q^{1/2}W^{-1/2}LW^{1/2} &= W^{1/2}(q^{-1/2}L - q^{1/2}W^{-1}LW)W^{-1/2} \\
&= W^{1/2}\Big(q^{-1/2}VL_0V^{-1} - q^{1/2}VW_0^{-1}L_0W_0V^{-1}\Big)W^{-1/2} \\
&= W^{1/2}VW_0^{-1/2}\Big(q^{-1/2}W_0^{1/2}L_0W_0^{-1/2} - q^{1/2}W_0^{-1/2}L_0W_0^{1/2}\Big)W_0^{1/2}V^{-1}W^{-1/2} \\
&= W^{1/2}VW_0^{-3/2}\dot{W}_0\,\mathbf{e}\otimes\mathbf{e}^T W_0^{1/2}V^{-1}W^{-1/2} \\
&= W^{1/2}VW_0^{-3/2}\dot{W}_0\,\mathbf{e}\otimes\mathbf{e}^T.
\end{aligned}$$

(The last equality follows from the condition (6.2).) Denoting $W^{3/2}VW_0^{-3/2}\dot{W}_0\,\mathbf{e} = \dot{W}\mathbf{e}$, we arrive at the desired commutation relation.

We are going to prove that the function (6.1) satisfies the bilinear equations (2.22) and (2.23) of the 2DTL hierarchy. We begin with equation (2.22):

$$
\begin{aligned}
&\mu\frac{\tau(x+\eta,\mathbf{t}+[\lambda^{-1}]-[\mu^{-1}])}{\tau(x+\eta,\mathbf{t})}-\lambda\frac{\tau(x,\mathbf{t}+[\lambda^{-1}]-[\mu^{-1}])}{\tau(x,\mathbf{t})}\\
&\quad+(\lambda-\mu)\frac{\tau(x+\eta,\mathbf{t}+[\lambda^{-1}])}{\tau(x+\eta,\mathbf{t})}\,\frac{\tau(x,\mathbf{t}-[\mu^{-1}])}{\tau(x,\mathbf{t})}=0.
\end{aligned}
\tag{6.4}
$$

(Here and below in the proof we put $\bar{\mathbf{t}}=0$ and identify τ and τ'.) The similarity transformation with the matrix V under the determinant in (6.1) allows one to write the following formulas:

$$
\tau(x,\mathbf{t})=\det(wI-W),
\tag{6.5}
$$

$$
\tau(x,\mathbf{t}+[\lambda^{-1}])=\det\left(wI-\frac{\lambda I-q^{1/2}L}{\lambda I-q^{-1/2}L}\,W\right)=\frac{\det\Big(w(\lambda I-q^{-1/2}L)-(\lambda I-q^{1/2}L)W\Big)}{\det\Big(\lambda I-q^{-1/2}L\Big)}
$$

$$
=\frac{\det\Big((wI-W)(\lambda I-q^{-1/2}L)-\tilde{E}\Big)}{\det\Big(\lambda I-q^{-1/2}L\Big)}=\tau(x,\mathbf{t})\left(1-\mathrm{tr}\left(\frac{1}{\lambda I-q^{-1/2}L}\,\frac{1}{wI-W}\,\tilde{E}\right)\right),
$$

where the matrix $\tilde{E}=W^{-1/2}\dot{W}EW^{1/2}$ has rank 1 and we have used the commutation relation (3.15) and the formula $\det(I+A)=1+\mathrm{tr}A$ valid for any rank 1 matrix A. Similar calculations yield

$$
\tau(x,\mathbf{t}-[\mu^{-1}])=\tau(x,\mathbf{t})\left(1+\mathrm{tr}\left(\frac{1}{wI-W}\,\frac{1}{\mu I-q^{1/2}L}\,\tilde{E}\right)\right),
$$

$$
\tau(x,\mathbf{t}+[\lambda^{-1}]-[\mu^{-1}])=\det\left(wI-\frac{\lambda I-q^{1/2}L}{\lambda I-q^{-1/2}L}\,\frac{\mu I-q^{-1/2}L}{\mu I-q^{1/2}L}\,W\right)
$$

$$
=\det\left(wI-\frac{\lambda I-q^{1/2}L}{\lambda I-q^{-1/2}L}\,W\frac{\mu I-q^{-1/2}L}{\mu I-q^{1/2}L}\right)
$$

$$
=\frac{\det\Big(w(\lambda I-q^{-1/2}L)(\mu I-q^{1/2}L)-(\lambda I-q^{1/2}L)W(\mu I-q^{-1/2}L)\Big)}{\det\Big(\lambda I-q^{-1/2}L\Big)\det\Big(\mu I-q^{1/2}L\Big)}
$$

$$
=\frac{\det\Big((\mu I-q^{1/2}L)(wI-W)(\lambda I-q^{-1/2}L)+(\lambda-\mu)\tilde{E}\Big)}{\det\Big(\lambda I-q^{-1/2}L\Big)\det\Big(\mu I-q^{1/2}L\Big)}
$$

$$
=\tau(x,\mathbf{t})\left[1+(\lambda-\mu)\,\mathrm{tr}\left(\frac{1}{\lambda I-q^{-1/2}L}\,\frac{1}{wI-W}\,\frac{1}{\mu I-q^{1/2}L}\,\tilde{E}\right)\right].
$$

Substituting everything into the left hand side of (6.4), we obtain:

$$
\text{LHS of (6.4)}\propto q^{-1}\mu\,\mathrm{tr}\left[\frac{1}{\lambda I-q^{-1/2}L}\,\frac{1}{wI-q^{-1}W}\,\frac{1}{\mu I-q^{1/2}L}\,\tilde{E}\right]
$$

$$-\lambda\,\mathrm{tr}\left[\frac{1}{\lambda I-q^{-1/2}L}\,\frac{1}{wI-W}\,\frac{1}{\mu I-q^{1/2}L}\,\tilde{E}\right]$$

$$+\mathrm{tr}\left[\frac{1}{wI-W}\,\frac{1}{\mu I-q^{1/2}L}\,\tilde{E}\right]-q^{-1}\mathrm{tr}\left[\frac{1}{\lambda I-q^{-1/2}L}\,\frac{1}{wI-q^{-1}W}\,\tilde{E}\right]$$

$$-q^{-1}\mathrm{tr}\left[\frac{1}{wI-W}\,\frac{1}{\mu I-q^{1/2}L}\,\tilde{E}\right]\mathrm{tr}\left[\frac{1}{\lambda I-q^{-1/2}L}\,\frac{1}{wI-q^{-1}W}\,\tilde{E}\right].$$

This expression is a rational function of w with simple poles at $w=w_i$ and $w=q^{-1}w_i$ vanishing at ∞. To prove that it actually vanishes identically it is enough to prove that the residues at the poles are zero. The residue at the pole at $w=w_i$ is equal to

$$-\lambda\sum_{j,k}\Big(\frac{1}{\lambda I-q^{-1/2}L}\Big)_{ji}\Big(\frac{1}{\mu I-q^{1/2}L}\Big)_{ik}w_k^{-1/2}\dot{w}_k w_j^{1/2}+\sum_j\Big(\frac{1}{\mu I-q^{1/2}L}\Big)_{ij}w_j^{-1/2}\dot{w}_j w_i^{1/2}$$

$$-\sum_{j,k,k'}\Big(\frac{1}{\mu I-q^{1/2}L}\Big)_{ij}w_j^{-1/2}\dot{w}_j w_i^{1/2}\Big(\frac{1}{\lambda I-q^{-1/2}L}\Big)_{kk'}\frac{w_{k'}^{-1/2}\dot{w}_{k'}w_k^{1/2}}{qw_i-w_{k'}}.$$

Recalling that $L_{ij}=q^{1/2}\frac{\dot{w}_i w_i^{-1/2}w_j^{1/2}}{w_i-qw_j}$, we can rewrite the last line (the triple sum) in the form

$$\sum_{j,k}\Big(\frac{1}{\mu I-q^{1/2}L}\Big)_{ij}w_j^{-1/2}\dot{w}_j w_k^{1/2}\Big(\frac{q^{-1/2}L}{\lambda I-q^{-1/2}L}\Big)_{ki}$$

from which it is seen that the residue is zero. The calculation for the residue at $w=q^{-1}w_i$ is similar.

The fact that the function (6.1) is a KP tau-function with respect to the times $\mathbf{t}$ follows also from the result of Kasman and Gekhtman [28]: for any matrices X, Y, Z such that the matrix $XZ-YX$ has rank 1 the function

$$\tau=\det\left(X\exp\Big(\sum_{k\geqslant 1}t_kZ^k\Big)+\exp\Big(\sum_{k\geqslant 1}t_kY^k\Big)\right) \tag{6.6}$$

is a tau-function of the KP hierarchy. In our case $X=-W_0$, $Z=q^{-1/2}L_0$, $Y=q^{1/2}L_0$ and the condition that $XZ-YX$ has rank 1 is equivalent to the commutation relation (3.15).

Let us pass to the proof of equation (2.23) which we write here in the equivalent form

$$\begin{aligned}&\frac{\tau'(x,\mathbf{t}+[\lambda^{-1}],\bar{\mathbf{t}}-[\nu])}{\tau'(x-\eta,\mathbf{t},\bar{\mathbf{t}})}-\frac{\nu}{\lambda}\,\frac{\tau'(x+\eta,\mathbf{t}+[\lambda^{-1}],\bar{\mathbf{t}}-[\nu])}{\tau'(x,\mathbf{t},\bar{\mathbf{t}})}\\&\quad-\Big(1-\frac{\nu}{\lambda}\Big)\frac{\tau'(x,\mathbf{t},\bar{\mathbf{t}}-[\nu])}{\tau'(x-\eta,\mathbf{t},\bar{\mathbf{t}})}\,\frac{\tau'(x,\mathbf{t}+[\lambda^{-1}])}{\tau'(x,\mathbf{t},\bar{\mathbf{t}})}=0.\end{aligned} \tag{6.7}$$

The calculations similar to the ones done above yield:

$$\tau'(x,\mathbf{t}+[\lambda^{-1}],\bar{\mathbf{t}})=\tau'(x,\mathbf{t},\bar{\mathbf{t}})\left(1-\mathrm{tr}\Big(\frac{1}{\lambda I-q^{-1/2}L}\,\frac{1}{wI-W}\,\tilde{E}\Big)\right),$$

$$\tau'(x,\mathbf{t},\bar{\mathbf{t}}-[\nu])=q^N\tau'(x-\eta,\mathbf{t},\bar{\mathbf{t}})\left(1+\mathrm{tr}\Big(\frac{q}{wI-qW}\,\frac{1}{\nu I-q^{1/2}L}\,\tilde{E}\Big)\right),$$

$$\tau'(x, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}} - [\nu]) = q^N \tau'(x - \eta, \mathbf{t}, \bar{\mathbf{t}})$$
$$\times \left(1 + (\lambda - \nu)\,\mathrm{tr}\left(\frac{1}{\lambda I - q^{-1/2}L}\,\frac{q}{wI - qW}\,\frac{1}{\nu I - q^{1/2}L}\,\tilde{E}\right)\right).$$

Using the results of the above calculation, it is not difficult to see that the substitution into (6.7) gives the identity, so the equation (2.23) is proved.

7. Conclusion

The main result of this paper is establishing the precise correspondence between trigonometric solutions of the 2D Toda lattice hierarchy and the hierarchy of the Hamiltonian equations for the integrable Ruijsenaars–Schneider model with higher Hamiltonians. The zeros of the tau-function move as particles of the Ruijsenaars–Schneider model. We have shown that the mth time flow t_m of the 2DTL hierarchy gives rise to the flow with the Hamiltonian H_m of the Ruijsenars–Schneider model proportional to $\mathrm{tr}\, L^m$, where L is the Lax matrix, while the time flow $\bar{t}_m$ gives rise to the Hamiltonian flow with the Hamiltonian $\bar{H}_m$ proportional to $\mathrm{tr}\, L^{-m}$. In some sense this correspondence is simpler and more natural then a similar correspondence between the KP hierarchy and trigonometric Calogero–Moser hierarchy [11], which in principle can be obtained from our results in the limit $\eta \to 0$.

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ORCID iDs

A Zabrodin https://orcid.org/0000-0002-4319-770X

References

[1] Airault H, McKean H P and Moser J 1977 Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem *Commun. Pure Appl. Math.* **30** 95–148
[2] Calogero F 1971 Solution of the one-dimensional *N*-body problems with quadratic and/or inversely quadratic pair potentials *J. Math. Phys.* **12** 419–36
[3] Calogero F 1975 Exactly solvable one-dimensional many-body systems *Lett. Nuovo Cimento* **13** 411–5
[4] Moser J 1975 Three integrable Hamiltonian systems connected with isospectral deformations *Adv. Math.* **16** 197–220
[5] Olshanetsky M A and Perelomov A M 1981 Classical integrable finite-dimensional systems related to Lie algebras *Phys. Rep.* **71** 313–400
[6] Krichever I M 1978 Rational solutions of the Kadomtsev–Petviashvili equation and integrable systems of *N* particles on a line *Funct. Anal. Appl.* **12** 59–61
[7] Chudnovsky D V and Chudnovsky G V 1977 Pole expansions of non-linear partial differential equations *Nuovo Cimento* **40B** 339–50

[8] Krichever I M 1980 Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles *Funk. Anal. i Ego Pril.* **14** 45–54 (in Russian)
Krichever I M 1980 *Funct. Anal. Appl.* 14 282–90 (Engl. transl.)
[9] Shiota T 1994 Calogero–Moser hierarchy and KP hierarchy *J. Math. Phys.* **35** 5844–9
[10] Haine L 2007 KP trigonometric solitons and an adelic flag manifold *SIGMA* **3** 015
[11] Zabrodin A 2019 KP hierarchy and trigonometric Calogero–Moser hierarchy (arXiv:1906.09846)
[12] Krichever I and Zabrodin A 1995 Spin generalization of the Ruijsenaars–Schneider model, non-abelian 2D Toda chain and representations of Sklyanin algebra *Usp. Math. Nauk* **50** 3–56
[13] Ruijsenaars S N M and Schneider H 1986 A new class of integrable systems and its relation to solitons *Ann. Phys.* **170** 370–405
[14] Iliev P 2007 Rational Ruijsenaars–Schneider hierarchy and bispectral difference operators *Physica* D **229** 184–90
[15] Zabrodin A 2014 The master *T*-operator for inhomogeneous *XXX* spin chain and mKP hierarchy *SIGMA* **10** 006
[16] Ueno K and Takasaki K 1984 Toda lattice hierarchy *Adv. Stud. Pure Math.* **4** 1–95
[17] Zabrodin A 2019 On matrix modified Kadomtsev–Petviashvili hierarchy *Teor. Mat. Fys.* **199** 343–56
Zabrodin A 2019 Matrix modified Kadomtsev–Petviashvili hierarchy *Theor. Math. Phys.* **199** 771–83 (Engl. transl.)
[18] Date E, Jimbo M, Kashiwara M and Miwa T 1983 *Transformation Groups for Soliton Equations: Nonlinear Integrable Systems—Classical Theory and Quantum Theory (Kyoto, 1981)* (Singapore: World Scientific) pp 39–119
[19] Jimbo M and Miwa T 1983 Solitons and infinite dimensional Lie algebras *Publ. Res. Inst. Math. Sci. Kyoto* **19** 943–1001
[20] Takasaki K and Takebe T 1995 Integrable hierarchies and dispersionless limit *Rev. Math. Phys.* **7** 743–808
[21] Ruijsenaars S N M 1987 Complete integrability of relativistic Calogero–Moser systems and elliptic function identities *Commun. Math. Phys.* **110** 191–213
[22] Wojciechowski S 1982 The analogue of the Bäcklund transformation for integrable many-body systems *J. Phys. A: Math. Gen.* **15** L653–7
[23] Abanov A, Bettelheim E and Wiegmann P 2009 Integrable hydrodynamics of Calogero-Sutherland model: bidirectional Benjamin–Ono equation *J. Phys. A: Math. Theor.* **42** 135201
[24] Suris Y 2003 *The Problem of Integrable Discretization: Hamiltonian Approach* (Basel, AG: Springer)
[25] Zabrodin A and Zotov A 2018 Self-dual form of Ruijsenaars–Schneider models and ILW equation with discrete laplacian *Nucl. Phys.* B **927** 550–65
[26] Zabrodin A 2019 Elliptic solutions to integrable nonlinear equations and many-body systems *J. Geom. Phys.* **146** 103506
[27] Nihhoff F W, Ragnisco O and Kuznetsov V 1996 Integrable time-discretization of the Ruijsenaars–Schneider model *Commun. Math. Phys.* **176** 681–700
[28] Kasman A and Gekhtman M 2001 Solitons and almost-intertwining matrices *J. Math. Phys.* **42** 3540–51

Appendix B

V.Prokofev, A Zabrodin "Matrix Kadomtsev-Petviashvili Hierarchy and Spin Generalization of Trigonometric Calogero Moser Hierarchy" Proceedings of the Steklov Institute of Mathematics, 2020.

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Matrix Kadomtsev–Petviashvili Hierarchy and Spin Generalization of Trigonometric Calogero–Moser Hierarchy

V. V. Prokofev[a,b] **and A. V. Zabrodin**[c]

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Abstract—We consider solutions of the matrix Kadomtsev–Petviashvili (KP) hierarchy that are trigonometric functions of the first hierarchical time $t_1 = x$ and establish the correspondence with the spin generalization of the trigonometric Calogero–Moser system at the level of hierarchies. Namely, the evolution of poles x_i and matrix residues at the poles $a_i^\alpha b_i^\beta$ of the solutions with respect to the kth hierarchical time of the matrix KP hierarchy is shown to be given by the Hamiltonian flow with the Hamiltonian which is a linear combination of the first k higher Hamiltonians of the spin trigonometric Calogero–Moser system with coordinates x_i and with spin degrees of freedom a_i^α and b_i^β. By considering the evolution of poles according to the discrete time matrix KP hierarchy, we also introduce the integrable discrete time version of the trigonometric spin Calogero–Moser system.

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1. INTRODUCTION

The matrix generalization of the Kadomtsev–Petviashvili (KP) hierarchy is an infinite set of compatible nonlinear differential equations with infinitely many independent (time) variables $\mathbf{t} = \{t_1, t_2, t_3, \dots\}$ and matrix dependent variables. It is a subhierarchy of the multicomponent KP hierarchy [5, 10, 23, 24]. Among all solutions to these equations, of special interest are solutions which have a finite number of poles in the variable $x = t_1$ in a fundamental domain of the complex plane. In particular, one can consider solutions which are trigonometric or hyperbolic functions of x with poles depending on the times $t_2, t_3, \dots$.

The dynamics of poles of singular solutions to nonlinear integrable equations is a well-known subject in mathematical physics [1, 4, 11, 12]. It has been shown that the poles of solutions to the KP equation as functions of the time t_2 move as particles of the integrable Calogero–Moser many-body system [2, 3, 15, 18]. Rational, trigonometric, and elliptic solutions correspond to rational, trigonometric, and elliptic Calogero–Moser systems, respectively.

The further progress was achieved in [21], where it was shown that the correspondence between rational solutions to the KP equation and the Calogero–Moser system with rational potential can be extended to the level of hierarchies. Namely, the evolution of poles with respect to the higher time t_m of the KP hierarchy was shown to be given by the higher Hamiltonian flow of the integrable Calogero–Moser system with the Hamiltonian $H_m = \operatorname{tr} L^m$, where L is the Lax matrix. Later this correspondence was generalized to trigonometric solutions of the KP hierarchy (see [9, 26]). It was shown that the dynamics of poles with respect to the higher time t_m is given by the Hamiltonian

[a] Moscow Institute of Physics and Technology (National Research University), Institutskii per. 9, Dolgoprudnyi, Moscow oblast, 141701 Russia.

[b] Skolkovo Institute of Science and Technology, Bol'shoi bul'var 30, str. 1, Moscow, 121205 Russia.

[c] Steklov Mathematical Institute of Russian Academy of Sciences, ul. Gubkina 8, Moscow, 119991 Russia.

E-mail addresses: vadim.prokofev@phystech.edu (V. V. Prokofev), zabrodin@itep.ru (A. V. Zabrodin).

flow with the Hamiltonian

$$\mathcal{H}_m = \frac{1}{2(m+1)\gamma}\,\mathrm{tr}\big((L+\gamma I)^{m+1} - (L-\gamma I)^{m+1}\big), \tag{1.1}$$

where I is the identity matrix and γ is a parameter such that $\pi i/\gamma$ is the period of the trigonometric or hyperbolic functions. Clearly, the Hamiltonian $\mathcal{H}_m$ is a linear combination of the Hamiltonians $H_k = \mathrm{tr}\, L^k$.

In the present paper we generalize this result to trigonometric solutions of the matrix KP hierarchy. The singular (in general, elliptic) solutions to the matrix KP equation were investigated in [13]. It was shown that the evolution of data of such solutions (positions of poles and some internal degrees of freedom) with respect to the time t_2 is isomorphic to the dynamics of a spin generalization of the Calogero–Moser system (the Gibbons–Hermsen system [8]). It is a system of $\mathcal{N}$ particles with coordinates x_i and with internal degrees of freedom given by N-dimensional column vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ which interact pairwise with each other. The Hamiltonian is

$$H = \sum_{i=1}^{\mathcal{N}} p_i^2 - \gamma^2 \sum_{i\neq k} \frac{(\mathbf{b}_i^{\mathrm{T}}\mathbf{a}_k)(\mathbf{b}_k^{\mathrm{T}}\mathbf{a}_i)}{\sinh^2(\gamma(x_i - x_k))} \tag{1.2}$$

(here $\mathbf{b}_i^{\mathrm{T}}$ is the transposed row vector) with the nonvanishing Poisson brackets $\{x_i, p_k\} = \delta_{ik}$ and $\{a_i^\alpha, b_k^\beta\} = \delta_{\alpha\beta}\delta_{ik}$. The model is known to be integrable, with the higher Hamiltonians (integrals of motion in involution) $H_k = \mathrm{tr}\, L^k$, where L is the Lax matrix of the model given by

$$L_{jk} = -p_j\delta_{jk} - (1-\delta_{jk})\frac{\gamma \mathbf{b}_j^{\mathrm{T}}\mathbf{a}_k}{\sinh(\gamma(x_j - x_k))}. \tag{1.3}$$

Our main result in this paper is that the dynamics of the poles x_i and vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ (which parametrize the matrix residues at the poles) with respect to the higher time t_m is given by the Hamiltonian flow with the Hamiltonian (1.1) and the Lax matrix (1.3). The corresponding result for rational solutions ($\gamma = 0$) was established in [19].

We use the method suggested by Krichever [12] for elliptic solutions of the KP equation. It consists in substituting the solution not in the KP equation itself but in an auxiliary linear problem for it (this implies a suitable pole ansatz for the wave function). This method allows one to obtain the equations of motion together with the Lax representation for them.

Another result of this paper is the time discretization of the trigonometric spin Calogero–Moser (Gibbons–Hermsen) model. (The time discretization of the rational spin Calogero–Moser system within the same approach was suggested in [25].) Because of the precise correspondence between the trigonometric solutions of the matrix KP hierarchy and the trigonometric spin Calogero–Moser hierarchy, the integrable time discretization of the Calogero–Moser system and its spin generalization can be obtained from the dynamics of poles of trigonometric solutions to semi-discrete soliton equations. ("Semi-discrete" means that the time becomes discrete while the space variable x, with respect to which one considers pole solutions, remains continuous.) At the same time, it is known that integrable discretizations of soliton equations can be regarded as belonging to the same hierarchy as their continuous counterparts. Namely, the discrete time step is equivalent to a special shift of infinitely many continuous hierarchical times. This fact underlies the method of generating discrete soliton equations developed in [6, 7]. For integrable time discretization of many-body systems, see [14, 16, 17, 20, 22]. In this paper, we derive equations of motion in discrete time p for the spin generalization of the trigonometric Calogero–Moser model:

$$\sum_j \coth\big(\gamma(x_i(p)-x_j(p+1))\big)(\mathbf{b}_i^{\mathrm T}(p)\mathbf{a}_j(p+1))(\mathbf{b}_j^{\mathrm T}(p+1)\mathbf{a}_i(p))$$
$$+\sum_j \coth\big(\gamma(x_i(p)-x_j(p-1))\big)(\mathbf{b}_i^{\mathrm T}(p)\mathbf{a}_j(p-1))(\mathbf{b}_j^{\mathrm T}(p-1)\mathbf{a}_i(p))$$
$$=2\sum_{j\ne i} \coth\big(\gamma(x_i(p)-x_j(p))\big)(\mathbf{b}_i^{\mathrm T}(p)\mathbf{a}_j(p))(\mathbf{b}_j^{\mathrm T}(p)\mathbf{a}_i(p)), \quad (1.4)$$

where $\mathbf{a}_i(p)$ and $\mathbf{b}_i(p)$ are spin variables. In the limit $\gamma\to 0$ the result of [25] is reproduced.

2. MATRIX KADOMTSEV–PETVIASHVILI HIERARCHY

Here we briefly review the main facts about the multicomponent and matrix KP hierarchies following [23, 24]. We start from the more general multicomponent KP hierarchy. The independent variables are N infinite sets of continuous "times"

$$\mathbf{t}=\{\mathbf{t}_1,\mathbf{t}_2,\dots,\mathbf{t}_N\},\qquad \mathbf{t}_\alpha=\{t_{\alpha,1},t_{\alpha,2},t_{\alpha,3},\dots\},\quad \alpha=1,\dots,N,$$

and N discrete integer variables $\mathbf{s}=\{s_1,s_2,\dots,s_N\}$ ("charges") constrained by the condition $\sum_{\alpha=1}^N s_\alpha=0$. In what follows, we will mostly put $s_\alpha=0$ since we are interested in the dynamics in the continuous times.

In the bilinear formalism, the dependent variable is the tau function $\tau(\mathbf{s};\mathbf{t})$. We also introduce the tau functions

$$\tau_{\alpha\beta}(\mathbf{t})=\tau(\mathbf{e}_\alpha-\mathbf{e}_\beta;\mathbf{t}), \quad (2.1)$$

where $\mathbf{e}_\alpha$ is the vector whose αth component is 1 and all the other entries are zero. The N-component KP hierarchy is the infinite set of bilinear equations for the tau functions which are encoded in the basic bilinear relation

$$\sum_{\nu=1}^N \epsilon_{\alpha\nu}\epsilon_{\beta\nu}\oint_{C_\infty} dz\, z^{\delta_{\alpha\nu}+\delta_{\beta\nu}-2}e^{\xi(\mathbf{t}_\nu-\mathbf{t}'_\nu,z)}\tau_{\alpha\nu}(\mathbf{t}-[z^{-1}]_\nu)\tau_{\nu\beta}(\mathbf{t}'+[z^{-1}]_\nu)=0, \quad (2.2)$$

which is valid for any $\mathbf{t}$ and $\mathbf{t}'$. Here $\epsilon_{\alpha\beta}$ is a sign factor: $\epsilon_{\alpha\beta}=1$ if $\alpha\le\beta$, and $\epsilon_{\alpha\beta}=-1$ if $\alpha>\beta$. In (2.2) we use the following standard notation:

$$\xi(\mathbf{t}_\gamma,z)=\sum_{k\ge1}t_{\gamma,k}z^k,\qquad (\mathbf{t}\pm[z^{-1}]_\gamma)_{\alpha k}=t_{\alpha,k}\pm\delta_{\alpha\gamma}\frac{z^{-k}}{k}.$$

The integration contour C_∞ is a large circle around ∞. Hereafter, we omit the variables $\mathbf{s}$ in the notation for the tau functions.

An important role in the theory of integrable hierarchies is played by the wave function. In the multicomponent KP hierarchy, the wave function $\Psi(\mathbf{t};z)$ and its adjoint $\Psi^\dagger(\mathbf{t};z)$ are $N\times N$ matrices with the components

$$\begin{aligned}\Psi_{\alpha\beta}(\mathbf{t};z)&=\epsilon_{\alpha\beta}\frac{\tau_{\alpha\beta}(\mathbf{t}-[z^{-1}]_\beta)}{\tau(\mathbf{t})}\,z^{\delta_{\alpha\beta}-1}e^{\xi(\mathbf{t}_\beta,z)},\\ \Psi^\dagger_{\alpha\beta}(\mathbf{t};z)&=\epsilon_{\beta\alpha}\frac{\tau_{\alpha\beta}(\mathbf{t}+[z^{-1}]_\alpha)}{\tau(\mathbf{t})}\,z^{\delta_{\alpha\beta}-1}e^{-\xi(\mathbf{t}_\alpha,z)}\end{aligned} \quad (2.3)$$

(here and below † does not mean the Hermitian conjugation). The complex variable z is called the spectral parameter. Around $z = \infty$, the wave function Ψ can be represented in the form of the series

$$\Psi_{\alpha\beta}(\mathbf{t}; z) = \Bigg(\delta_{\alpha\beta} + \sum_{k\geq 1} \frac{w^{(k)}_{\alpha\beta}(\mathbf{t})}{z^k}\Bigg) e^{\xi(\mathbf{t}_\beta, z)}, \tag{2.4}$$

where $w^{(k)}(\mathbf{t})$ are some matrix functions. In terms of the wave functions, the bilinear relation (2.2) can be written as

$$\oint_{C_\infty} dz\, \Psi(\mathbf{t}; z)\Psi^\dagger(\mathbf{t}'; z) = 0. \tag{2.5}$$

Another (equivalent) approach to the multicomponent KP hierarchy is based on matrix pseudodifferential operators. The hierarchy can be understood as an infinite set of evolution equations in the times $\mathbf{t}$ for matrix functions of a variable x. For example, the coefficients $w^{(k)}$ of the wave function can be taken as such matrix functions, the evolution being $w^{(k)}(x) \to w^{(k)}(x, \mathbf{t})$. In what follows we denote $\tau(x, \mathbf{t})$ and $w^{(k)}(x, \mathbf{t})$ simply by $\tau(\mathbf{t})$ and $w^{(k)}(\mathbf{t})$, suppressing the dependence on x. Let us introduce the matrix pseudodifferential "wave operator" $\mathcal{W}$ with matrix elements

$$\mathcal{W}_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{k\geq 1} w^{(k)}_{\alpha\beta}(\mathbf{t})\,\partial_x^{-k}, \tag{2.6}$$

where $w^{(k)}_{\alpha\beta}(\mathbf{t})$ are the same matrix functions as in (2.4). The wave function is a result of the action of the wave operator on the exponential function:

$$\Psi(\mathbf{t}; z) = \mathcal{W} \exp\Bigg(xzI + \sum_{\alpha=1}^{N} E_\alpha \xi(\mathbf{t}_\alpha, z)\Bigg), \tag{2.7}$$

where E_α is the $N \times N$ matrix with the components $(E_\alpha)_{\beta\gamma} = \delta_{\alpha\beta}\delta_{\alpha\gamma}$. The adjoint wave function can be written as

$$\Psi^\dagger(\mathbf{t}; z) = \exp\Bigg(-xzI - \sum_{\alpha=1}^{N} E_\alpha \xi(\mathbf{t}_\alpha, z)\Bigg)\mathcal{W}^{-1}. \tag{2.8}$$

Here the operators ∂_x involved in $\mathcal{W}^{-1}$ act to the left (the left action is defined as $f\partial_x \equiv -\partial_x f$).

It is proved in [24] that the wave function and its adjoint satisfy the linear equations

$$\partial_{t_{\alpha,m}} \Psi(\mathbf{t}; z) = B_{\alpha m}\Psi(\mathbf{t}; z), \qquad -\partial_{t_{\alpha,m}} \Psi^\dagger(\mathbf{t}; z) = \Psi^\dagger(\mathbf{t}; z)B_{\alpha m}, \tag{2.9}$$

where $B_{\alpha m}$ is the differential operator $B_{\alpha m} = (\mathcal{W}E_\alpha \partial_x^m \mathcal{W}^{-1})_+$. The notation $(\cdot)_+$ means the differential part of a pseudodifferential operator, i.e., the sum of all terms with ∂_x^k, where $k \geq 0$. Again, the operator $B_{\alpha m}$ in the second equation in (2.9) acts to the left. In particular, it follows from (2.9) at $m = 1$ that

$$\sum_{\alpha=1}^{N} \partial_{t_{\alpha,1}} \Psi(\mathbf{t}; z) = \partial_x \Psi(\mathbf{t}; z), \qquad \sum_{\alpha=1}^{N} \partial_{t_{\alpha,1}} \Psi^\dagger(\mathbf{t}; z) = \partial_x \Psi^\dagger(\mathbf{t}; z), \tag{2.10}$$

so the vector field ∂_x can be identified with the vector field $\sum_\alpha \partial_{t_{\alpha,1}}$.

The matrix KP hierarchy is a subhierarchy of the multicomponent KP one which is obtained by a restriction of the time variables in the following manner: $t_{\alpha,m} = t_m$ for all α and m. The corresponding vector fields are related as $\partial_{t_m} = \sum_{\alpha=1}^{N} \partial_{t_{\alpha,m}}$. The wave function for the matrix KP hierarchy has the expansion

$$\Psi_{\alpha\beta}(\mathbf{t}; z) = \big(\delta_{\alpha\beta} + w^{(1)}_{\alpha\beta}(\mathbf{t})z^{-1} + O(z^{-2})\big)e^{xz+\xi(\mathbf{t}, z)}, \tag{2.11}$$

where $\xi(\mathbf{t}, z) = \sum_{k\ge 1} t_k z^k$. The coefficient $w^{(1)}_{\alpha\beta}(\mathbf{t})$ plays an important role in what follows. Equations (2.9) imply that the wave function of the matrix KP hierarchy and its adjoint satisfy the linear equations

$$\partial_{t_m}\Psi(\mathbf{t}; z) = B_m\Psi(\mathbf{t}; z), \qquad -\partial_{t_m}\Psi^\dagger(\mathbf{t}; z) = \Psi^\dagger(\mathbf{t}; z)B_m, \qquad m \ge 1, \tag{2.12}$$

where B_m is the differential operator $B_m = (\mathcal{W}\partial_x^m\mathcal{W}^{-1})_+$. At $m = 1$ we have $\partial_{t_1}\Psi = \partial_x\Psi$, so we can identify ∂_x and $\partial_{t_1} = \sum_{\alpha=1}^N \partial_{t_{\alpha,1}}$ and the evolution in t_1 is simply a shift of the variable x: $w^{(k)}(x, t_1, t_2, \ldots) = w^{(k)}(x + t_1, t_2, \ldots)$. At $m = 2$ equations (2.12) turn into the linear problems

$$\partial_{t_2}\Psi = \partial_x^2\Psi + V(\mathbf{t})\Psi, \qquad -\partial_{t_2}\Psi^\dagger = \partial_x^2\Psi^\dagger + \Psi^\dagger V(\mathbf{t}), \tag{2.13}$$

which have the form of the matrix nonstationary Schrödinger equations with

$$V(\mathbf{t}) = -2\partial_x w^{(1)}(\mathbf{t}). \tag{2.14}$$

Let us derive a useful corollary of the bilinear relation (2.2). Differentiating it with respect to t_m and putting $\mathbf{t}' = \mathbf{t}$ after this, we obtain

$$\frac{1}{2\pi i}\sum_{\nu=1}^N \oint\limits_{C_\infty} dz\, z^m \Psi_{\alpha\nu}(\mathbf{t}; z)\Psi^\dagger_{\nu\beta}(\mathbf{t}; z) = -\partial_{t_m} w^{(1)}_{\alpha\beta}(\mathbf{t}), \tag{2.15}$$

or, equivalently,

$$\operatorname*{res}_\infty\bigl(z^m\Psi_{\alpha\nu}\Psi^\dagger_{\nu\beta}\bigr) = -\partial_{t_m} w^{(1)}_{\alpha\beta}. \tag{2.16}$$

Here and below the summation from 1 to N over repeated Greek indices is implied. The residue at infinity is defined according to $\operatorname{res}_\infty(z^{-n}) = \delta_{n1}$.

To conclude this section, let us make some remarks on the discrete time version of the matrix KP hierarchy. The discrete time evolution is defined as a special shift of the infinite number of continuous time variables according to the rule [6, 7]

$$\tau^p = \tau\left(\mathbf{t} - p\sum_{\alpha=1}^N [\mu^{-1}]_\alpha\right), \qquad \Psi^p = \Psi\left(\mathbf{t} - p\sum_{\alpha=1}^N [\mu^{-1}]_\alpha; z\right).$$

Here p is the discrete time variable and μ is a continuous parameter. Every μ corresponds to its own discrete time flow. The limit $\mu \to \infty$ is the continuous limit. One can show, using explicit expressions for the wave functions in terms of the tau function and some corollaries of the bilinear relation (see [25]), that the corresponding linear problems have the form

$$\mu\Psi^p_{\alpha\beta} - \mu\Psi^{p+1}_{\alpha\beta} = \partial_x\Psi^p_{\alpha\beta} + \bigl(w^{(1)}_{\alpha\nu}(p+1) - w^{(1)}_{\alpha\nu}(p)\bigr)\Psi^p_{\nu\beta}, \tag{2.17}$$

$$\mu\Psi^{\dagger p}_{\alpha\beta} - \mu\Psi^{\dagger\, p-1}_{\alpha\beta} = -\partial_x\Psi^{\dagger p}_{\alpha\beta} + \Psi^{\dagger p}_{\alpha\nu}\bigl(w^{(1)}_{\nu\beta}(p) - w^{(1)}_{\nu\beta}(p-1)\bigr). \tag{2.18}$$

3. TRIGONOMETRIC SOLUTIONS OF THE MATRIX KADOMTSEV–PETVIASHVILI HIERARCHY: DYNAMICS OF POLES IN t_2

Our aim is to study solutions to the matrix KP hierarchy which are trigonometric functions of the variable x (and, therefore, t_1). For the trigonometric solutions, the tau function has the form

$$\tau = C\prod_{i=1}^{\mathcal{N}}\bigl(e^{2\gamma x} - e^{2\gamma x_i}\bigr), \tag{3.1}$$

where γ is a parameter. The period of the function is $\pi i/\gamma$. Real (respectively, imaginary) γ correspond to hyperbolic (respectively, trigonometric) functions. In the limit $\gamma\to 0$ one obtains rational solutions. The $\mathcal{N}$ roots x_i (assumed to be distinct) depend on the times $\mathbf{t}$. It is convenient to pass to the exponentiated variables

$$w=e^{2\gamma x},\qquad w_i=e^{2\gamma x_i};\tag{3.2}$$

then the tau function becomes a polynomial with the roots w_i: $\tau=C\prod_i(w-w_i)$. Clearly, we have $\partial_x=2\gamma w\partial_w$ and $\partial_x^2=4\gamma^2(w^2\partial_w^2+w\partial_w)$.

It is clear from (2.3) that the wave functions Ψ and $\Psi^\dagger$ (and thus the coefficient $w^{(1)}$), as functions of x, have simple poles at $x=x_i$. It is shown in [19] that the residues at these poles are matrices of rank 1. We parametrize them through the column vectors $\mathbf{a}_i=(a_i^1,a_i^2,\dots,a_i^N)^{\mathrm T}$, $\mathbf{b}_i=(b_i^1,b_i^2,\dots,b_i^N)^{\mathrm T}$, and $\mathbf{c}_i=(c_i^1,c_i^2,\dots,c_i^N)^{\mathrm T}$ (T means transposition) and the row vector $\mathbf{c}_i^*=(c_i^{*1},c_i^{*2},\dots,c_i^{*N})$:

$$\Psi_{\alpha\beta}=e^{xz+\xi(\mathbf{t},z)}\left(C_{\alpha\beta}+\sum_i\frac{2\gamma w_i^{1/2}a_i^\alpha c_i^\beta}{w-w_i}\right),\tag{3.3}$$

$$\Psi^\dagger_{\alpha\beta}=e^{-xz-\xi(\mathbf{t},z)}\left(C^{-1}_{\alpha\beta}+\sum_i\frac{2\gamma w_i^{1/2}c_i^{*\alpha}b_i^\beta}{w-w_i}\right),\tag{3.4}$$

where the matrix $C_{\alpha\beta}$ does not depend on x. Note that the constant term in the adjoint wave function is the inverse matrix $C^{-1}_{\alpha\beta}$. This follows from (2.8). For the matrices $w^{(1)}$ and $V=-2\partial_x w^{(1)}$ we have

$$w^{(1)}_{\alpha\beta}=S_{\alpha\beta}-\sum_i\frac{2\gamma w_i a_i^\alpha b_i^\beta}{w-w_i},\qquad V_{\alpha\beta}=-8\gamma^2\sum_i\frac{ww_ia_i^\alpha b_i^\beta}{(w-w_i)^2},\tag{3.5}$$

where the matrix $S_{\alpha\beta}$ does not depend on x. Letting $w\to\infty$ in (2.16), one concludes that $\partial_{t_m}S_{\alpha\beta}=0$ for all $m\ge 1$, so the matrix $S_{\alpha\beta}$ does not depend on any times. The components of the vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ are going to be spin variables of the Gibbons–Hermsen model.

We first consider the dynamics of poles with respect to the time t_2. The procedure is similar to that in the rational case [19]. Following Krichever's approach, we consider the linear problems (2.13),

$$\partial_{t_2}\Psi_{\alpha\beta}=\partial_x^2\Psi_{\alpha\beta}-8\gamma^2\sum_{i=1}^{\mathcal{N}}\frac{ww_ia_i^\alpha b_i^\nu}{(w-w_i)^2}\Psi_{\nu\beta},\qquad -\partial_{t_2}\Psi^\dagger_{\alpha\beta}=\partial_x^2\Psi^\dagger_{\alpha\beta}-8\gamma^2\Psi^\dagger_{\alpha\nu}\sum_{i=1}^{\mathcal{N}}\frac{ww_ia_i^\nu b_i^\beta}{(w-w_i)^2},$$

and substitute here the pole ansatz (3.3), (3.4) for the wave functions. Consider first the equation for Ψ. First of all, comparing the behavior of both sides as $w\to\infty$, we conclude that $\partial_{t_2}C_{\alpha\beta}=0$, so $C_{\alpha\beta}$ does not depend on t_2 (in a similar way, from the higher linear problems one can see that $C_{\alpha\beta}$ does not depend on any times t_m). After the substitution, we see that the expression has poles at $w=w_i$ up to the third order. Equating the coefficients of the poles of different orders at $w=w_i$, we get the following conditions:

- at $(w-w_i)^{-3}$,

$$b_i^\nu a_i^\nu=1;$$

- at $(w-w_i)^{-2}$,

$$-\frac12\dot x_ic_i^\beta-2\gamma\sum_{k\ne i}\frac{w_i^{1/2}w_k^{1/2}b_i^\nu a_k^\nu c_k^\beta}{w_i-w_k}-(z-\gamma)c_i^\beta=w_i^{1/2}\tilde b_i^\beta;$$

- at $(w-w_i)^{-1}$,

$$\partial_{t_2}\big(w_i^{1/2}a_i^\alpha c_i^\beta\big)=2\gamma w_i^{1/2}\dot x_i a_i^\alpha c_i^\beta+8\gamma^2\sum_{k\ne i}\frac{w_i^2w_k^{1/2}a_i^\alpha b_i^\nu a_k^\nu c_k^\beta}{(w_i-w_k)^2}-8\gamma^2\sum_{k\ne i}\frac{w_i^{3/2}w_k a_k^\alpha b_k^\nu a_i^\nu c_i^\beta}{(w_i-w_k)^2},$$

where $\widetilde b_i^\beta=b_i^\nu C_{\nu\beta}$ and $\dot x_i=\partial_{t_2}x_i$.

Similar calculations for the linear problem for $\Psi^\dagger$ lead to the following conditions:

- at $(w-w_i)^{-3}$,

$$b_i^\nu a_i^\nu=1;$$

- at $(w-w_i)^{-2}$,

$$-\frac12\dot x_i c_i^{*\alpha}-2\gamma\sum_{k\ne i}\frac{w_i^{1/2}w_k^{1/2}c_k^{*\alpha}b_k^\nu a_i^\nu}{w_k-w_i}-(z+\gamma)c_i^{*\alpha}=-w_i^{1/2}\widetilde a_i^\alpha;$$

- at $(w-w_i)^{-1}$,

$$\partial_{t_2}\big(w_i^{1/2}c_i^{*\alpha}b_i^\beta\big)=-2\gamma w_i^{1/2}\dot x_i c_i^{*\alpha}b_i^\beta+8\gamma^2\sum_{k\ne i}\frac{w_i^2w_k^{1/2}c_k^{*\alpha}a_i^\nu b_k^\nu b_i^\beta}{(w_i-w_k)^2}-8\gamma^2\sum_{k\ne i}\frac{w_i^{3/2}w_k c_i^{*\alpha}a_k^\nu b_i^\nu b_k^\beta}{(w_i-w_k)^2},$$

where $\widetilde a_i^\alpha=C^{-1}_{\alpha\nu}a_i^\nu$.

The conditions coming from the third-order poles are constraints on the vectors $\mathbf a_i$ and $\mathbf b_i$. The other conditions can be written in the matrix form

$$\begin{cases}(zI-(L+\gamma I))\mathbf c^\beta=-W^{1/2}\widetilde{\mathbf b}^\beta,\\ \dot{\mathbf c}^\beta=M\mathbf c^\beta,\end{cases}\tag{3.6}$$

$$\begin{cases}\mathbf c^{*\alpha}(zI-(L-\gamma I))=\widetilde{\mathbf a}^{\alpha\mathrm T}W^{1/2},\\ \dot{\mathbf c}^{*\alpha}=\mathbf c^{*\alpha}\widetilde M,\end{cases}\tag{3.7}$$

where $\mathbf c^\beta=(c_1^\beta,\dots,c_{\mathcal N}^\beta)^{\mathrm T}$, $\mathbf c^{*\alpha}=(c_1^{*\alpha},\dots,c_{\mathcal N}^{*\alpha})$, $\widetilde{\mathbf b}^\beta=(\widetilde b_1^\beta,\dots,\widetilde b_{\mathcal N}^\beta)^{\mathrm T}$, and $\widetilde{\mathbf a}^\alpha=(\widetilde a_1^\alpha,\dots,\widetilde a_{\mathcal N}^\alpha)$ are $\mathcal N$-dimensional vectors, I is the identity matrix, $W=\operatorname{diag}(w_1,w_2,\dots,w_{\mathcal N})$, and L, M, and $\widetilde M$ are $\mathcal N\times\mathcal N$ matrices of the form

$$L_{ik}=-\frac12\,\dot x_i\delta_{ik}-2\gamma(1-\delta_{ik})\frac{w_i^{1/2}w_k^{1/2}b_i^\nu a_k^\nu}{w_i-w_k},\tag{3.8}$$

$$M_{ik}=(\gamma\dot x_i-\Lambda_i)\delta_{ik}+8\gamma^2(1-\delta_{ik})\frac{w_i^{3/2}w_k^{1/2}b_i^\nu a_k^\nu}{(w_i-w_k)^2},\tag{3.9}$$

$$\widetilde M_{ik}=(\gamma\dot x_i+\Lambda_i^*)\delta_{ik}-8\gamma^2(1-\delta_{ik})\frac{w_i^{1/2}w_k^{3/2}b_i^\nu a_k^\nu}{(w_i-w_k)^2}.\tag{3.10}$$

Here

$$\Lambda_i=\frac{\dot a_i^\alpha}{a_i^\alpha}+8\gamma^2\sum_{k\ne i}\frac{w_iw_ka_k^\alpha b_k^\nu a_i^\nu}{a_i^\alpha(w_i-w_k)^2},\qquad -\Lambda_i^*=\frac{\dot b_i^\alpha}{b_i^\alpha}-8\gamma^2\sum_{k\ne i}\frac{w_iw_kb_i^\nu a_k^\nu b_k^\alpha}{b_i^\alpha(w_i-w_k)^2}\tag{3.11}$$

do not depend on the index α. In fact one can see that $\Lambda_i=\Lambda_i^*$. Indeed, multiplying equations (3.11) by $a_i^\alpha b_i^\alpha$ (no summation here!), summing over α, and summing the two equations, we get $\Lambda_i-\Lambda_i^*=\partial_{t_2}(a_i^\alpha b_i^\alpha)=0$ by virtue of the constraint $a_i^\alpha b_i^\alpha=1$.

Differentiating the first equation in (3.6) with respect to t_2, we get, after some calculations, the compatibility condition of equations (3.6):

$$(\dot{L} + [L, M])\mathbf{c}^{\beta} = 0. \tag{3.12}$$

Taking into account equations (3.11), which we write here in the form

$$\dot{a}_i^{\alpha} = \Lambda_i a_i^{\alpha} - 2\gamma^2 \sum_{k\neq i} \frac{a_k^{\alpha} b_k^{\nu} a_i^{\nu}}{\sinh^2(\gamma(x_i - x_k))}, \qquad \dot{b}_i^{\alpha} = -\Lambda_i b_i^{\alpha} + 2\gamma^2 \sum_{k\neq i} \frac{b_i^{\nu} a_k^{\nu} b_k^{\alpha}}{\sinh^2(\gamma(x_i - x_k))} \tag{3.13}$$

(in this form they are equations of motion for the spin degrees of freedom), one can see that the off-diagonal entries of the matrix $\dot{L} + [L, M]$ are equal to zero. The vanishing of the diagonal entries yields equations of motion for the poles x_i:

$$\ddot{x}_i = -8\gamma^3 \sum_{k\neq i} \frac{\cosh(\gamma(x_i - x_k))}{\sinh^3(\gamma(x_i - x_k))}\, b_i^{\mu} a_k^{\mu} b_k^{\nu} a_i^{\nu}. \tag{3.14}$$

The gauge transformation $a_i^{\alpha} \to a_i^{\alpha} q_i$, $b_i^{\alpha} \to b_i^{\alpha} q_i^{-1}$ with $q_i = \exp\bigl(\int^{t_2} \Lambda_i\, dt\bigr)$ eliminates the terms with Λ_i in (3.13), so we can put $\Lambda_i = 0$. This gives the equations of motion

$$\dot{a}_i^{\alpha} = -2\gamma^2 \sum_{k\neq i} \frac{a_k^{\alpha} b_k^{\nu} a_i^{\nu}}{\sinh^2(\gamma(x_i - x_k))}, \qquad \dot{b}_i^{\alpha} = 2\gamma^2 \sum_{k\neq i} \frac{b_i^{\nu} a_k^{\nu} b_k^{\alpha}}{\sinh^2(\gamma(x_i - x_k))}. \tag{3.15}$$

Together with (3.14) they are equations of motion of the trigonometric Gibbons–Hermsen model. Their Lax representation is given by the matrix equation $\dot{L} = [M, L]$. It states that the time evolution of the Lax matrix is an isospectral transformation. It follows that the quantities $H_k = \operatorname{tr} L^k$ are integrals of motion. In particular,

$$H_2 = \sum_{i=1}^{\mathcal{N}} p_i^2 - \gamma^2 \sum_{i\neq k} \frac{b_i^{\mu} a_k^{\mu} b_k^{\nu} a_i^{\nu}}{\sinh^2(\gamma(x_i - x_k))} = \operatorname{tr} L^2 \tag{3.16}$$

is the Hamiltonian of the Gibbons–Hermsen model. The equations of motion (3.15) and (3.14) are equivalent to the Hamiltonian equations

$$\dot{x}_i = \frac{\partial H_2}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H_2}{\partial x_i}, \qquad \dot{a}_i^{\alpha} = \frac{\partial H_2}{\partial b_i^{\alpha}}, \qquad \dot{b}_i^{\alpha} = -\frac{\partial H_2}{\partial a_i^{\alpha}}. \tag{3.17}$$

4. DYNAMICS OF POLES IN THE HIGHER TIMES

The main tool for the analysis of the dynamics in the higher times is the relation (2.16), which after the substitution of (3.3)–(3.5) takes the form

$$\operatorname*{res}_{\infty}\left[z^m \left(C_{\alpha\nu} + \sum_i \frac{2\gamma w_i^{1/2} a_i^{\alpha} c_i^{\nu}}{w - w_i}\right)\left(C^{-1}_{\nu\beta} + \sum_k \frac{2\gamma w_k^{1/2} c_k^{*\nu} b_k^{\beta}}{w - w_k}\right)\right]$$
$$= 2\gamma \sum_i \frac{\partial_{t_m}(w_i a_i^{\alpha} b_i^{\beta})}{w - w_i} + 4\gamma^2 \sum_i \frac{\partial_{t_m} x_i\, w_i^2 a_i^{\alpha} b_i^{\beta}}{(w - w_i)^2}. \tag{4.1}$$

Both sides are rational functions of w with poles at $w = w_i$ that vanish at infinity. Identifying the coefficients in front of the second-order poles, we obtain

$$\partial_{t_m} x_i = \operatorname*{res}_{\infty}\bigl(z^m c_i^{\nu} w_i^{-1} c_i^{*\nu}\bigr). \tag{4.2}$$

Solving the linear equations (3.6) and (3.7), we get

$$c_i^{\nu} = -\sum_k (zI - (L + \gamma I))^{-1}_{ik} w_k^{1/2} \widetilde{b}_k^{\nu}, \qquad c_i^{*\nu} = \sum_k \widetilde{a}_k^{\nu} w_k^{1/2} (zI - (L - \gamma I))^{-1}_{ki}, \tag{4.3}$$

and, therefore, (4.2) reads

$$\partial_{t_m} x_i = -\operatorname*{res}_{\infty} \sum_{k,k'} \left(z^m \widetilde{a}_k^\nu \widetilde{b}_{k'}^\nu w_k^{1/2} \left(\frac{1}{zI - (L - \gamma I)} \right)_{ki} w_i^{-1} \left(\frac{1}{zI - (L + \gamma I)} \right)_{ik'} w_{k'}^{1/2} \right)$$

$$= -\operatorname*{res}_{\infty} \operatorname{tr} \left(z^m W^{1/2} R W^{1/2} \frac{1}{zI - (L - \gamma I)} W^{-1} E_i \frac{1}{zI - (L + \gamma I)} \right),$$

where E_i is the diagonal matrix with entries $(E_i)_{jk} = \delta_{ij}\delta_{ik}$ and R is the $\mathcal{N} \times \mathcal{N}$ matrix with

$$R_{ik} = \widetilde{b}_i^\nu \widetilde{a}_k^\nu = b_i^\nu a_k^\nu. \tag{4.4}$$

The following commutation relation can be checked directly:

$$[L, W] = 2\gamma \big(W^{1/2} R W^{1/2} - W \big). \tag{4.5}$$

Note that $E_i = -\partial L / \partial p_i$. The rest of the calculation is similar to that in [26]. Using (4.5), we have

$$\partial_{t_m} x_i = \frac{1}{2\gamma} \operatorname*{res}_{\infty} \operatorname{tr} \left(z^m \big(LW - WL + 2\gamma W \big) \frac{1}{zI - (L - \gamma I)} W^{-1} \frac{\partial L}{\partial p_i} \frac{1}{zI - (L + \gamma I)} \right)$$

$$= \frac{1}{2\gamma} \operatorname*{res}_{\infty} \operatorname{tr} \left(z^m \left(\frac{\partial L}{\partial p_i} \frac{1}{zI - (L + \gamma I)} - \frac{\partial L}{\partial p_i} \frac{1}{zI - (L - \gamma I)} \right) \right)$$

$$= \frac{1}{2\gamma} \operatorname{tr} \left(\frac{\partial L}{\partial p_i} (L + \gamma I)^m - \frac{\partial L}{\partial p_i} (L - \gamma I)^m \right)$$

$$= \frac{1}{2(m+1)\gamma} \frac{\partial}{\partial p_i} \operatorname{tr} \big((L + \gamma I)^{m+1} - (L - \gamma I)^{m+1} \big) = \frac{\partial \mathcal{H}_m}{\partial p_i},$$

where $\mathcal{H}_m$ is given by (1.1). Note that $\mathcal{H}_2 = H_2 + \text{const}$. We have obtained one part of the Hamiltonian equations for the higher time flows. In the case $\gamma \to 0$ (rational solutions) the result of the paper [19] is reproduced.

In order to obtain another part of the Hamiltonian equations, let us differentiate (4.2) with respect to t_2:

$$\partial_{t_m} \dot{x}_i = -2\gamma \operatorname*{res}_{\infty} \big(z^m c_i^{*\nu} \dot{x}_i w_i^{-1} c_i^\nu \big) + \operatorname*{res}_{\infty} \big(z^m \big(c_i^\nu w_i^{-1} \partial_{t_2} c_i^\nu + \partial_{t_2} c_i^{*\nu} w_i^{-1} c_i^\nu \big) \big)$$

$$= \operatorname*{res}_{\infty} \sum_k \big(z^m \big(c_i^{*\nu} w_i^{-1} B_{ik} c_k^\nu - c_k^{*\nu} w_k^{-1} B_{ki} c_i^\nu \big) \big),$$

where

$$B_{jk} = 8\gamma^2 (1 - \delta_{jk}) \frac{w_j^{3/2} w_k^{1/2} b_j^\nu a_k^\nu}{(w_i - w_k)^2}.$$

Therefore, using (4.3), we have

$$\partial_{t_m} p_i = \frac{1}{2} \partial_{t_m} \dot{x}_i = -\operatorname*{res}_{\infty} \left[z^m \operatorname{tr} \left(W^{1/2} R W^{1/2} \frac{1}{zI - (L - \gamma I)} G^{(i)} \frac{1}{zI - (L + \gamma I)} \right) \right],$$

where the matrix $G^{(i)}$ is given by

$$G_{jk}^{(i)} = 4\gamma^2 (\delta_{ij} - \delta_{ik}) \frac{w_j^{1/2} w_k^{1/2} b_j^\nu a_k^\nu}{(w_i - w_k)^2}.$$

It is straightforward to check the identities

$$\big(WG^{(i)}-G^{(i)}W\big)_{jk}=-2\gamma L_{jk}(\delta_{ij}-\delta_{ik}),\qquad WG^{(i)}+G^{(i)}W=2\frac{\partial L}{\partial x_i}.$$

A direct calculation which literally repeats the one in [26] shows that

$$\begin{aligned}\partial_{t_m}p_i&=-\frac{1}{2\gamma}\operatorname*{res}_{\infty}\bigg[z^m\operatorname{tr}\bigg(\big(LW-WL+2\gamma W\big)\frac{1}{zI-(L-\gamma I)}G^{(i)}\frac{1}{zI-(L+\gamma I)}\bigg)\bigg]\\&=-\frac{1}{2\gamma}\operatorname*{res}_{\infty}\bigg[z^m\operatorname{tr}\bigg(WG^{(i)}\frac{1}{zI-(L+\gamma I)}-G^{(i)}W\frac{1}{zI-(L-\gamma I)}\bigg)\bigg]\\&=-\frac{1}{2\gamma}\operatorname*{res}_{\infty}\bigg[z^m\operatorname{tr}\bigg(\frac{\partial L}{\partial x_i}\bigg(\frac{1}{zI-(L+\gamma I)}-\frac{1}{zI-(L-\gamma I)}\bigg)\bigg)\bigg]\\&=-\frac{1}{2\gamma}\operatorname{tr}\bigg(\frac{\partial L}{\partial x_i}(L+\gamma I)^m-\frac{\partial L}{\partial x_i}(L-\gamma I)^m\bigg)=-\frac{\partial\mathcal{H}_m}{\partial x_i}.\end{aligned}$$

We have established the remaining part of the Hamiltonian equations for the higher time dynamics of the x_i's.

5. DYNAMICS OF SPIN VARIABLES IN THE HIGHER TIMES

A comparison of the first-order poles in (4.1) gives the following relation:

$$\begin{aligned}&\partial_{t_m}(w_ia_i^\alpha b_i^\beta)\\&\quad=\operatorname*{res}_{\infty}\bigg[z^m\bigg(w_i^{1/2}C_{\alpha\nu}c_i^{*\nu}b_i^\beta+w_i^{1/2}a_i^\alpha c_i^\nu C^{-1}_{\nu\beta}+2\gamma\sum_{k\neq i}\frac{w_i^{1/2}w_k^{1/2}}{w_i-w_k}\big(a_i^\alpha b_k^\beta c_i^\nu c_k^{*\nu}+a_k^\alpha b_i^\beta c_k^\nu c_i^{*\nu}\big)\bigg)\bigg].\end{aligned}$$

Using (4.3), we can rewrite it in the form

$$\begin{aligned}&b_i^\beta\bigg[-\partial_{t_m}a_i^\alpha+\operatorname*{res}_{\infty}\bigg(z^m\bigg(\sum_k a_k^\alpha w_i^{-1/2}w_k^{1/2}\bigg(\frac{1}{zI-(L-\gamma I)}\bigg)_{ki}\\&\qquad-2\gamma\sum_{k\neq i}\sum_{l,n}\frac{w_i^{-1/2}w_k^{1/2}}{w_i-w_k}a_k^\alpha a_l^\nu w_l^{1/2}\bigg(\frac{1}{zI-(L-\gamma I)}\bigg)_{li}\bigg(\frac{1}{zI-(L+\gamma I)}\bigg)_{kn}w_n^{1/2}b_n^\nu\bigg)\bigg)\bigg]\\&-a_i^\alpha\bigg[\partial_{t_m}b_i^\beta+\operatorname*{res}_{\infty}\bigg(z^m\bigg(\sum_k b_k^\beta w_i^{-1/2}w_k^{1/2}\bigg(\frac{1}{zI-(L+\gamma I)}\bigg)_{ik}\\&\qquad+2\gamma\sum_{k\neq i}\sum_{l,n}\frac{w_i^{-1/2}w_k^{1/2}}{w_i-w_k}b_k^\beta a_l^\nu w_l^{1/2}\bigg(\frac{1}{zI-(L-\gamma I)}\bigg)_{lk}\bigg(\frac{1}{zI-(L+\gamma I)}\bigg)_{in}w_n^{1/2}b_n^\nu\bigg)\bigg)\bigg]\\&\qquad\qquad=2\gamma\partial_{t_m}x_i\,a_i^\alpha b_i^\beta.\end{aligned}$$

Separating the terms with $k=i$ in the sums over k in the first and third lines and taking into account that

$$\begin{aligned}2\gamma\partial_{t_m}x_i&=\operatorname*{res}_{\infty}\operatorname{tr}\bigg[z^mE_i\bigg(\frac{1}{zI-(L-\gamma I)}-\frac{1}{zI-(L+\gamma I)}\bigg)\bigg]\\&=\operatorname*{res}_{\infty}\bigg[z^m\bigg(\frac{1}{zI-(L-\gamma I)}\bigg)_{ii}-z^m\bigg(\frac{1}{zI-(L+\gamma I)}\bigg)_{ii}\bigg],\end{aligned}$$

we represent this equation as follows:

$$b_i^\beta P_i^\alpha - a_i^\alpha Q_i^\beta = 0, \tag{5.1}$$

where

$$P_i^\alpha = -\partial_{t_m} a_i^\alpha + \operatorname*{res}_\infty \Bigg[z^m \Bigg(\sum_{k\ne i} a_k^\alpha w_i^{-1/2} w_k^{1/2} \bigg(\frac{1}{zI-(L-\gamma I)} \bigg)_{ki}$$
$$+ \operatorname{tr}\bigg(W^{1/2} R W^{1/2} \frac{1}{zI-(L-\gamma I)} W^{-1} \frac{\partial L}{\partial b_i^\alpha} \frac{1}{zI-(L+\gamma I)} \bigg) \Bigg) \Bigg],$$

$$Q_i^\beta = \partial_{t_m} b_i^\beta + \operatorname*{res}_\infty \Bigg[z^m \Bigg(\sum_{k\ne i} b_k^\beta w_i^{-1/2} w_k^{1/2} \bigg(\frac{1}{zI-(L+\gamma I)} \bigg)_{ik}$$
$$+ \operatorname{tr}\bigg(W^{1/2} R W^{1/2} \frac{1}{zI-(L-\gamma I)} \frac{\partial L}{\partial a_i^\beta} W^{-1} \frac{1}{zI-(L+\gamma I)} \bigg) \Bigg) \Bigg].$$

Here we took into account that

$$\frac{\partial L_{jk}}{\partial b_i^\alpha} = -2\gamma \delta_{ij}(1-\delta_{jk}) \frac{w_i^{1/2} w_k^{1/2} a_k^\alpha}{w_i - w_k}, \qquad \frac{\partial L_{jk}}{\partial a_i^\alpha} = -2\gamma \delta_{ik}(1-\delta_{jk}) \frac{w_j^{1/2} w_i^{1/2} b_j^\alpha}{w_j - w_i}.$$

It then follows from (5.1) that

$$\frac{P_i^\alpha}{a_i^\alpha} = \frac{Q_i^\beta}{b_i^\beta} = -\Lambda_i^{(m)} \tag{5.2}$$

does not depend on the indices α and β.

Let us transform the expressions for P_i^α and Q_i^β using the commutation relation (4.5), i.e., substituting

$$W^{1/2} R W^{1/2} = \frac{1}{2\gamma}\big(LW - WL + 2\gamma W \big).$$

We have

$$P_i^\alpha = -\partial_{t_m} a_i^\alpha + \frac{1}{2\gamma} \operatorname*{res}_\infty \Bigg[z^m \Bigg(\operatorname{tr}\bigg(\frac{\partial L}{\partial b_i^\alpha} \frac{1}{zI-(L+\gamma I)} - \frac{\partial L}{\partial b_i^\alpha} \frac{1}{zI-(L-\gamma I)} \bigg)$$
$$+ 2\gamma \sum_{k\ne i} a_k^\alpha w_i^{-1/2} w_k^{1/2} \bigg(\frac{1}{zI-(L-\gamma I)} \bigg)_{ki} + \operatorname{tr}\bigg(\frac{\partial L}{\partial b_i^\alpha} - W^{-1} \frac{\partial L}{\partial b_i^\alpha} W \bigg) \frac{1}{zI-(L-\gamma I)} \Bigg) \Bigg].$$

However,

$$\bigg(\frac{\partial L}{\partial b_i^\alpha} - W^{-1} \frac{\partial L}{\partial b_i^\alpha} W \bigg)_{jk} = -2\gamma \delta_{ij}(1-\delta_{jk}) w_i^{-1/2} w_k^{1/2} a_k^\alpha,$$

and so the second line vanishes. We are left with

$$P_i^\alpha = -\partial_{t_m} a_i^\alpha + \frac{\partial \mathcal{H}_m}{\partial b_i^\alpha}. \tag{5.3}$$

A similar calculation for Q_i^α yields

$$Q_i^\alpha = \partial_{t_m} b_i^\alpha + \frac{\partial \mathcal{H}_m}{\partial a_i^\alpha}. \tag{5.4}$$

Therefore, from (5.2) we have the equations of motion

$$\partial_{t_m} a_i^\alpha = \frac{\partial \mathcal{H}_m}{\partial b_i^\alpha} + \Lambda_i^{(m)} a_i^\alpha, \qquad \partial_{t_m} b_i^\alpha = -\frac{\partial \mathcal{H}_m}{\partial a_i^\alpha} - \Lambda_i^{(m)} b_i^\alpha.$$

The gauge transformation $a_i^\alpha \to a_i^\alpha q_i^{(m)}$, $b_i^\alpha \to b_i^\alpha \big(q_i^{(m)}\big)^{-1}$ with $q_i^{(m)} = \exp\big(\int^{t_m} \Lambda_i^{(m)} dt\big)$ eliminates the terms with $\Lambda_i^{(m)}$, and so we can put $\Lambda_i^{(m)} = 0$. In this way we obtain the Hamiltonian equations of motion for spin variables in the higher times:

$$\partial_{t_m} a_i^\alpha = \frac{\partial \mathcal{H}_m}{\partial b_i^\alpha}, \qquad \partial_{t_m} b_i^\alpha = -\frac{\partial \mathcal{H}_m}{\partial a_i^\alpha}, \tag{5.5}$$

with $\mathcal{H}_m$ given by (1.1).

6. TIME DISCRETIZATION OF THE TRIGONOMETRIC GIBBONS–HERMSEN MODEL

Our strategy is to substitute the pole ansatz for the discrete time wave functions

$$\Psi^p_{\alpha\beta} = \Big(1 - \frac{z}{\mu}\Big)^p e^{xz} \left(C_{\alpha\beta} + \sum_i \frac{2\gamma w_i^{1/2}(p) a_i^\alpha(p) c_i^\beta(p)}{w - w_i(p)} \right), \tag{6.1}$$

$$\Psi^{\dagger p}_{\alpha\beta} = \Big(1 - \frac{z}{\mu}\Big)^{-p} e^{-xz} \left(C^{-1}_{\alpha\beta} + \sum_i \frac{2\gamma w_i^{1/2}(p) c_i^{*\alpha}(p) b_i^\beta(p)}{w - w_i(p)} \right), \tag{6.2}$$

and $w^{(1)}_{\alpha\beta}$ (see (3.5)) into the linear problems (2.17) and (2.18) and identify the coefficients in front of the poles $(w - w_i(p))^{-2}$, $(w - w_i(p \pm 1))^{-1}$, and $(w - w_i(p))^{-1}$. (Note that the constant term $S_{\alpha\beta}$ in $w^{(1)}_{\alpha\beta}(p)$ cancels in the combination $w^{(1)}_{\alpha\beta}(p+1) - w^{(1)}_{\alpha\beta}(p)$ because the shift $p \to p+1$ is equivalent to a shift of times and $S_{\alpha\beta}$ does not depend on the times.) We begin with the linear problem (2.17) for Ψ. From the cancellation of different poles we have the following conditions:

- at $(w - w_i(p))^{-2}$,

$$b_i^\nu(p) a_i^\nu(p) = 1;$$

- at $(w - w_i(p+1))^{-1}$,

$$(z-\mu) c_i^\beta(p+1) = -w_i^{1/2}(p) \widetilde{b}_i^\beta(p+1) - 2\gamma \sum_j \frac{w_i^{1/2}(p) w_j^{1/2}(p) b_i^\nu(p+1) a_j^\nu(p) c_j^\beta(p)}{w_i(p+1) - w_j(p)};$$

- at $(w - w_i(p))^{-1}$,

$$(z - \mu - 2\gamma) a_i^\alpha(p) c_i^\beta(p) + w_i^{1/2}(p) a_i^\alpha(p) \widetilde{b}_i^\beta(p) - 2\gamma \sum_j \frac{w_j(p+1) a_j^\alpha(p+1) b_j^\nu(p+1) a_i^\nu(p) c_i^\beta(p)}{w_i(p) - w_j(p+1)}$$
$$+ 2\gamma \sum_{j\ne i} \frac{w_i^{1/2}(p) w_j^{1/2}(p) a_i^\alpha(p) b_i^\nu(p) a_j^\nu(p) c_j^\beta(p)}{w_i(p) - w_j(p)} + 2\gamma \sum_{j \ne i} \frac{w_j(p) a_j^\alpha(p) b_j^\nu(p) a_i^\nu(p) c_i^\beta(p)}{w_i(p) - w_j(p)} = 0.$$

Introduce the matrices

$$L_{ij}(p) = -\delta_{ij} \frac{\dot{x}_i(p)}{2} - 2\gamma(1 - \delta_{ij}) \frac{w_i^{1/2}(p) w_j^{1/2}(p) b_i^\nu(p) a_j^\nu(p)}{w_i(p) - w_j(p)} \tag{6.3}$$

(the same Lax matrix as in (3.8)) and

$$M_{ij}(p)=2\gamma\frac{w_i^{1/2}(p+1)w_j^{1/2}(p)b_i^{\nu}(p+1)a_j^{\nu}(p)}{w_i(p+1)-w_j(p)}. \tag{6.4}$$

Then the above conditions can be written as

$$\begin{gathered}(z-\mu)c_i^{\beta}(p+1)=-w_i^{1/2}(p+1)\widetilde{b}_i^{\beta}(p+1)-\sum_j M_{ij}(p)c_j^{\beta}(p),\\ a_i^{\alpha}(p)\underbrace{\biggl[\sum_j\bigl((z-\gamma)\delta_{ij}-L_{ij}(p)\bigr)c_j^{\beta}(p)+w_i^{1/2}(p)\widetilde{b}_i^{\beta}(p)\biggr]}_{=0}\\ +c_i^{\beta}(p)\biggl[\sum_j a_j^{\alpha}(p+1)\bigl(W^{1/2}(p+1)M(p)W^{-1/2}(p)\bigr)_{ji}\\ +\sum_j a_j^{\alpha}(p)\bigl(W^{1/2}(p)L(p)W^{-1/2}(p)\bigr)_{ji}-(\mu+\gamma)a_i^{\alpha}(p)\biggr]=0.\end{gathered} \tag{6.5}$$

The first line in the second equation vanishes by virtue of (3.6). Therefore, we have the following equations:

$$(z-\mu)\mathbf{c}^{\beta}(p+1)=-W^{1/2}(p+1)\widetilde{\mathbf{b}}^{\beta}(p+1)-M(p)\mathbf{c}^{\beta}(p), \tag{6.6}$$

$$\mathbf{a}^{\alpha\mathrm{T}}(p+1)W^{1/2}(p+1)M(p)W^{-1/2}(p)+\mathbf{a}^{\alpha\mathrm{T}}(p)W^{1/2}(p)L(p)W^{-1/2}(p)=(\mu+\gamma)\mathbf{a}^{\alpha\mathrm{T}}(p). \tag{6.7}$$

A similar solution of the linear problem (2.18) for $\Psi^{\dagger}$ gives the equations

$$(z-\mu)\mathbf{c}^{*\alpha}(p-1)=\widetilde{\mathbf{a}}^{\alpha\mathrm{T}}(p-1)W^{1/2}(p-1)-\mathbf{c}^{*\alpha}(p)M(p-1), \tag{6.8}$$

$$W^{-1/2}(p)M(p-1)W^{1/2}(p-1)\mathbf{b}^{\beta}(p-1)+W^{-1/2}(p)L(p)W^{1/2}(p)\mathbf{b}^{\beta}(p)=(\mu-\gamma)\mathbf{b}^{\beta}(p). \tag{6.9}$$

A simple calculation similar to that in [25] shows that the compatibility condition of equations (3.6) and (6.6) is the discrete Lax equation

$$L(p+1)M(p)=M(p)L(p), \tag{6.10}$$

which holds true provided equations (6.7) and (6.9) are satisfied.

Equations (6.7) and (6.9) are equations of motion of the discrete time trigonometric Gibbons–Hermsen model. Let us consider equation (6.7) and represent it in a somewhat more convenient form. In order to do this, we write it in the form

$$\begin{gathered}2\gamma\sum_k\frac{w_i(p)a_k^{\alpha}(p+1)b_k^{\nu}(p+1)a_i^{\nu}(p)}{w_k(p+1)-w_i(p)}+2\gamma\sum_{k\neq i}\frac{w_i(p)a_k^{\alpha}(p)b_k^{\nu}(p)a_i^{\nu}(p)}{w_i(p)-w_k(p)}\\ +2\gamma\sum_k a_k^{\alpha}(p+1)b_k^{\nu}(p+1)a_i^{\nu}(p)-2\gamma\sum_{k\neq i}a_k^{\alpha}(p)b_k^{\nu}(p)a_i^{\nu}(p)-(\mu+\gamma)a_i^{\alpha}(p)-\frac{\dot{x}_i(p)}{2}\,a_i^{\alpha}(p)=0\end{gathered}$$

and add it to the original equation, taking into account that

$$\sum_k a_k^{\alpha}(p+1)b_k^{\nu}(p+1)=\sum_k a_k^{\alpha}(p)b_k^{\nu}(p).$$

This follows from the fact that $\sum_i a_i^\alpha b_i^\beta$ is an integral of motion, i.e., $\partial_{t_m}\bigl(\sum_i a_i^\alpha b_i^\beta\bigr) = 0$ for all m. Indeed, we have

$$\partial_{t_m}\left(\sum_i a_i^\alpha b_i^\beta\right) = \sum_i \left(b_i^\beta \frac{\partial \mathcal{H}_m}{\partial b_i^\alpha} - a_i^\alpha \frac{\partial \mathcal{H}_m}{\partial a_i^\beta}\right),$$

and this is zero because $\mathcal{H}_m$ is a linear combination of $H_k = \operatorname{tr} L^k$ and

$$\sum_i \left(b_i^\beta \operatorname{tr}\left(\frac{\partial L}{\partial b_i^\alpha} L^{m-1}\right) - a_i^\alpha \operatorname{tr}\left(\frac{\partial L}{\partial a_i^\beta} L^{m-1}\right)\right) = \sum_i \sum_{j,k}\left(b_i^\beta \frac{\partial L_{jk}}{\partial b_i^\alpha} L_{kj}^{m-1} - a_i^\alpha \frac{\partial L_{jk}}{\partial a_i^\beta} L_{kj}^{m-1}\right)$$
$$= 2\gamma \sum_i \sum_{j\neq k} (\delta_{ik} - \delta_{ij}) \frac{w_j^{1/2} w_k^{1/2} b_j^\beta a_k^\alpha}{w_j - w_k} L_{kj}^{m-1} = 0.$$

As a result, we obtain the equation

$$\gamma \sum_k \coth\bigl(\gamma(x_k(p+1) - x_i(p))\bigr) a_k^\alpha(p+1) b_k^\nu(p+1) a_i^\nu(p)$$
$$= \gamma \sum_{k\neq i} \coth\bigl(\gamma(x_k(p) - x_i(p))\bigr) a_k^\alpha(p) b_k^\nu(p) a_i^\nu(p) + \frac{\dot{x}_i(p)}{2} a_i^\alpha(p) + \mu a_i^\alpha(p). \quad (6.11)$$

A similar transformation of equation (6.9) leads to the equation

$$\gamma \sum_k \coth\bigl(\gamma(x_i(p) - x_k(p-1))\bigr) b_k^\alpha(p-1) b_i^\nu(p) a_k^\nu(p-1)$$
$$= \gamma \sum_{k\neq i} \coth\bigl(\gamma(x_i(p) - x_k(p))\bigr) b_k^\alpha(p) b_i^\nu(p) a_k^\nu(p) + \frac{\dot{x}_i(p)}{2} b_i^\alpha(p) + \mu b_i^\alpha(p). \quad (6.12)$$

We multiply the first equation by $b_i^\alpha(p)$ and sum over α; then we multiply the second equation by $a_i^\alpha(p)$, sum over α, and take into account the constraint $b_i^\nu a_i^\nu = 1$. Subtracting the resulting equations, we eliminate $\dot{x}_i(p)$ and obtain the equations of motion (1.4):

$$\sum_j \coth\bigl(\gamma(x_i(p) - x_j(p+1))\bigr) b_i^\nu(p) a_j^\nu(p+1) b_j^\beta(p+1) a_i^\beta(p)$$
$$+ \sum_j \coth\bigl(\gamma(x_i(p) - x_j(p-1))\bigr) b_i^\nu(p) a_j^\nu(p-1) b_j^\beta(p-1) a_i^\beta(p)$$
$$= 2 \sum_{j\neq i} \coth\bigl(\gamma(x_i(p) - x_j(p))\bigr) b_i^\nu(p) a_j^\nu(p) b_j^\beta(p) a_i^\beta(p). \quad (6.13)$$

These equations of motion generalize those for the rational Gibbons–Hermsen model obtained in [25]. They look like the Bethe ansatz equations for the quantum trigonometric Gaudin model "dressed" by the spin variables. In the continuum limit the equations of motion (3.14) are reproduced.

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REFERENCES

1. H. Airault, H. P. McKean, and J. Moser, "Rational and elliptic solutions of the Korteweg–de Vries equation and a related many-body problem," Commun. Pure Appl. Math. **30** (1), 95–148 (1977).
2. F. Calogero, "Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials," J. Math. Phys. **12** (3), 419–436 (1971).
3. F. Calogero, "Exactly solvable one-dimensional many-body problems," Lett. Nuovo Cimento **13** (11), 411–416 (1975).
4. D. V. Choodnovsky and G. V. Choodnovsky, "Pole expansions of nonlinear partial differential equations," Nuovo Cimento B **40** (2), 339–350 (1977).
5. E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, "Operator approach to the Kadomtsev–Petviashvili equation—Transformation groups for soliton equations. III," J. Phys. Soc. Japan **50** (11), 3806–3812 (1981).
6. E. Date, M. Jinbo, and T. Miwa, "Method for generating discrete soliton equations. I," J. Phys. Soc. Japan **51** (12), 4116–4124 (1982).
7. E. Date, M. Jinbo, and T. Miwa, "Method for generating discrete soliton equations. II," J. Phys. Soc. Japan **51** (12), 4125–4131 (1982).
8. J. Gibbons and T. Hermsen, "A generalisation of the Calogero–Moser system," Physica D **11** (3), 337–348 (1984).
9. L. Haine, "KP trigonometric solitons and an adelic flag manifold," SIGMA, Symmetry Integrability Geom. Methods Appl. **3**, 015 (2007).
10. V. Kac and J. van de Leur, "The n-component KP hierarchy and representation theory," in *Important Developments in Soliton Theory*, Ed. by A. S. Fokas and V. E. Zakharov (Springer, Berlin, 1993), pp. 302–343.
11. I. M. Krichever, "Rational solutions of the Kadomtsev–Petviashvili equation and integrable systems of N particles on a line," Funct. Anal. Appl. **12** (1), 59–61 (1978) [transl. from Funkts. Anal. Prilozh. **12** (1), 76–78 (1978)].
12. I. M. Krichever, "Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles," Funct. Anal. Appl. **14** (4), 282–290 (1980) [transl. from Funkts. Anal. Prilozh. **14** (4), 45–54 (1980)].
13. I. Krichever, O. Babelon, E. Billey, and M. Talon, "Spin generalization of the Calogero–Moser system and the matrix KP equation," in *Topics in Topology and Mathematical Physics* (Am. Math. Soc., Providence, RI, 1995), AMS Transl., Ser. 2, **170**, pp. 83–119.
14. I. Krichever, P. Wiegmann, and A. Zabrodin, "Elliptic solutions to difference non-linear equations and related many-body problems," Commun. Math. Phys. **193** (2), 373–396 (1998).
15. J. Moser, "Three integrable Hamiltonian systems connected with isospectral deformations," Adv. Math. **16**, 197–220 (1975).
16. F. W. Nijhoff and G.-D. Pang, "A time-discretized version of the Calogero–Moser model," Phys. Lett. A **191** (1–2), 101–107 (1994).
17. F. W. Nijhoff, O. Ragnisco, and V. B. Kuznetsov, "Integrable time-discretization of the Ruijsenaars–Schneider model," Commun. Math. Phys. **176** (3), 681–700 (1996).
18. M. A. Olshanetsky and A. M. Perelomov, "Classical integrable finite-dimensional systems related to Lie algebras," Phys. Rep. **71** (5), 313–400 (1981).
19. V. Pashkov and A. Zabrodin, "Spin generalization of the Calogero–Moser hierarchy and the matrix KP hierarchy," J. Phys. A: Math. Theor. **51** (21), 215201 (2018).
20. O. Ragnisco and Yu. B. Suris, "Integrable discretizations of the spin Ruijsenaars-Schneider models," J. Math. Phys. **38** (9), 4680–4691 (1997).
21. T. Shiota, "Calogero–Moser hierarchy and KP hierarchy," J. Math. Phys. **35** (11), 5844–5849 (1994).
22. Yu. B. Suris, *The Problem of Integrable Discretization: Hamiltonian Approach* (Birkhäuser, Basel, 2003), Prog. Math. **219**.
23. K. Takasaki and T. Takebe, "Universal Whitham hierarchy, dispersionless Hirota equations and multicomponent KP hierarchy," Physica D **235** (1–2), 109–125 (2007).
24. L.-P. Teo, "The multicomponent KP hierarchy: Differential Fay identities and Lax equations," J. Phys. A: Math. Theor. **44** (22), 225201 (2011).
25. A. Zabrodin, "Time discretization of the spin Calogero–Moser model and the semi-discrete matrix KP hierarchy," J. Math. Phys. **60** (3), 033502 (2019).
26. A. Zabrodin, "KP hierarchy and trigonometric Calogero–Moser hierarchy," J. Math. Phys. **61** (4), 043502 (2020); arXiv: 1906.09846 [math-ph].

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Appendix C

V.Prokofev, A Zabrodin "Elliptic solutions to the KP hierarchy and elliptic Calogero–Moser model"

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Contribution: I conducted all calculations independently. I suggested the main idea of introduction of parameters k, λ in Baker-Akhiezer functions, when they are written as a sum of Lame functions (30), (31) and connected them with z through system (33) by comparing Bloch multipliers. Another step was transition from second equation of system (33) to equation (38) which was based on the known fact, that parameters of spectral curve for Calogero-Moser system have zero Poisson brackets.

The main difficulty of elliptic case was that systems (44), (47) are homogeneous unlike their analogs in rational and trigonometric cases. Because of that it becomes impossible to express right hand side of (52) through known variables since it can be multiplied by any function. My idea was to use equation (38) which allows one to fix this degree of freedom. It made it possible to determine Hamiltionians for the higher times.

I also found rational and trigonometric limits and first five Hamiltonians.

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Elliptic solutions to the KP hierarchy and elliptic Calogero–Moser model

V Prokofev[1,2,*] and **A Zabrodin**[2,3,4]

[1] Moscow Institute of Physics and Technology, Dolgoprudny, Institutsky per., 9, Moscow region, 141700, Russia
[2] Skolkovo Institute of Science and Technology, 143026 Moscow, Russia
[3] National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia
[4] Institute of Biochemical Physics, Kosygina str. 4, 119334, Moscow, Russia, Russia

E-mail: vadim.prokofev@phystech.edu and zabrodin@itep.ru

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Abstract

We consider solutions of the Kadomtsev–Petviashvili hierarchy which are elliptic functions of $x = t_1$. It is known that their poles as functions of t_2 move as particles of the elliptic Calogero–Moser model. We extend this correspondence to the level of hierarchies and find the Hamiltonian H_k of the elliptic Calogero–Moser model which governs the dynamics of poles with respect to the kth hierarchical time. The Hamiltonians H_k are obtained as coefficients of the expansion of the spectral curve near the marked point in which the Baker–Akhiezer function has essential singularity.

Keywords: integrable systems, elliptic Calogero–Moser model, Kadomtsev–Petviashvili hierarchy

1. Introduction

The investigation of dynamics of poles of singular solutions to nonlinear integrable equations was initiated in the seminal paper [1], where it was shown that poles of elliptic and rational solutions to the Korteweg–de Vries and Boussinesq equations move as particles of the integrable many-body Calogero–Moser system [2–5] with some restrictions in the phase space. As it was proved in [6, 7], this connection becomes most natural for the more general Kadomtsev–Petviashvili (KP) equation, in which case there are no restrictions in the phase space for the Calogero–Moser dynamics of poles.

The KP equation is the first member of an infinite hierarchy of consistent integrable equations with infinitely many independent variables (times) $\mathbf{t} = \{t_1, t_2, t_3, \ldots\}$ (the KP hierarchy). In [8], Shiota has shown that the correspondence between rational solutions to the

[*] Author to whom correspondence should be addressed.

KP equation and the Calogero–Moser system with rational potential can be extended to the level of hierarchies: the evolution of poles with respect to the higher times t_k of the KP hierarchy was shown to be governed by the higher Hamiltonians $H_k = \mathrm{Tr}\, L^k$ of the integrable Calogero–Moser system, where L is the Lax matrix. Later this correspondence was generalized to trigonometric solutions of the KP hierarchy (see [9, 10]).

A natural generalization of rational and trigonometric solutions are elliptic (double periodic in the complex plane) solutions. Elliptic solutions to the KP equation

$$3u_{t_2t_2} = \left(4u_{t_3} - 12uu_x - u_{xxx}\right)_x \tag{1}$$

(where $x = t_1$) were studied by Krichever in [11], where it was shown that poles x_i of the elliptic solutions

$$u = -\sum_{i=1}^{N} \wp(x - x_i) + 2c \tag{2}$$

as functions of t_2 move according to the equations of motion

$$\ddot{x}_i = 4\sum_{k\neq i} \wp'(x_i - x_k) \tag{3}$$

of the Calogero–Moser system of particles with the elliptic interaction potential $\wp(x_i - x_j)$ ($\wp$ is the Weierstrass $\wp$-function). Here dot means derivative with respect to the time t_2. See also the review [12]. The Calogero–Moser system is Hamiltonian with the Hamiltonian

$$H = \sum_i p_i^2 - 2\sum_{i<j} \wp(x_i - x_j) \tag{4}$$

and the Poisson brackets $\{x_i, p_k\} = \delta_{ik}$. Note that $\dot{x}_i = \partial H/\partial p_i = 2p_i$. It is known [13] that the elliptic Calogero–Moser system is integrable, i.e., there are N independent integrals of motion H_k in involution.

The aim of this paper is to establish the precise correspondence between the flows of the KP hierarchy parametrized by the times t_m and the Hamiltonian flows of the hierarchy of the elliptic Calogero–Moser systems. In short, the result is as follows. Let the function $\lambda(z)$ be determined from the equation of the Calogero–Moser spectral curve in the form

$$\det_{N\times N} ((z + \zeta(\lambda))I - L(\lambda)) = 0, \tag{5}$$

where I is the unity matrix, $L(\lambda)$ is the Lax matrix of the Calogero–Moser system depending on the spectral parameter λ and $\zeta(\lambda)$ is the Weierstrass ζ-function. We show that the function $\lambda(z)$ expanded as $z \to \infty$ as

$$\lambda(z) = -Nz^{-1} + \sum_{m\geqslant 1} H_m z^{-m-1} \tag{6}$$

is the generating function for the Calogero–Moser Hamiltonians H_m corresponding to the flows t_m of the KP hierarchy. We find first few Hamiltonians explicitly. In the rational and trigonometric limit it is possible to find them for any m in terms of traces of the Lax matrix and the result coincides with what was previously known (see [8, 10]).

2. The KP hierarchy

We begin with a short review of the KP hierarchy (see [14] for more details). The KP hierarchy is an infinite set of evolution equations in the times $\mathbf{t} = \{t_1, t_2, t_3, \ldots\}$ for functions of a variable x. In the Lax formulation of the hierarchy, the main object is the pseudo-differential operator

$$\mathcal{L} = \partial_x + \sum_{k\geqslant 1} u_k \partial_x^{-k}, \tag{7}$$

where the coefficient functions u_k are functions of x and $\mathbf{t}$. The equations of the KP hierarchy are encoded in the Lax equations

$$\partial_{t_m}\mathcal{L} = [\mathcal{A}_m, \mathcal{L}], \qquad \mathcal{A}_m = (\mathcal{L}^m)_+, \tag{8}$$

where $(\ldots)_+$ means taking the purely differential part of a pseudo-differential operator. In particular, we have $\partial_{t_1}\mathcal{L} = \partial_x\mathcal{L}$, i.e., $\partial_{t_1}u_k = \partial_x u_k$ for all $k \geqslant 1$. This means that the evolution in t_1 is simply a shift of x: $u_k(x, \mathbf{t}) = u_k(x + t_1, t_2, t_3, \ldots)$.

An equivalent formulation of the KP hierarchy is through the zero curvature (Zakharov–Shabat) equations

$$\partial_{t_n}\mathcal{A}_m - \partial_{t_m}\mathcal{A}_n + [\mathcal{A}_m, \mathcal{A}_n] = 0. \tag{9}$$

The simplest nontrivial equation, i.e., (1) is obtained for $u = u_1$ with $m = 2, n = 3$.

A common solution to the KP hierarchy is provided by the tau-function $\tau = \tau(x, \mathbf{t})$. The coefficient functions u_k of the Lax operator can be expressed through the tau-function. For example,

$$u_1(x, \mathbf{t}) = u(x, \mathbf{t}) = \partial_x^2 \log \tau(x, \mathbf{t}). \tag{10}$$

The whole hierarchy is encoded in the bilinear relation [14, 15]

$$\oint_\infty e^{(x-x')z+\xi(\mathbf{t},z)-\xi(\mathbf{t}',z)}(e^{-D(z)}\tau(x, \mathbf{t}))(e^{D(z)}\tau(x', \mathbf{t}'))dz = 0 \tag{11}$$

valid for all $x, x', \mathbf{t}, \mathbf{t}'$, where

$$\xi(\mathbf{t}, z) = \sum_{k\geqslant 1} t_k z^k$$

and $D(z)$ is the differential operator

$$D(z) = \sum_{k\geqslant 1} \frac{z^{-k}}{k}\partial_{t_k}. \tag{12}$$

The integration contour is a big circle around infinity separating the singularities coming from the exponential factor from those coming from the tau-functions.

Let us point out an important corollary of the bilinear relation. Applying the operator $D'(\mu) = -\sum_{k\geqslant 1}\mu^{-k-1}\partial_{t_k}$ to (11) and putting $x = x'$, $\mathbf{t} = \mathbf{t}'$ after that, we obtain

$$-\sum_{k\geqslant 1}\oint_\infty \mu^{-k-1}z^k(e^{-D(z)}\tau)(e^{D(z)}\tau)dz + \oint_\infty D'(\mu)(e^{-D(z)}\tau)(e^{D(z)}\tau)dz = 0$$

or

$$\frac{1}{2\pi i}\oint_{\infty}\frac{z}{\mu(z-\mu)}(\mathrm{e}^{-D(z)}\tau)(\mathrm{e}^{D(z)}\tau)\mathrm{d}z = D'(\mu)\partial_x\tau\,\tau - D'(\mu)\tau\partial_x\tau.$$

Taking the residues in the left-hand side, we get the equation

$$\frac{(\mathrm{e}^{D(\mu)}\tau)(\mathrm{e}^{-D(\mu)}\tau)}{\tau^2} = 1 - D'(\mu)\partial_x\,\log\,\tau. \tag{13}$$

The zero curvature equation (9) are compatibility conditions of the auxiliary linear problems

$$\partial_{t_k}\psi = \mathcal{A}_k\psi \tag{14}$$

for the wave function $\psi = \psi(x, \mathbf{t}, z)$ depending on the spectral parameter z. In particular, at $k = 2$ we have the equation

$$\partial_{t_2}\psi = \partial_x^2\psi + 2u\psi. \tag{15}$$

One can also introduce the adjoint wave function ψ^* satisfying the adjoint equation (14):

$$-\partial_{t_k}\psi^* = \mathcal{A}_k^{\dagger}\psi^*, \tag{16}$$

where the †-operation is defined as $(f(x)\circ\partial_x^n)^{\dagger} = (-\partial_x)^n\circ f(x)$. In [14, 15] it is shown that the wave functions can be expressed through the tau-function in the following way:

$$\psi(x, \mathbf{t}, z) = \mathrm{e}^{xz+\xi(\mathbf{t},z)}\frac{\mathrm{e}^{-D(z)}\tau(x, \mathbf{t})}{\tau(x, \mathbf{t})}, \tag{17}$$

$$\psi^*(x, \mathbf{t}, z) = \mathrm{e}^{-xz-\xi(\mathbf{t},z)}\frac{\mathrm{e}^{D(z)}\tau(x, \mathbf{t})}{\tau(x, \mathbf{t})}. \tag{18}$$

Note that in terms of the wave functions the equation (13) can be written in the form

$$\partial_{t_m}\partial_{t_1}\,\log\,\tau(x, \mathbf{t}) = \operatorname*{res}_{\infty}\left(z^m\psi(x, \mathbf{t}, z)\psi^*(x, \mathbf{t}, z)\right), \tag{19}$$

where $\operatorname*{res}_{\infty}$ is defined as $\operatorname*{res}_{\infty}(z^{-n}) = \delta_{n1}$.

3. Elliptic solutions

The ansatz for the tau-function of elliptic (double-periodic in the complex plane) solutions to the KP hierarchy is

$$\tau = \mathrm{e}^{Q(x,\mathbf{t})}\prod_{i=1}^{N}\sigma(x - x_i(\mathbf{t})), \tag{20}$$

where

$$Q(x, \mathbf{t}) = c(x + t_1)^2 + (x + t_1)A(t_2, t_3, \ldots) + B(t_2, t_3, \ldots)$$

with a constant c, a linear function

$$A(t_2, t_3, \ldots) = A_0 + \sum_{j\geqslant 2}a_j t_j \tag{21}$$

and some function $B(t_2, t_3, \ldots)$. The coefficients c, a_j here are not arbitrary but are determined by algebro-geometric data of the spectral curve, and, in particular, by the choice of the local parameter around the marked point of the curve. The function σ in equation (20) is the Weierstrass σ-function

$$\sigma(x) = \sigma(x|\omega, \omega') = x\prod_{s\neq 0}\left(1 - \frac{x}{s}\right)\mathrm{e}^{\frac{x}{s} + \frac{x^2}{2s^2}},$$

$$s = 2\omega m + 2\omega' m' \quad \text{with integer } m, m'$$

with quasi-periods 2ω, $2\omega'$ such that $\mathrm{Im}(\omega'/\omega) > 0$. It is connected with the Weierstrass ζ- and $\wp$-functions by the formulas $\zeta(x) = \sigma'(x)/\sigma(x)$, $\wp(x) = -\zeta'(x) = -\partial_x^2 \log \sigma(x)$. The monodromy properties of the function $\sigma(x)$ are

$$\sigma(x + 2\omega) = -\mathrm{e}^{2\eta(x+\omega)}\sigma(x), \qquad \sigma(x + 2\omega') = -\mathrm{e}^{2\eta'(x+\omega')}\sigma(x), \tag{22}$$

where the constants $\eta = \zeta(\omega)$, $\eta' = \zeta(\omega')$ are related by $\eta\omega' - \eta'\omega = \pi i/2$. The roots x_i are assumed to be all distinct. Correspondingly, the function $u = \partial_x^2 \log \tau$ is an elliptic function with double poles at the points x_i:

$$u = -\sum_{i=1}^{N} \wp(x - x_i) + 2c. \tag{23}$$

The poles depend on the times $t_1, t_2, t_3, \ldots$. The dependence on t_1 is especially simple: since the solution must depend on $x + t_1$, we have $\partial_{t_1} x_i = -1$.

Let $\Delta(\mu)$ be the difference operator

$$\Delta(\mu) = \mathrm{e}^{D(\mu)} + \mathrm{e}^{-D(\mu)} - 2. \tag{24}$$

Substituting the ansatz (20) into equation (13), we get:

$$\mathrm{e}^{G+(x+t_1)\Delta(\mu)A}\prod_i \frac{\sigma(x - \mathrm{e}^{D(\mu)}x_i)\sigma(x - \mathrm{e}^{-D(\mu)}x_i)}{\sigma^2(x - x_i)}$$

$$= 1 + 2c\mu^{-2} - D'(\mu)A - \sum_k D'(\mu)x_k\wp(x - x_k),$$

where

$$G = 2c\mu^{-2} + \mu^{-1}(\mathrm{e}^{D(\mu)} - \mathrm{e}^{-D(\mu)})A + \Delta(\mu)B.$$

The right-hand side is an elliptic function of x with periods 2ω, $2\omega'$. Therefore, for the left-hand side be also an elliptic function of x with the same periods the following relations have to be satisfied:

$$\begin{cases} \exp\left(-2\eta\Delta(\mu)\sum_k x_k + 2\omega\Delta(\mu)A\right) = 1 \\ \exp\left(-2\eta'\Delta(\mu)\sum_k x_k + 2\omega'\Delta(\mu)A\right) = 1 \end{cases}$$

from which it follows that

$$\Delta(\mu)A = 2n'\eta - 2n\eta', \qquad \Delta(\mu)\sum_k x_k = 2n'\omega - 2n\omega' \quad \text{with integer } n, n'.$$

The right hand sides do not depend on μ. Expanding the equalities in powers of μ, one sees that the left hand sides are $O(\mu^{-2})$ as $\mu \to \infty$, therefore, $n = n' = 0$ and we have

$$\Delta(\mu)A = 0, \qquad \Delta(\mu)\sum_k x_k = 0. \tag{25}$$

The first equation is satisfied if A is a linear function of times as in (21). The second equation means that

$$(1 - \mathrm{e}^{-D(\mu)})\sum_i x_i = -(1 - \mathrm{e}^{D(\mu)})\sum_i x_i. \tag{26}$$

Note that the functions (17) and (18) with τ as in (20) are *double-Bloch functions*, i.e., they satisfy the monodromy properties $\psi(x + 2\omega) = B\psi(x)$, $\psi(x + 2\omega') = B'\psi(x)$ with some Bloch multipliers B, B'. Any non-trivial double-Bloch function (i.e. not an exponential function) must have poles in x in the fundamental domain. The Bloch multipliers of the function (17) are

$$\begin{aligned} B &= \mathrm{e}^{2\omega(z-\alpha(z))-2\zeta(\omega)(\mathrm{e}^{-D(z)}-1)\sum_i x_i}, \\ B' &= \mathrm{e}^{2\omega'(z-\alpha(z))-2\zeta(\omega')(\mathrm{e}^{-D(z)}-1)\sum_i x_i}, \end{aligned} \tag{27}$$

where

$$\alpha(z) = 2cz^{-1} + \sum_{j\geqslant 2} \frac{a_j}{j} z^{-j} \tag{28}$$

and the coefficients a_j are those entering (21). Equation (26) means that the Bloch multipliers of the adjoint wave function ψ^* are B^{-1} and B'^{-1}.

Let us introduce the elementary double-Bloch function $\Phi(x, \lambda)$ defined as

$$\Phi(x, \lambda) = \frac{\sigma(x + \lambda)}{\sigma(\lambda)\sigma(x)}\, \mathrm{e}^{-\zeta(\lambda)x} \tag{29}$$

($\zeta(\lambda)$ is the Weierstrass ζ-function). The monodromy properties of the function Φ are

$$\Phi(x + 2\omega, \lambda) = \mathrm{e}^{2(\zeta(\omega)\lambda-\zeta(\lambda)\omega)}\Phi(x, \lambda),$$

$$\Phi(x + 2\omega', \lambda) = \mathrm{e}^{2(\zeta(\omega')\lambda-\zeta(\lambda)\omega')}\Phi(x, \lambda),$$

so it is indeed a double-Bloch function. The function Φ has a simple pole at $x = 0$ with residue 1:

$$\Phi(x, \lambda) = \frac{1}{x} + \alpha_1 x + \alpha_2 x^2 + \cdots, \quad x \to 0,$$

where $\alpha_1 = -\frac{1}{2}\wp(\lambda)$, $\alpha_2 = -\frac{1}{6}\wp'(\lambda)$. We will often suppress the second argument of Φ writing simply $\Phi(x) = \Phi(x, \lambda)$. We will also need the x-derivative $\Phi'(x, \lambda) = \partial_x\Phi(x, \lambda)$.

Equations (17), (18) and (20) imply that the wave functions ψ, ψ^* have simple poles at the points x_i. One can expand the wave functions using the elementary double-Bloch functions as follows:

$$\psi = \mathrm{e}^{xk+t_1(k-z)+\xi(\mathbf{t},z)}\sum_i c_i\Phi(x-x_i,\lambda) \tag{30}$$

$$\psi^* = \mathrm{e}^{-xk-t_1(k-z)-\xi(\mathbf{t},z)}\sum_i c_i^*\Phi(x-x_i,-\lambda) \tag{31}$$

(this is similar to expansion of a rational function in a linear combination of simple fractions). Here c_i, c_i^* are expansion coefficients which do not depend on x and k is an additional spectral parameter. Note that the normalization of the functions (17) and (18) implies that c_i and c_i^* are $O(\lambda)$ as $\lambda\to 0$. One can see that (30) is a double-Bloch function with Bloch multipliers

$$B = \mathrm{e}^{2\omega(k-\zeta(\lambda))+2\zeta(\omega)\lambda}, \qquad B' = \mathrm{e}^{2\omega'(k-\zeta(\lambda))+2\zeta(\omega')\lambda} \tag{32}$$

and (31) has Bloch multipliers B^{-1} and B'^{-1}. These Bloch multipliers should coincide with (27).

Therefore, comparing (27) with (32), we get

$$2\omega(k-\zeta(\lambda)-z+\alpha(z))+2\zeta(\omega)\left(\lambda+(\mathrm{e}^{-D(z)}-1)\sum_i x_i\right)=2\pi i n,$$

$$2\omega'(k-\zeta(\lambda)-z+\alpha(z))+2\zeta(\omega')\left(\lambda+(\mathrm{e}^{-D(z)}-1)\sum_i x_i\right)=2\pi i n'$$

with some integer n, n'. Regarding these equations as a linear system, we obtain the solution

$$k-z+\alpha(z)-\zeta(\lambda)=2n'\zeta(\omega)-2n\zeta(\omega'),$$

$$\lambda+(\mathrm{e}^{-D(z)}-1)\sum_i x_i = 2n\omega'-2n'\omega.$$

Shifting λ by a suitable vector of the lattice spanned by 2ω, $2\omega'$, one gets zeros in the right hand sides of these equalities, so we can write

$$\begin{aligned} k &= z-\alpha(z)+\zeta(\lambda),\\ \lambda &= (1-\mathrm{e}^{-D(z)})\sum_i x_i. \end{aligned} \tag{33}$$

These two equations for three variables k, z, λ determine the spectral curve. Below we will obtain another description of the spectral curve as the spectral curve of the Calogero–Moser system (given by the characteristic polynomial of the Lax matrix $L(\lambda)$ for the Calogero–Moser system). It appears in the form $R(k,\lambda)=0$, where $R(k,\lambda)$ is a polynomial in k whose coefficients are elliptic functions of λ (see below in section 5). These coefficients are integrals of motion in involution. The spectral curve in the form $R(k,\lambda)=0$ appears if one excludes z from the equation (33). Equivalently, one can represent the spectral curve as a relation connecting two variables z and λ:

$$R(z-\alpha(z)+\zeta(\lambda),\lambda)=0. \tag{34}$$

Let us write the second equation in (33) as the expansion in powers of z:

$$\lambda = -\sum_{m\geqslant 1} z^{-m}\hat{h}_m X, \qquad X := \sum_i x_i, \tag{35}$$

where $\hat{h}_k$ are differential operators of the form

$$\hat{h}_m = -\frac{1}{m}\partial_{t_m} + \text{ higher order operators in } \partial_{t_1}, \partial_{t_2}, \ldots, \partial_{t_{m-1}}. \tag{36}$$

For example, the first few are

$$\hat{h}_1 = -\partial_{t_1}, \qquad \hat{h}_2 = \frac{1}{2}(\partial_{t_1}^2 - \partial_{t_2}), \qquad \hat{h}_3 = \frac{1}{6}(-\partial_{t_1}^3 + 3\partial_{t_1}\partial_{t_2} - 2\partial_{t_3}).$$

As is explained above, the coefficients in the expansion (35) are integrals of motion, i.e., $\partial_{t_j}\hat{h}_m X = 0$ for all j, m. It then follows from the equation $\partial_{t_1} x_i = -1$ and from the explicit form of the operators $\hat{h}_m$ that $\partial_{t_j}\partial_{t_2} X = 0$ and $\partial_{t_j}\partial_{t_3} X = 0$. A simple inductive argument then shows that $\partial_{t_j}\partial_{t_m} X = 0$ for all j, m. This means that $-\hat{h}_m X = \frac{1}{m}\partial_{t_m} X$ and X is a linear function of the times:

$$X = \sum_i x_i = X_0 - Nt_1 + \sum_{m\geqslant 2} V_m t_m \tag{37}$$

with some constants V_m (velocities of the 'center of masses' of the points x_i multiplied by N). Therefore, the second equation in (33) can be written as

$$\lambda = D(z)\sum_i x_i = -Nz^{-1} + \sum_{j\geqslant 2} \frac{z^{-j}}{j} V_j. \tag{38}$$

In what follows we will show that $H_m = -\frac{1}{m+1}V_{m+1}$ are Hamiltonians for the dynamics of the poles in t_m, with H_2 being the standard Calogero–Moser Hamiltonian.

4. Dynamics of poles with respect to t_2

The coefficient u in the linear problem (15)

$$\partial_{t_2}\psi - \partial_x^2\psi - 2u\psi = 0 \tag{39}$$

is an elliptic function of x of the form (23). Therefore, one can find solutions which are double-Bloch functions of the form (30).

The next procedure is standard after the work [11]. We substitute u in the form (23) and ψ in the form (30) into the left-hand side of (39) and cancel the poles at the points $x = x_i$. The highest poles are of third order but it is easy to see that they cancel identically. It is a matter of direct calculation to see that the conditions of cancellation of second and first order poles have the form

$$c_i\dot{x}_i = -2kc_i - 2\sum_{j\neq i} c_j\Phi(x_i - x_j), \tag{40}$$

$$\dot{c}_i = (k^2 - z^2 + 4c - 2\alpha_1)c_i - 2\sum_{j\neq i} c_j\Phi'(x_i - x_j) - 2c_i\sum_{j\neq i}\wp(x_i - x_j), \tag{41}$$

where dot means the t_2-derivative. Introducing $N \times N$ matrices

$$L_{ij} = -\frac{1}{2}\delta_{ij}\dot{x}_i - (1-\delta_{ij})\Phi(x_i - x_j), \tag{42}$$

$$M_{ij} = \delta_{ij}(k^2 - z^2 + \wp(\lambda) + 4c) - 2\delta_{ij}\sum_{k\neq i}\wp(x_i - x_k) - 2(1-\delta_{ij})\Phi'(x_i - x_j), \tag{43}$$

we can write the above conditions as a system of linear equations for the vector $\mathbf{c} = (c_1, \ldots, c_N)^{\mathrm{T}}$:

$$\begin{cases} L(\lambda)\mathbf{c} = k\mathbf{c} \\ \dot{\mathbf{c}} = M(\lambda)\mathbf{c}. \end{cases} \tag{44}$$

Differentiating the first equation in (44) with respect to t_2, we arrive at the compatibility condition of the linear problems (44):

$$\left(\dot{L} + [L, M]\right)\mathbf{c} = 0. \tag{45}$$

The Lax equation $\dot{L} + [L, M] = 0$ is equivalent to the equations of motion of the elliptic Calogero–Moser system (see [12] for the detailed calculation). Our matrix M differs from the standard one by the term $\delta_{ij}(k^2 - z^2)$ but it does not affect the compatibility condition. It follows from the Lax representation that the time evolution is an isospectral transformation of the Lax matrix L, so all traces $\mathrm{Tr}\, L^m$ and the characteristic polynomial $\det(L - kI)$, where I is the unity matrix, are integrals of motion. Note that the Lax matrix is written in terms of the momenta p_i as follows:

$$L_{ij} = -\delta_{ij}p_i - (1-\delta_{ij})\Phi(x_i - x_j). \tag{46}$$

A similar calculation shows that the adjoint linear problem for the function (31) leads to the equations

$$\begin{cases} \mathbf{c}^{*\mathrm{T}}L(\lambda) = k\mathbf{c}^{*\mathrm{T}} \\ \dot{\mathbf{c}}^{*\mathrm{T}} = -\mathbf{c}^{*\mathrm{T}}M(\lambda) \end{cases} \tag{47}$$

with the compatibility condition $\mathbf{c}^{*\mathrm{T}}\left(\dot{L} + [L, M]\right) = 0$.

5. The spectral curve

The first of the equation (44) determines a connection between the spectral parameters k, λ which is the equation of the spectral curve:

$$R(k, \lambda) = \det(kI - L(\lambda)) = 0. \tag{48}$$

As it was already mentioned, the spectral curve is an integral of motion. The matrix $L = L(\lambda)$, which has an essential singularity at $\lambda = 0$, can be represented in the form $L = V\tilde{L}V^{-1}$, where matrix elements of $\tilde{L}$ do not have essential singularities and V is the diagonal matrix $V_{ij} = \delta_{ij}\,\mathrm{e}^{-\zeta(\lambda)x_i}$. Therefore,

$$R(k, \lambda) = \sum_{m=0}^{N} R_m(\lambda)k^m,$$

where the coefficients $R_m(\lambda)$ are elliptic functions of λ with poles at $\lambda = 0$. The functions $R_m(\lambda)$ can be represented as linear combinations of the $\wp$-function and its derivatives. Coefficients of this expansion are integrals of motion. Fixing values of these integrals, we obtain via the equation $R(k, \lambda) = 0$ an algebraic curve Γ which is an N-sheet covering of the initial elliptic curve $\mathcal{E}$ realized as a factor of the complex plane with respect to the lattice generated by $2\omega, 2\omega'$.

Example ($N = 2$):

$$\det_{2\times 2}(kI - L(\lambda)) = k^2 + k(p_1 + p_2) + p_1 p_2 + \wp(x_1 - x_2) - \wp(\lambda).$$

Example ($N = 3$):

$$\begin{aligned}\det_{3\times 3}(kI - L(\lambda)) &= k^3 + k^2(p_1 + p_2 + p_3)\\ &\quad + k\left(p_1p_2 + p_1p_3 + p_2p_3 + \wp(x_{12}) + \wp(x_{13}) + \wp(x_{23}) - 3\wp(\lambda)\right)\\ &\quad + p_1p_2p_3 + p_1\wp(x_{23}) + p_2\wp(x_{13}) + p_3\wp(x_{12}) - \wp(\lambda)(p_1 + p_2 + p_3) - \wp'(\lambda),\end{aligned}$$

where $x_{ik} = x_i - x_k$.

In a neighborhood of $\lambda = 0$ the matrix $\tilde{L}$ can be written as

$$\tilde{L} = -\lambda^{-1}(E - I) + O(1),$$

where E is the rank 1 matrix with matrix elements $E_{ij} = 1$ for all $i, j = 1, \dots, N$. The matrix E has eigenvalue 0 with multiplicity $N - 1$ and another eigenvalue equal to N. Therefore, we can write $R(k, \lambda)$ in the form

$$\begin{aligned}R(k,\lambda) &= \det\left(kI + \lambda^{-1}(E - I) + O(1)\right)\\ &= \left(k + (N-1)\lambda^{-1} - f_N(\lambda)\right)\prod_{i=1}^{N-1}(k - \lambda^{-1} - f_i(\lambda)),\end{aligned} \tag{49}$$

where f_i are regular functions of λ at $\lambda = 0$: $f_i(\lambda) = O(1)$ as $\lambda \to 0$. This means that the function k has simple poles on all sheets at the points P_j ($j = 1, \dots, N$) of the curve Γ located above $\lambda = 0$. Its expansion in the local parameter λ on the sheets near these points is given by the multipliers in the right-hand side of (49):

$$\begin{aligned}&k = \lambda^{-1} + f_j(\lambda) \quad \text{near } P_j, \quad j = 1, \dots, N-1,\\ &k = -(N-1)\lambda^{-1} + f_N(\lambda) \quad \text{near } P_N.\end{aligned} \tag{50}$$

The Nth sheet is distinguished, as it can be seen from (50). As in [11], we call it the upper sheet. Note that equations (33) and (38) imply

$$k(\lambda) = -\frac{N-1}{\lambda} + O(1) \quad \text{as } \lambda \to 0,$$

so the expansion (38) is the expansion of $\lambda(z)$ on the upper sheet of the spectral curve in a neighborhood of the point P_N.

6. Dynamics in higher times

Our basic tool is equation (19). Substituting $\tau(x, \mathbf{t})$ in the form (20) and ψ, ψ^* in the form (30) and (31) in it, we have:

$$\sum_i \partial_{t_m} x_i \wp(x - x_i) + C(t_2, t_3, \ldots) = \operatorname*{res}_{\infty} \left(z^m \sum_{i,j} c_i c_j^* \Phi(x - x_i, \lambda)\Phi(x - x_j, -\lambda) \right). \qquad (51)$$

Equating the coefficients in front of the second order poles at $x = x_i$, we obtain

$$\partial_{t_m} x_i = \operatorname*{res}_{\infty} \left(z^m c_i^* c_i \right) = \operatorname*{res}_{\infty} \left(z^m \mathbf{c}^{*\mathrm{T}} E_i \mathbf{c} \right), \qquad (52)$$

where E_i is the diagonal matrix with 1 at the place ii and zeros otherwise. At $m = 1$ this reads $\operatorname*{res}_{\infty} \left(z\mathbf{c}^{*\mathrm{T}} E_i \mathbf{c} \right) = -1$ or

$$\operatorname*{res}_{\infty} \left(z\mathbf{c}^{*\mathrm{T}} \mathbf{c} \right) = -N.$$

Summing the equation (52) over i, we get

$$\operatorname*{res}_{\infty} \left(z^m \mathbf{c}^{*\mathrm{T}} \mathbf{c} \right) = \partial_{t_m} \sum_i x_i.$$

It then follows from these equations that

$$(\mathbf{c}^{*\mathrm{T}} \mathbf{c}) = -N/z^2 + \sum_m z^{-m-1} \partial_{t_m} \sum_i x_i = -\lambda'(z). \qquad (53)$$

The absence of terms with non-negative powers of z in the right-hand side (which would not change the residue) follows from the above mentioned fact that $\mathbf{c}$ and $\mathbf{c}^*$ are $O(\lambda) = O(z^{-1})$ as $z \to \infty$. The last equality in (53) follows from (38). Equation (53) is an important non-trivial relation which will allow us to identify the Hamiltonians for the higher flows t_m.

Now let us note that according to (46) $E_i = -\partial_{p_i} L$. Therefore, we can continue the chain of equalities (52) as follows:

$$\begin{aligned}
\partial_{t_m} x_i &= \operatorname*{res}_{\infty} \left(z^m \mathbf{c}^{*\mathrm{T}} E_i \mathbf{c} \right) = -\operatorname*{res}_{\infty} \left(z^m \mathbf{c}^{*\mathrm{T}} \partial_{p_i} L \mathbf{c} \right) \\
&= -\partial_{p_i} \operatorname*{res}_{\infty} \left(z^m \mathbf{c}^{*\mathrm{T}} L \mathbf{c} \right) + \operatorname*{res}_{\infty} \left(z^m \partial_{p_i} \mathbf{c}^{*\mathrm{T}} L \mathbf{c} \right) + \operatorname*{res}_{\infty} \left(z^m \mathbf{c}^{*\mathrm{T}} L \partial_{p_i} \mathbf{c} \right) \\
&= -\partial_{p_i} \operatorname*{res}_{\infty} \left(z^m \mathbf{c}^{*\mathrm{T}} L \mathbf{c} \right) + \operatorname*{res}_{\infty} \left(z^m \partial_{p_i} \mathbf{c}^{*\mathrm{T}} k \mathbf{c} \right) + \operatorname*{res}_{\infty} \left(z^m \mathbf{c}^{*\mathrm{T}} k \partial_{p_i} \mathbf{c} \right) \\
&= -\partial_{p_i} \operatorname*{res}_{\infty} \left(z^m \mathbf{c}^{*\mathrm{T}} L \mathbf{c} \right) + \partial_{p_i} \operatorname*{res}_{\infty} \left(z^m \mathbf{c}^{*\mathrm{T}} k \mathbf{c} \right) - \operatorname*{res}_{\infty} \left(z^m \partial_{p_i} k \mathbf{c}^{*\mathrm{T}} \mathbf{c} \right) \\
&= -\partial_{p_i} \operatorname*{res}_{\infty} \left(z^m \mathbf{c}^{*\mathrm{T}} (L - kI) \mathbf{c} \right) - \operatorname*{res}_{\infty} \left(z^m \partial_{p_i} k \mathbf{c}^{*\mathrm{T}} \mathbf{c} \right) \\
&= \operatorname*{res}_{\infty} \left(z^m \lambda'(z) \partial_{p_i} k \right)
\end{aligned}$$

(the last equality follows from (44) and (53)). Here $\partial_{p_i} k = \partial_{p_i} k(\lambda, \mathbf{I})|_{\lambda=\text{const.}}$, where $\mathbf{I}$ is the full set of integrals of motion. From (33) we see that

$$\partial_{p_i} k = (1 - \alpha'(z))\partial_{p_i} z(\lambda, \mathbf{I})|_{\lambda=\text{const.}}. \tag{54}$$

We consider z as an independent variable, so we can write

$$0 = \frac{\mathrm{d}z}{\mathrm{d}p_i} = \partial_{p_i} z\Big|_{\lambda=\text{const}} + \partial_\lambda z\Big|_{\mathbf{I}=\text{const.}} \partial_{p_i}\lambda$$

or

$$\partial_{p_i} z = -\frac{\partial_{p_i}\lambda}{\lambda'(z)}.$$

Therefore, we have the first set of the Hamiltonian equations

$$\partial_{t_m} x_i = -\underset{\infty}{\text{res}}\left(z^m(1 - \alpha'(z))\partial_{p_i}\lambda\right) = \partial_{p_i}\mathcal{H}_m, \tag{55}$$

where the Hamiltonian $\mathcal{H}_m$ is

$$\mathcal{H}_m = H_m^{(\alpha)} + 2cH_{m-2}^{(\alpha)} + \sum_{j=2}^{m-1} a_j H_{m-j-1}^{(\alpha)} \tag{56}$$

(see (28)). It is the linear combination of the Hamiltonians

$$H_m^{(\alpha)} = -\underset{\infty}{\text{res}}\,(z^m\lambda(z)) \tag{57}$$

with constant coefficients. The latter implicitly depend on $\alpha(z)$ through the parametrization of the spectral curve (34).

In their turn, the Hamiltonians $H_m^{(\alpha)}$ are linear combinations of the basic Hamiltonians H_m defined at $\alpha(z) = 0$ by

$$H_m = -\underset{\infty}{\text{res}}\,(z^m\lambda_0(z)), \tag{58}$$

where $\lambda_0(z)$ is defined through the equation of the spectral curve

$$R(z + \zeta(\lambda_0), \lambda_0) = \det\left((z + \zeta(\lambda_0))I - L(\lambda_0)\right) = 0. \tag{59}$$

Then

$$\lambda(z) = \lambda_0(z - \alpha(z)) = \lambda_0(z) - \alpha(z)\lambda_0'(z) + \frac{1}{2}\alpha^2(z)\lambda_0''(z) + \cdots$$

and so we see that the Hamiltonians (57) are indeed linear combinations of the H_m's with constant coefficients.

The remaining set of Hamiltonian equations can be obtained by differentiating (52) with respect to t_2 and using (44) and (47):

$$\begin{aligned} 2\partial_{t_m} p_i = \partial_{t_m}\dot{x}_i &= \underset{\infty}{\text{res}}\left(z^m\dot{\mathbf{c}}^{*\mathrm{T}}E_i\mathbf{c}\right) + \underset{\infty}{\text{res}}\left(z^m\mathbf{c}^{*\mathrm{T}}E_i\dot{\mathbf{c}}\right) \\ &= \underset{\infty}{\text{res}}\left(z^m\mathbf{c}^{*\mathrm{T}}[E_i, M]\mathbf{c}\right) \end{aligned}$$

.Now, it is a matter of direct verification to see that

$$[E_i, M] = 2\partial_{x_i} L. \tag{60}$$

Therefore, we can write

$$\partial_{t_m} p_i = \operatorname*{res}_{\infty} \left(z^m \mathbf{c}^{*\mathrm{T}} \partial_{x_i} L \mathbf{c} \right).$$

Repeating the transformations presented above in detail, we have:

$$\partial_{t_m} p_i = \operatorname*{res}_{\infty} \left(z^m \mathbf{c}^{*\mathrm{T}} \mathbf{c} \partial_{x_i} k \right) = -\operatorname*{res}_{\infty} \left(z^m \lambda'(z) \partial_{x_i} k \right)$$

The same argument as above shows that

$$\partial_{x_i} k = (1 - \alpha'(z)) \partial_{x_i} z(\lambda, \mathbf{I})\big|_{\lambda=\mathrm{const.}} \tag{61}$$

and

$$\partial_{x_i} z = -\frac{\partial_{x_i} \lambda}{\lambda'(z)}.$$

Therefore, we obtain the second set of Hamiltonian equations for the dynamics of poles:

$$\partial_{t_m} p_i = \operatorname*{res}_{\infty} \left(z^m (1 - \alpha'(z)) \partial_{x_i} \lambda \right) = -\partial_{x_i} \mathcal{H}_m. \tag{62}$$

Let us find $\mathcal{H}_m$ explicitly in terms of H_m for the case when $a_j = 0$, $c \neq 0$. In this case $\alpha(z) = 2cz^{-1}$ and we have

$$\lambda(z) = \lambda_0(z) - \sum_{j,n \geqslant 1} (2c)^n \binom{n+j-1}{n} H_{j-1} z^{-j-2n}, \tag{63}$$

$\mathcal{H}_m = H_m^{(\alpha)} + 2cH_{m-2}^{(\alpha)}$ and from (63) we see that

$$H_m^{(\alpha)} = H_m + \sum_{j=1}^{[m/2]} (2c)^j \binom{m-j}{j} H_{m-2j}.$$

Therefore,

$$\mathcal{H}_m = H_m + \sum_{j=1}^{[m/2]} (2c)^j \left[\binom{m-j}{j} + \binom{m-j+1}{j-1} \right] H_{m-2j}. \tag{64}$$

In particular, $\mathcal{H}_3 = H_3 + 6cH_1$ which agrees with the result of the paper [12].

7. Calculation of the Hamiltonians

In order to find the Hamiltonians explicitly, we use the description of the spectral curve given in the paper [16]:

$$\sum_{j=0}^{N} I_j T_{N-j}(k|\lambda) = 0, \tag{65}$$

where $T_j(k|\lambda)$ are polynomials in k of degree N such that

$$\partial_k T_n(k|\lambda) = nT_{n-1}(k|\lambda) \tag{66}$$

and I_j are integrals of motion. The first few are

$$\begin{aligned}
I_0 &= 1,\\
I_1 &= \sum_i p_i,\\
I_2 &= \sum{}' \left(\tfrac{1}{2!}p_i p_j + \tfrac{1}{2!}\wp(x_{ij})\right),\\
I_3 &= \sum{}' \left(\tfrac{1}{3!}p_i p_j p_k + \tfrac{1}{2!}p_i\wp(x_{jk})\right),\\
I_4 &= \sum{}' \left(\tfrac{1}{4!}p_i p_j p_k p_l + \tfrac{1}{2!\cdot 2!}p_i p_j\wp(x_{kl}) + \tfrac{1}{2\cdot(2!)^2}\wp(x_{ij})\wp(x_{kl})\right),\\
I_5 &= \sum{}' \left(\tfrac{1}{5!}p_i p_j p_k p_l p_r + \tfrac{1}{2!\cdot 3!}p_i p_j p_r\wp(x_{kl}) + \tfrac{1}{2\cdot(2!)^2}p_r\wp(x_{ij})\wp(x_{kl})\right),
\end{aligned} \tag{67}$$

where $\sum'$ means summation over distinct indices. Recalling the equation of the spectral curve in terms of z and λ, let us also introduce $S_n(z|\lambda) = T_n(z+\zeta(\lambda)|\lambda)$, then

$$\partial_z S_n(z|\lambda) = nS_{n-1}(z|\lambda). \tag{68}$$

For example,

$$T_5(k|\lambda) = k^5 - 10\wp(\lambda)k^3 - 10\wp'(\lambda)k^2 - 5(\wp''(\lambda) - 3\wp^2(\lambda))k - 2\wp'(\lambda)\wp(\lambda).$$

We have

$$\zeta(\lambda) = \frac{1}{\lambda} - \frac{g_2\lambda^3}{2^2\cdot 3\cdot 5} + O(z^{-5}),$$

where

$$g_2 = 60\sum_{s\neq 0}\frac{1}{s^4}, \qquad s = 2m\omega + 2m'\omega', \quad m, m' \in \mathbb{Z}.$$

Expanding $S_5(z|\lambda)$ in z using the above formula for T_5, we get:

$$S_5(z|\lambda) = z^5 + \frac{5z^4}{\lambda} - \frac{g_2}{2}\left(\frac{1}{\lambda} + 5z + 5z^2\lambda + \frac{10}{3}z^3\lambda^2 + \frac{1}{6}z^4\lambda^3\right) + O(z^{-1}).$$

Note that if we introduce the gradation such that $\deg z = 1$, $\deg \lambda = -1$, then $\deg g_2 = 4$, $\deg S_n = n$. Note also that in the rational limit $g_2 = 0$ and the equation of the spectral curve becomes linear in λ^{-1} (see below in the next section). This can be only in the case if $S_n(z|\lambda) = z^n - nz^{n-1}\lambda^{-1}$ in the rational limit (the coefficient is found from the condition (68)).

In the non-degenerate case we have

$$S_n(z|\lambda) = z^n - \frac{nz^{n-1}}{\lambda} + g_2 O(z^{n-4}) \tag{69}$$

or

$$S_n(z|\lambda) = z^n - \frac{nz^{n-1}}{\lambda} + g_2 A_n z^{n-4} + g_2 B_n I_1 z^{n-5} + O(z^{n-6}), \tag{70}$$

where A_n and B_n are some constant coefficients. (I_1 comes from the expansion $\lambda = -Nz^{-1} - \frac{I_1}{2}z^{-2} + O(z^{-3})$.) Therefore, we can write

$$\begin{aligned}
S_N(z|\lambda) &= z^N - \frac{Nz^{N-1}}{\lambda} + g_2(A_N z^{N-4} + B_N I_1 z^{N-5}) + O(z^{N-6}),\\
S_{N-1}(z|\lambda) &= z^{N-1} - \frac{(N-1)z^{N-2}}{\lambda} + g_2 A_{N-1} z^{N-5} + O(z^{N-6}),\\
S_{N-j}(z|\lambda) &= z^{N-j} - \frac{(N-j)z^{N-j-1}}{\lambda} + O(z^{N-6}), \quad j = 2, 3, 4, 5.
\end{aligned} \tag{71}$$

and the equation of the spectral curve (65) acquires the form

$$\begin{aligned}
z^N &+ \sum_{i=1}^{5} I_i z^{N-i} + g_2 A_N z^{N-4} + g_2(A_{N-1} + B_N) I_1 z^{N-5} + O(z^{N-6})\\
&= -\frac{1}{\lambda}\left(Nz^{N-1} + \sum_{i=1}^{5}(N-i) I_i z^{N-i}\right).
\end{aligned}$$

Expressing λ as a function of z from here, we have:

$$\begin{aligned}
H_1 &= -I_1,\\
H_2 &= I_1^2 - 2I_2,\\
H_3 &= -I_1^3 + 3I_1 I_2 - 3I_3,\\
H_4 &= I_1^4 - 4I_1^2 I_2 + 2I_2^2 + 4I_1 I_3 - 4I_4 + \text{const.},\\
H_5 &= -I_1^5 + 5I_1^3 I_2 - 5I_1^2 I_3 + 5I_2 I_3 - 5I_1 I_2^2 + 5I_1 I_4 - 5I_5 + g_2 K I_1,
\end{aligned} \tag{72}$$

where $K = (N+1)A_N - N(A_{N-1} + B_N)$, or, explicitly,

$$
\begin{aligned}
H_1 &= -\sum_i p_i, \\
H_2 &= \sum_i p_i^2 - \sum_{i\neq j} \wp(x_{ij}), \\
H_3 &= -\sum_i p_i^3 + 3\sum_{i\neq j} p_i \wp(x_{ij}), \\
H_4 &= \sum_i p_i^4 - 2\sum_{i\neq j} p_i p_j \wp(x_{ij}) - 4\sum_{i\neq j} p_i^2 \wp(x_{ij}) \\
&\quad + \sum_{i\neq j} \wp^2(x_{ij}) + 2\sum{}' \wp(x_{ij})\wp(x_{jk}) + \text{const.}, \\
H_5 &= -\sum_i p_i^5 + 5\sum_{i\neq j}(p_i^3 + p_i^2 p_j)\wp(x_{ij}) - 5\sum_{i\neq j} p_i \wp^2(x_{ij}) \\
&\quad - 5\sum{}' p_i \wp(x_{ij})\wp(x_{ik}) - 5\sum{}' p_i \wp(x_{ij})\wp(x_{jk}) + \text{const.} \cdot \sum_i p_i.
\end{aligned}
\tag{73}
$$

These are indeed the Hamiltonians of the elliptic Calogero–Moser model. It is easy to see that they satisfy the property

$$H_{m-1} = -\frac{1}{m}\sum_i \partial_{p_i} H_m. \tag{74}$$

Indeed, we have

$$\lambda(z) = -Nz^{-1} + \sum_{m\geqslant 2} \frac{z^{-m}}{m} V_m = -Nz^{-1} + \sum_{m\geqslant 1} z^{-m-1} H_m$$

so

$$V_m = \partial_{t_m} \sum_i x_i = \sum_i \partial_{p_i} H_m = -mH_{m-1}.$$

One can see that the higher Hamiltonians will consist from the principal part and other terms as follows:

$$H_n = (-1)^n \sum_{|\mu|=n} C^n_\mu I_\mu + g_2 \sum_{|\nu|=n-4} B^n_\nu I_\nu + \cdots, \tag{75}$$

where the first sum is taken over Young diagrams μ of $n = |\mu|$ boxes, $I_\mu = I_{\mu_1} I_{\mu_2} \ldots I_{\ell(\mu)}$, where $\ell(\mu)$ is the number of non-empty rows of the diagram μ and C^n_μ is the matrix of the transition from the basis of elementary symmetric polynomials to the basis of power sums.

8. Rational and trigonometric limits

In the rational limit $\omega_1, \omega_2 \to \infty$, $\sigma(\lambda) = \lambda$, $\Phi(x, \lambda) = (x^{-1} + \lambda^{-1})e^{-x/\lambda}$ and the equation of the spectral curve becomes

$$\det\left(L_{\text{rat}} - (E - I)\lambda^{-1} - (z + \lambda^{-1})I\right) = 0, \tag{76}$$

where

$$(L_{\text{rat}})_{ij} = -\delta_{ij}p_i - \frac{1-\delta_{ij}}{x_i - x_j} \tag{77}$$

is the Lax matrix of the rational Calogero–Moser model. Rewriting the equation of the spectral curve in the form

$$\det\left(I - E\frac{\lambda^{-1}}{L_{\text{rat}} - zI}\right) = 0$$

and using the property $\det(I+Y) = 1 + \operatorname{Tr} Y$ for any matrix Y of rank 1, we get

$$\lambda = -\text{tr}\left(E\frac{1}{zI - L_{\text{rat}}}\right) = -\sum_{n\geqslant 0} z^{-n-1}\,\text{tr}\,L_{\text{rat}}^n, \tag{78}$$

where we use the well known property $\text{tr}(EL_{\text{rat}}^n) = \text{tr}(L_{\text{rat}}^n)$. So the Hamiltonians are $H_m = \text{tr}\,L_{\text{rat}}^m$ which agrees with Shiota's result [8].

The trigonometric limit is more tricky. Let $\pi i/\gamma$ be period of the trigonometric (or hyperbolic) functions (the second period tends to infinity). The Weierstrass functions in this limit become

$$\sigma(x) = \gamma^{-1}\,e^{-\frac{1}{6}\gamma^2x^2}\,\sinh(\gamma x), \qquad \zeta(x) = \gamma\,\coth(\gamma x) - \frac{1}{3}\gamma^2 x.$$

The tau-function for trigonometric solutions is

$$\tau(x, \mathbf{t}) = \prod_{i=1}^{N}\left(e^{2\gamma x} - e^{2\gamma x_i(\mathbf{t})}\right), \tag{79}$$

so we should consider

$$\tau(x, \mathbf{t}) = \prod_{i=1}^{N}\sigma(x - x_i)e^{\frac{1}{6}\,\gamma^2(x-x_i)^2 + \gamma(x+x_i)}. \tag{80}$$

With this choice, equation (33) acquires the form $k = z + \zeta(\lambda) + \frac{1}{3}\gamma^2\lambda$ or

$$k = z + \gamma\,\coth(\gamma\lambda). \tag{81}$$

The trigonometric limit of the function $\Phi(x, \lambda)$ is

$$\Phi(x, \lambda) = \gamma\,(\coth(\gamma x) + \coth(\gamma\lambda))\,e^{-\gamma x\,\coth(\gamma\lambda)}.$$

Therefore, the equation of the spectral curve can be written in the form

$$\det\left(W^{1/2}LW^{-1/2} + \gamma(1 - \coth(\gamma\lambda))(E - I) - (z + \gamma\,\coth(\gamma\lambda))I\right) = 0, \tag{82}$$

where $W = \text{diag}(w_1, w_2, \ldots, w_N)$ and

$$L_{ij} = -\delta_{ij}p_i - \frac{(1-\delta_{ij})\gamma}{\sinh(\gamma(x_i - x_j))} = -\delta_{ij}p_i - 2\gamma(1-\delta_{ij})\frac{w_i^{1/2}w_j^{1/2}}{w_i - w_j} \tag{83}$$

is the Lax matrix of the trigonometric Calogero–Moser model. Here and below we use the notation $w_i = e^{2\gamma x_i}$.

After the transformations similar to the rational case equation (82) can be brought to the form

$$\gamma(1-\coth(\gamma\lambda))\mathrm{tr}\left[W^{-1/2}EW^{1/2}\frac{1}{zI-(L-\gamma I)}\right]=1$$

or

$$\lambda=\frac{1}{2\gamma}\,\log\left[1-2\gamma\,\mathrm{tr}\left(W^{-1/2}EW^{1/2}\frac{1}{zI-(L-\gamma I)}\right)\right]. \tag{84}$$

Applying the formula $\det(I+Y)=1+\ \mathrm{tr}\,Y$ for any matrix Y of rank 1 in the opposite direction, we have

$$\lambda=\frac{1}{2\gamma}\,\log\,\det\left[I-2\gamma W^{-1/2}EW^{1/2}\frac{1}{zI-(L-\gamma I)}\right]. \tag{85}$$

Now we are going to use the identity

$$[L,W]=2\gamma(W^{1/2}EW^{1/2}-W) \tag{86}$$

which can be easily checked. With the help of this identity, we can transform (85) as follows:

$$\begin{aligned}
2\gamma\lambda &= \log\,\det\Bigg(I-W^{-1}LW\frac{1}{zI-(L-\gamma I)}+\frac{L}{zI-(L-\gamma I)}\\
&\quad -\frac{2\gamma}{zI-(L-\gamma I)}\Bigg)\\
&= \log\,\det\Bigg[\left(I-\frac{2\gamma}{zI-(L-\gamma I)}\right)\frac{zI-(L-\gamma I)}{zI-(L+\gamma I)}\\
&\quad\times\left(I-W^{-1}LW\frac{1}{zI-(L-\gamma I)}+\frac{L}{zI-(L-\gamma I)}-\frac{2\gamma}{zI-(L-\gamma I)}\right)\Bigg]\\
&= \log\,\det\Bigg[\left(I-\frac{2\gamma}{zI-(L-\gamma I)}\right)\Bigg(I-W^{-1}LW\frac{1}{zI-(L+\gamma I)}\\
&\quad +\frac{L}{zI-(L+\gamma I)}\Bigg)\Bigg]\\
&= \log\,\det\Bigg[\left(I-\frac{2\gamma}{zI-(L-\gamma I)}\right)\frac{1}{zI-(L+\gamma I)}\Big(zI-(L+\gamma I)\\
&\quad -W^{-1}LW+L\Big)\Bigg]\\
&= \log\,\det\frac{zI-(L+\gamma I)}{zI-(L-\gamma I)}.
\end{aligned}$$

Therefore, we get

$$\lambda = \frac{1}{2\gamma}\,\mathrm{tr}\left(\log(I - z^{-1}(L+\gamma I)) - \log(I - z^{-1}(L-\gamma I))\right)$$
$$= -\frac{1}{2\gamma}\,\mathrm{tr}\sum_{m\geqslant 1}\frac{z^{-m}}{m}\left((L+\gamma I)^m - (L-\gamma I)^m\right) \quad (87)$$

and

$$H_m = \frac{1}{2\gamma(m+1)}\,\mathrm{tr}\left((L+\gamma I)^{m+1} - (L-\gamma I)^{m-1}\right) \quad (88)$$

which agrees with the result of paper [10].

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

ORCID iDs

V Prokofev https://orcid.org/0000-0001-5120-1783
A Zabrodin https://orcid.org/0000-0002-4319-770X

References

[1] Airault H, McKean H P and Moser J 1977 Rational and elliptic solutions of the Korteweg–de Vries equation and a related many-body problem *Commun. Pure Appl. Math.* **30** 95–148
[2] Calogero F 1971 Solution of the one-dimensional *N*-body problems with quadratic and/or inversely quadratic pair potentials *J. Math. Phys.* **12** 419–36
[3] Calogero F 1975 Exactly solvable one-dimensional many-body problems *Lett. Nuovo Cimento* **13** 411–6
[4] Moser J 1975 Three integrable Hamiltonian systems connected with isospectral deformations *Adv. Math.* **16** 197–220
[5] Olshanetsky M A and Perelomov A M 1981 Classical integrable finite-dimensional systems related to Lie algebras *Phys. Rep.* **71** 313–400
[6] Krichever I M 1978 Rational solutions of the Kadomtsev–Petviashvili equation and integrable systems of N particles on a line *Funct. Anal. Appl.* **12** 59–61
[7] Chudnovsky D V and Chudnovsky G V 1977 Pole expansions of non-linear partial differential equations *Nuovo Cimento* B **40** 339–50
[8] Shiota T 1994 Calogero–Moser hierarchy and KP hierarchy *J. Math. Phys.* **35** 5844–9
[9] Haine L 2007 KP trigonometric solitons and an adelic flag manifold *Symmetry, Integrability Geometry Methods Appl.* **3** 015
[10] Zabrodin A 2020 KP hierarchy and trigonometric Calogero–Moser hierarchy *J. Math. Phys.* **61** 043502
[11] Krichever I M 1980 Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles *Funk. Anal. i Ego Pril.* **14** 45–54 (in Russian)
Krichever I M 1980 Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles *Funct. Anal. Appl.* **14** 282–90 (Engl. transl.)

[12] Zabrodin A 2019 Elliptic solutions to integrable nonlinear equations and many-body systems *J. Geom. Phys.* **146** 103506
[13] Perelomov A 1990 *Integrable Systems of Classical Mechanics and Lie Algebras* (Basel: Birkhäuser)
[14] Date E, Jimbo M, Kashiwara M and Miwa T 1983 Transformation Groups for Soliton Equations *Nonlinear Integrable Systems–Classical Theory and Quantum Theory* (Kyoto 1981) (Singapore: World Scientific) pp 39–119
[15] Jimbo M and Miwa T 1983 Solitons and infinite-dimensional Lie algebras *Publ. Res. Inst. Math. Sci.* **19** 943–1001
[16] Itoyama H and Morozov A 1996 Integrability and Seiberg–Witten theory. Curves and periods *Nucl. Phys.* B **477** 855–77

Appendix D

V.Prokofev, A Zabrodin "Elliptic solutions to Toda lattice hierarchy and elliptic Ruijsenaars-Schneider model"

Theoretical and Mathematical Physics, 2021.

Contribution: I conducted all calculations independently. Basically the main idea of calculations the same as in elliptic case for KP. However there was quite cumbersome calculations especially ones for proving (6.11). I suggested the way to conduct them.

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ELLIPTIC SOLUTIONS OF THE TODA LATTICE HIERARCHY AND THE ELLIPTIC RUIJSENAARS–SCHNEIDER MODEL

V. V. Prokofev[∗†] **and A. V. Zabrodin**[†‡§]

We consider solutions of the 2D Toda lattice hierarchy that are elliptic functions of the "zeroth" time $t_0 = x$. It is known that their poles as functions of t_1 move as particles of the elliptic Ruijsenaars–Schneider model. The goal of this paper is to extend this correspondence to the level of hierarchies. We show that the Hamiltonians that govern the dynamics of poles with respect to the mth hierarchical times t_m and $\bar{t}_m$ of the 2D Toda lattice hierarchy are obtained from the expansion of the spectral curve for the Lax matrix of the Ruijsenaars–Schneider model at the marked points.

Keywords: Toda lattice hierarchy, Ruijsenaars–Schneider model, elliptic solutions

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1. Introduction

1.1. Motivation and result. The 2D Toda lattice (2DTL) hierarchy [1] is an infinite set of compatible nonlinear differential–difference equations involving infinitely many time variables $\mathbf{t} = \{t_1, t_2, t_3, \dots\}$ ("positive" times), $\bar{\mathbf{t}} = \{\bar{t}_1, \bar{t}_2, \bar{t}_3, \dots\}$ ("negative" times), in which the equations are differential, and the "zeroth" time $t_0 = x$, in which the equations are difference. Among all solutions of these equations, of special interest are solutions that have a finite number of poles in the variable x in a fundamental domain of the complex plane. In particular, one can consider solutions that are elliptic (double-periodic in the complex plane) functions of x with poles depending on the times.

The investigation of the dynamics of poles of singular solutions of nonlinear integrable equations was initiated in the seminal paper [2], where elliptic and rational solutions of the Korteweg–de Vries and Boussinesq equations were studied. It was shown that the poles move as particles of the integrable Calogero–Moser many-body system [3]–[6] with some restrictions in the phase space. As was proved in [7], [8], this connection becomes most natural for the more general Kadomtsev–Petviashvili (KP) equation, in which case there are no restrictions in the phase space for the Calogero–Moser dynamics of poles. The method suggested by Krichever [9] for elliptic solutions of the KP equation consists in substituting the solution not in the KP equation itself but in the auxiliary linear problem for it (this implies a suitable pole ansatz for the wave function). This method allows obtaining the equations of motion together with their Lax representation.

[∗]Moscow Institute for Physics and Technology (National Research University), Dolgoprudny, Moscow Region, Russia, e-mail: vadim.prokofev@phystech.edu.

[†]Skolkovo Institute of Science and Technology, Moscow, Russia.

[‡]National Research University "Higher School of Economics," Moscow, Russia.

[§]Alikhanov Institute for Theoretical and Experimental Physics, National Research Center "Kurchatov Institute," Moscow, Russia, e-mail: zabrodin@itep.ru (corresponding author).

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Later, Shiota showed [10] that the correspondence between rational solutions of the KP equation and the Calogero–Moser system with a rational potential can be extended to the level of hierarchies. The evolution of poles with respect to the higher times t_k of the KP hierarchy was shown to be governed by the higher Hamiltonians $H_k = \operatorname{tr} L^k$ of the integrable Calogero–Moser system, where L is the Lax matrix. More recently, this correspondence was generalized to trigonometric and elliptic solutions of the KP hierarchy (see [11], [12] for the trigonometric solutions and [13] for the elliptic solutions).

The dynamics of poles of elliptic solutions of the 2DTL and modified KP (mKP) equations was studied in [14] (also see [15]). It was proved that the poles move as particles of the integrable Ruijsenaars–Schneider many-body system [16], [17], which is a relativistic generalization of the Calogero–Moser system. The extension to the level of hierarchies for rational solutions of the mKP equation was achieved in [18] (also see [19]): again, the evolution of poles with respect to the higher times t_k of the mKP hierarchy is governed by the higher Hamiltonians $\operatorname{tr} L^k$ of the Ruijsenaars–Schneider system. Recently, this result was generalized to trigonometric solutions [20]. However, the corresponding result for more general elliptic solutions was missing in the literature.

In this paper, we study the correspondence of the 2DTL hierarchy and the Ruijsenaars–Schneider hierarchy for elliptic solutions of the former. Our method consists in solving the auxiliary linear problems for the wave function and its adjoint using a suitable pole ansatz. The tau function of the 2DTL hierarchy for elliptic solutions has the form

$$\tau(x, \mathbf{t}, \bar{\mathbf{t}}) = \exp\biggl(-\sum_{k\geq 1} k t_k \bar{t}_k\biggr) \prod_{i=1}^{N} \sigma(x - x_i(\mathbf{t}, \bar{\mathbf{t}})), \tag{1.1}$$

where $\sigma(x)$ is the Weierstrass σ-function with quasiperiods 2ω, $2\omega'$ such that $\operatorname{Im}(\omega'/\omega) > 0$ (the definition is given in Sec. 3 below). The zeros x_i of the tau function are poles of the solution. They are assumed to be all distinct.

We show that the dynamics of poles in the times $\mathbf{t}$, $\bar{\mathbf{t}}$ is Hamiltonian; we then identify the corresponding Hamiltonians, which turn out to be higher Hamiltonians of the elliptic Ruijsenaars–Schneider system. The generating function of the Hamiltonians is $\lambda(z)$, where the spectral parameters λ and z are connected by the equation of the spectral curve

$$\det_{N\times N}(z e^{\eta\zeta(\lambda)} I - L(\lambda)) = 0, \tag{1.2}$$

where $\zeta(\lambda)$ is the Weierstrass ζ function, I is the unity matrix, and $L(\lambda)$ is the Lax matrix. Any point of the spectral curve is $P = (z, \lambda)$, where z, λ are connected by Eq. (1.2). There are to distinguished points on the spectral curve: $P_\infty = (\infty, 0)$ and $P_0 = (0, N\eta)$. The Hamiltonians corresponding to the positive-time flows $\mathbf{t}$ are the coefficients of the expansion of the function $\lambda(z)$ in negative powers of z around the point P_∞, while the Hamiltonians corresponding to the negative-time flows $\bar{\mathbf{t}}$ are coefficients of the expansion of the function $\lambda(z)$ in positive powers of z around the point P_0. This is the main result of this paper.

1.2. Elliptic Ruijsenaars–Schneider model. Here, we collect the main facts on the elliptic Ruijsenaars–Schneider system following [17].

The N-particle elliptic Ruijsenaars–Schneider system is a completely integrable model. It can be regarded as a relativistic extension of the Calogero–Moser system. The dynamical variables are the coordinates x_i and momenta p_i with canonical Poisson brackets $\{x_i, p_j\} = \delta_{ij}$. The integrals of motion in involution have the form

$$I_k = \sum_{I\subset\{1,\dots,N\},\, |I|=k} \exp\biggl(\sum_{i\in I} p_i\biggr) \prod_{i\in I,\, j\notin I} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)}, \qquad k = 1, \dots, N, \tag{1.3}$$

where η is a parameter (the inverse speed of light). In particular,

$$I_1 = \sum_i e^{p_i} \prod_{j\neq i} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)}, \qquad I_N = \exp\Big(\sum_{i=1}^N p_i\Big). \tag{1.4}$$

Comparing to [17], our formulas differ by the canonical transformation

$$e^{p_i} \to e^{p_i} \prod_{j\neq i} \Big(\frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j - \eta)}\Big)^{1/2}, \qquad x_i \to x_i,$$

which allows eliminating square roots in [17]. The Hamiltonian of the model is $H_1 = I_1$.

The velocities of the particles are

$$\dot{x}_i = \frac{\partial H_1}{\partial p_i} = e^{p_i} \prod_{j\neq i} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)}. \tag{1.5}$$

The Hamiltonian equations $\dot{p}_i = -\partial H_1/\partial x_i$ are equivalent to the equations of motion

$$\begin{aligned} \ddot{x}_i &= -\sum_{k\neq i} \dot{x}_i \dot{x}_k (\zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k)) = \\ &= \sum_{k\neq i} \dot{x}_i \dot{x}_k \frac{\wp'(x_i - x_k)}{\wp(\eta) - \wp(x_i - x_k)}, \end{aligned} \tag{1.6}$$

where $\wp(x)$ is the Weierstrass $\wp$ functions.

We can also introduce integrals of motion I_{-k} as

$$I_{-k} = I_N^{-1} I_{N-k} = \sum_{I\subset\{1,\dots,N\},\, |I|=k} \exp\Big(-\sum_{i\in I} p_i\Big) \prod_{i\in I,\, j\notin I} \frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)}, \tag{1.7}$$

In particular,

$$I_{-1} = \sum_i e^{-p_i} \prod_{j\neq i} \frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)}. \tag{1.8}$$

It is natural to set $I_0 = 1$. It can be easily verified that the equations of motion in the time $\bar{t}_1$ corresponding to the Hamiltonian $\bar{H}_1 = I_{-1}$ are the same equations (1.6).

This paper is organized as follows. In Sec. 2 we recall the main facts about the 2DTL hierarchy. We recall the Lax formulation in terms of pseudodifference operators, the bilinear identity for the tau function, and auxiliary linear problems for the wave function. Section 3 is devoted to solutions that are elliptic functions of $x = t_0$. We introduce double-Bloch solutions of the auxiliary linear problem and express them as a linear combination of elementary double-Bloch functions having just one simple pole in the fundamental domain. In Sec. 4, we obtain equations of motion for the poles as functions of the time t_1 together with their Lax representation. The properties of the spectral curve are discussed in Sec. 5. In Sec. 6, we consider the dynamics of poles with respect to the higher times and derive the corresponding Hamiltonian equations. Rational and trigonometric limits are addressed in Sec. 7. Explicit examples of the Hamiltonians are given in Sec. 8.

2. The 2D Toda lattice hierarchy

Here we very briefly review the 2DTL hierarchy (see [1]). We consider the pseudodifference Lax operators

$$\mathcal{L} = e^{\eta\,\partial_x} + \sum_{k\geq 0} U_k(x) e^{-k\eta\,\partial_x}, \qquad \overline{\mathcal{L}} = c(x)e^{-\eta\partial_x} + \sum_{k\geq 0} \overline{U}_k(x) e^{k\eta\,\partial_x}, \tag{2.1}$$

where $e^{\eta\,\partial_x}$ is the shift operator acting as $e^{\pm\eta\,\partial_x} f(x) = f(x \pm \eta)$ and the coefficient functions U_k, $\overline{U}_k$ are functions of x, $\mathbf{t}$, and $\overline{\mathbf{t}}$. The equations of the hierarchy are differential–difference equations for the functions c, U_k, and $\overline{U}_k$. They are encoded in the Lax equations

$$\begin{aligned} &\partial_{t_m}\mathcal{L} = [\mathcal{B}_m, \mathcal{L}], \qquad \partial_{t_m}\overline{\mathcal{L}} = [\mathcal{B}_m, \overline{\mathcal{L}}], \qquad \mathcal{B}_m = (\mathcal{L}^m)_{\geq 0}, \\ &\partial_{\bar{t}_m}\mathcal{L} = [\overline{\mathcal{B}}_m, \mathcal{L}], \qquad \partial_{\bar{t}_m}\overline{\mathcal{L}} = [\overline{\mathcal{B}}_m, \overline{\mathcal{L}}], \qquad \overline{\mathcal{B}}_m = (\overline{\mathcal{L}}^m)_{<0}, \end{aligned} \tag{2.2}$$

where

$$\Bigl(\sum_{k\in\mathbb{Z}} U_k e^{k\eta\,\partial_x}\Bigr)_{\geq 0} = \sum_{k\geq 0} U_k e^{k\eta\,\partial_x}, \qquad \Bigl(\sum_{k\in\mathbb{Z}} U_k e^{k\eta\,\partial_x}\Bigr)_{<0} = \sum_{k<0} U_k e^{k\eta\,\partial_x}.$$

For example, $\mathcal{B}_1 = e^{\eta\,\partial_x} + U_0(x)$, $\overline{\mathcal{B}}_1 = c(x)e^{-\eta\,\partial_x}$.

An equivalent formulation is provided by the zero-curvature (Zakharov–Shabat) equations

$$\partial_{t_n}\mathcal{B}_m - \partial_{t_m}\mathcal{B}_n + [\mathcal{B}_m, \mathcal{B}_n] = 0, \tag{2.3}$$

$$\partial_{\bar{t}_n}\mathcal{B}_m - \partial_{t_m}\overline{\mathcal{B}}_n + [\mathcal{B}_m, \overline{\mathcal{B}}_n] = 0, \tag{2.4}$$

$$\partial_{\bar{t}_n}\overline{\mathcal{B}}_m - \partial_{\bar{t}_m}\overline{\mathcal{B}}_n + [\overline{\mathcal{B}}_m, \overline{\mathcal{B}}_n] = 0. \tag{2.5}$$

For example, from (2.4) with $m = n = 1$, we have

$$\begin{aligned} &\partial_{t_1}\ln c(x) = v(x) - v(x-\eta), \\ &\partial_{\bar{t}_1} v(x) = c(x) - c(x+\eta), \end{aligned}$$

where $v = U_0$. Eliminating $v(x)$, we obtain a second-order differential–difference equation for $c(x)$,

$$\partial_{t_1}\partial_{\bar{t}_1}\ln c(x) = 2c(x) - c(x+\eta) - c(x-\eta),$$

which is one of the forms of the 2D Toda equation. After the change of variables $c(x) = e^{\varphi(x)-\varphi(x-\eta)}$, it acquires the most familiar form

$$\partial_{t_1}\partial_{\bar{t}_1}\varphi(x) = e^{\varphi(x)-\varphi(x-\eta)} - e^{\varphi(x+\eta)-\varphi(x)}. \tag{2.6}$$

The zero-curvature equations are compatibility conditions for the auxiliary linear problems

$$\partial_{t_m}\psi = \mathcal{B}_m(x)\psi, \qquad \partial_{\bar{t}_m}\psi = \overline{\mathcal{B}}_m(x)\psi, \tag{2.7}$$

where the wave function ψ depends on a spectral parameter z: $\psi = \psi(z; \mathbf{t}, \overline{\mathbf{t}})$. The wave function has the following expansion in powers of z:

$$\psi = z^{x/\eta} e^{\xi(\mathbf{t},z)}\left(1 + \frac{\xi_1(x,\mathbf{t},\overline{\mathbf{t}})}{z} + \frac{\xi_2(x,\mathbf{t},\overline{\mathbf{t}})}{z^2} + \cdots\right), \tag{2.8}$$

where

$$\xi(\mathbf{t}, z) = \sum_{k\geq 1} t_k z^k.$$

The wave operator is the pseudodifference operator of the form

$$\mathcal{W}(x) = 1 + \xi_1(x)e^{-\eta\partial_x} + \xi_2(x)e^{-2\eta\,\partial_x} + \cdots \tag{2.9}$$

with the same coefficient functions ξ_k as in (2.8). The wave function can then be written as

$$\psi = \mathcal{W}(x)z^{x/\eta}e^{\xi(\mathbf{t},z)}. \tag{2.10}$$

The adjoint wave function $\psi^\dagger$ is defined by the formula

$$\psi^\dagger = (\mathcal{W}^\dagger(x-\eta))^{-1}z^{-x/\eta}e^{-\xi(\mathbf{t},z)} \tag{2.11}$$

(see, e.g., [21]), where the adjoint difference operator is defined according to the rule $(f(x)\circ e^{n\eta\,\partial_x})^\dagger = e^{-n\eta\,\partial_x}\circ f(x)$. The auxiliary linear problems for the adjoint wave function have the form

$$-\partial_{t_m}\psi^\dagger = \mathcal{B}_m^\dagger(x-\eta)\psi^\dagger. \tag{2.12}$$

In particular, we have

$$\begin{aligned} \partial_{t_1}\psi(x) &= \psi(x+\eta) + v(x)\psi(x), \\ -\partial_{t_1}\psi^\dagger(x) &= \psi^\dagger(x-\eta) + v(x-\eta)\psi^\dagger(x), \\ \partial_{\bar{t}_1}\psi(x) &= c(x)\psi(x-\eta). \end{aligned} \tag{2.13}$$

The general solution of the 2DTL hierarchy is provided by the tau function $\tau = \tau(x, \mathbf{t}, \bar{\mathbf{t}})$ [22], [23]. The tau function satisfies the bilinear relation

$$\begin{aligned} &\oint_\infty z^{(x-x')/\eta-1}e^{\xi(\mathbf{t},z)-\xi(\mathbf{t}',z)}\tau(x, \mathbf{t}-[z^{-1}], \bar{\mathbf{t}})\tau(x'+\eta, \mathbf{t}'+[z^{-1}], \bar{\mathbf{t}}')\,dz = \\ &= \oint_0 z^{(x-x')/\eta-1}e^{\xi(\bar{\mathbf{t}},z^{-1})-\xi(\bar{\mathbf{t}}',z^{-1})}\tau(x+\eta, \mathbf{t}, \bar{\mathbf{t}}-[z])\tau(x', \mathbf{t}', \bar{\mathbf{t}}'+[z])\,dz, \end{aligned} \tag{2.14}$$

valid for all x, x', $\mathbf{t}$, $\mathbf{t}'$, $\bar{\mathbf{t}}$, $\bar{\mathbf{t}}'$, where

$$\mathbf{t} \pm [z] = \left\{t_1 \pm z, t_2 \pm \frac{1}{2}z^2, t_3 \pm \frac{1}{3}z^3, \ldots\right\}.$$

The integration contour in the left-hand side of (2.14) is a big circle around infinity separating the singularities coming from the exponential factor from those coming from the tau functions. The integration contour in the right-hand side of (2.14) is a small circle around zero separating the singularities coming from the exponential factor from those coming from the tau functions.

The coefficient functions of the Lax operators can be expressed through the tau function. In particular,

$$U_0(x) = v(x) = \partial_{t_1}\ln\frac{\tau(x+\eta)}{\tau(x)}, \qquad c(x) = \frac{\tau(x+\eta)\tau(x-\eta)}{\tau^2(x)}. \tag{2.15}$$

In terms of the tau function, Toda equation (2.6) becomes

$$\partial_{t_1}\partial_{\bar t_1}\ln\tau(x) = -\frac{\tau(x+\eta)\tau(x-\eta)}{\tau^2(x)}. \tag{2.16}$$

The wave function and its adjoint are expressed through the tau function as

$$\psi = z^{x/\eta}e^{\xi(\mathbf{t},z)}\frac{\tau(x,\mathbf{t}-[z^{-1}],\bar{\mathbf{t}})}{\tau(x,\mathbf{t},\bar{\mathbf{t}})}, \tag{2.17}$$

$$\psi^\dagger = z^{-x/\eta}e^{-\xi(\mathbf{t},z)}\frac{\tau(x,\mathbf{t}+[z^{-1}],\bar{\mathbf{t}})}{\tau(x,\mathbf{t},\bar{\mathbf{t}})}. \tag{2.18}$$

We can also introduce the complementary wave functions $\bar\psi$, $\bar\psi^\dagger$ by the formulas

$$\begin{aligned}\bar\psi &= z^{x/\eta}e^{\xi(\bar{\mathbf{t}},z^{-1})}\frac{\tau(x+\eta,\mathbf{t},\bar{\mathbf{t}}-[z])}{\tau(x,\mathbf{t},\bar{\mathbf{t}})},\\ \bar\psi^\dagger &= z^{-x/\eta}e^{-\xi(\bar{\mathbf{t}},z^{-1})}\frac{\tau(x-\eta,\mathbf{t},\bar{\mathbf{t}}+[z])}{\tau(x,\mathbf{t},\bar{\mathbf{t}})}.\end{aligned} \tag{2.19}$$

They satisfy the same auxiliary linear problems as the wave functions ψ and $\psi^\dagger$. It is more convenient for us to work with the renormalized wave functions

$$\begin{aligned}\phi(x) &= \frac{\tau(x)}{\tau(x+\eta)}\bar\psi(x) = z^{x/\eta}e^{\xi(\bar{\mathbf{t}},z^{-1})}\frac{\tau(x+\eta,\mathbf{t},\bar{\mathbf{t}}-[z])}{\tau(x+\eta,\mathbf{t},\bar{\mathbf{t}})},\\ \phi^\dagger(x) &= \frac{\tau(x)}{\tau(x-\eta)}\bar\psi^\dagger(x) = z^{-x/\eta}e^{-\xi(\bar{\mathbf{t}},z^{-1})}\frac{\tau(x-\eta,\mathbf{t},\bar{\mathbf{t}}+[z])}{\tau(x-\eta,\mathbf{t},\bar{\mathbf{t}})}.\end{aligned} \tag{2.20}$$

They satisfy the linear equations

$$\partial_{\bar t_1}\phi(x) = \phi(x-\eta) - \bar v(x)\phi(x), \qquad -\partial_{\bar t_1}\phi^\dagger(x) = \phi^\dagger(x+\eta) - \bar v(x-\eta)\phi^\dagger(x), \tag{2.21}$$

where $\bar v(x) = \partial_{\bar t_1}\ln[\tau(x+\eta)/\tau(x)]$.

Finally, we note the useful corollaries of bilinear relation (2.14). Differentiating it with respect to t_m and setting $x = x'$, $\mathbf{t} = \mathbf{t}'$ and $\bar{\mathbf{t}} = \bar{\mathbf{t}}'$ after that, we obtain

$$\begin{aligned}&\frac{1}{2\pi i}\oint_\infty z^{m-1}\tau(x,\mathbf{t}-[z^{-1}])\tau(x+\eta,\mathbf{t}+[z^{-1}])\,dz =\\ &= \partial_{t_m}\tau(x+\eta,\mathbf{t})\tau(x,\mathbf{t}) - \partial_{t_m}\tau(x,\mathbf{t})\tau(x+\eta,\mathbf{t})\end{aligned} \tag{2.22}$$

or

$$\operatorname*{res}_\infty(z^m\psi(x)\psi^\dagger(x+\eta)) = \partial_{t_m}\ln\frac{\tau(x+\eta)}{\tau(x)}, \tag{2.23}$$

where $\operatorname*{res}_\infty$ is defined in accordance with the convention $\operatorname*{res}_\infty(z^{-n}) = \delta_{n1}$. Equivalently, Eq. (2.23) can be written in the form

$$\psi(x)\psi^\dagger(x+\eta) = 1 + \sum_{m\ge1} z^{-m-1}\,\partial_{t_m}\ln\frac{\tau(x+\eta)}{\tau(x)}. \tag{2.24}$$

In a similar way, differentiating bilinear relation (2.14) with respect to $\bar t_m$ and setting $x = x'$, $\mathbf{t} = \mathbf{t}'$ and $\bar{\mathbf{t}} = \bar{\mathbf{t}}'$ after that, we obtain the relation

$$\operatorname*{res}_0(z^{-m}\phi(x)\phi^\dagger(x+\eta)) = -\partial_{\bar t_m}\ln\frac{\tau(x+\eta)}{\tau(x)}. \tag{2.25}$$

Here, $\operatorname*{res}_0$ is defined according to the convention $\operatorname*{res}_0(z^{-n}) = \delta_{n1}$.

3. Elliptic solutions of the Toda equation and the double-Bloch wave functions

The ansatz for the tau function of elliptic (double-periodic in the complex plane) solutions of the 2DTL hierarchy is given by (1.1), where

$$\sigma(x)=\sigma(x\mid\omega,\omega')=x\prod_{s\neq 0}\left(1-\frac{x}{s}\right)e^{x/s+x^2/2s^2},\quad s=2\omega m+2\omega' m' \text{ with integer } m,m',$$

is the Weierstrass σ function with quasiperiods 2ω and $2\omega'$ such that $\operatorname{Im}(\omega'/\omega)>0$. It is connected with the Weierstrass ζ and $\wp$ functions by the formulas $\zeta(x)=\sigma'(x)/\sigma(x)$, $\wp(x)=-\zeta'(x)=-\partial_x^2\ln\sigma(x)$. The monodromy properties of $\sigma(x)$ are

$$\sigma(x+2\omega)=-e^{2\zeta(\omega)(x+\omega)}\sigma(x),\qquad \sigma(x+2\omega')=-e^{2\zeta(\omega')(x+\omega')}\sigma(x),\tag{3.1}$$

where the constants $\zeta(\omega)$, $\zeta(\omega')$ are related by $\zeta(\omega)\omega'-\zeta(\omega')\omega=\pi i/2$.

For elliptic solutions,

$$v(x)=\sum_i \dot{x}_i(\zeta(x-x_i)-\zeta(x-x_i+\eta))\tag{3.2}$$

is an elliptic function, and we can therefore find *double-Bloch* solutions of linear problem (2.13). The double-Bloch function satisfies the monodromy properties $\psi(x+2\omega)=B\psi(x)$ and $\psi(x+2\omega')=B'\psi(x)$ with some Bloch multipliers B and B'. Any nontrivial (i.e., not exponential) double-Bloch function must have poles in x in the fundamental domain. The Bloch multipliers of wave function (2.17) are

$$\begin{aligned}B&=z^{2\omega/\eta}\exp\left(-2\zeta(\omega)\sum_i(e^{-D(z)}-1)x_i\right),\\ B'&=z^{2\omega'/\eta}\exp\left(-2\zeta(\omega')\sum_i(e^{-D(z)}-1)x_i\right),\end{aligned}\tag{3.3}$$

where we define the differential operator

$$D(z)=\sum_{k\geq 1}\frac{z^{-k}}{k}\,\partial_{t_k}.\tag{3.4}$$

It follows from Eq. (2.24) that the Bloch multipliers of the function $\psi^\dagger$ are $1/B$ and $1/B'$. Indeed, the right-hand side of (2.24) is an elliptic function of x, and therefore the left-hand side must be also an elliptic function.

We introduce the elementary double-Bloch function $\Phi(x,\lambda)$ defined as

$$\Phi(x,\lambda)=\frac{\sigma(x+\lambda)}{\sigma(\lambda)\sigma(x)}e^{-\zeta(\lambda)x}.\tag{3.5}$$

The monodromy properties of Φ are

$$\begin{aligned}\Phi(x+2\omega,\lambda)&=e^{2(\zeta(\omega)\lambda-\zeta(\lambda)\omega)}\Phi(x,\lambda),\\ \Phi(x+2\omega',\lambda)&=e^{2(\zeta(\omega')\lambda-\zeta(\lambda)\omega')}\Phi(x,\lambda),\end{aligned}$$

and therefore it is indeed a double-Bloch function. The function Φ has a simple pole at $x=0$ with residue 1,

$$\Phi(x,\lambda)=\frac{1}{x}+\alpha_1x+\cdots,\qquad x\to 0,$$

where $\alpha_1=-\wp(\lambda)/2$. When this does not lead to misunderstanding, we suppress the second argument of Φ, writing simply $\Phi(x)=\Phi(x,\lambda)$. We also need the x-derivative $\Phi'(x,\lambda)=\partial_x\Phi(x,\lambda)$.

Below, we need the following identities satisfied by Φ:

$$\Phi(x,-\lambda) = -\Phi(-x,\lambda), \tag{3.6}$$

$$\Phi'(x) = \Phi(x)(\zeta(x+\lambda) - \zeta(\lambda) - \zeta(x)), \tag{3.7}$$

$$\Phi(x)\Phi(y) = \Phi(x+y)(\zeta(x) + \zeta(y) + \zeta(\lambda) - \zeta(x+y+\lambda)). \tag{3.8}$$

Equations (2.17) and (2.18) imply that the wave functions ψ and $\psi^\dagger$ have simple poles at the points x_i. We can expand the wave functions using the elementary double-Bloch functions as

$$\psi = k^{x/\eta} e^{\xi(\mathbf{t},z)} \sum_i c_i \Phi(x - x_i, \lambda), \tag{3.9}$$

$$\psi^\dagger = k^{-x/\eta} e^{-\xi(\mathbf{t},z)} \sum_i c_i^* \Phi(x - x_i, -\lambda), \tag{3.10}$$

where c_i, c_i^* are expansion coefficients, which do not depend on x, and k is an additional spectral parameter. We note that the normalization of functions (2.17) and (2.18) implies that c_i and c_i^* are $O(\lambda)$ as $\lambda \to 0$. We can see that (3.9) is a double-Bloch function with the Bloch multipliers

$$B = e^{(2\omega/\eta)(\ln k - \eta\zeta(\lambda)) + 2\zeta(\omega)\lambda}, \qquad B' = e^{(2\omega'/\eta)(\ln k - \eta\zeta(\lambda)) + 2\zeta(\omega')\lambda}, \tag{3.11}$$

and (3.10) has the Bloch multipliers B^{-1} and $B^{'-1}$. These Bloch multipliers should coincide with (3.3). Therefore, comparing Eqs. (3.3) and (3.11), we have

$$\frac{2\omega}{\eta}\left(\ln\frac{k}{z} - \eta\zeta(\lambda)\right) + 2\zeta(\omega)\left(\lambda + (e^{-D(z)} - 1)\sum_i x_i\right) = 2\pi i n,$$

$$\frac{2\omega'}{\eta}\left(\ln\frac{k}{z} - \eta\zeta(\lambda)\right) + 2\zeta(\omega')\left(\lambda + (e^{-D(z)} - 1)\sum_i x_i\right) = 2\pi i n'$$

with some integers n and n'. Regarding these equations as a linear system, we obtain the solution

$$\ln\frac{k}{z} - \eta\zeta(\lambda) = 2n'\eta\zeta(\omega) - 2n\eta\zeta(\omega'),$$

$$\lambda + (e^{-D(z)} - 1)\sum_i x_i = 2n\omega' - 2n'\omega.$$

Shifting λ by a suitable vector of the lattice spanned by 2ω, $2\omega'$, we obtain zeros in the right-hand sides of these equalities, and we can therefore represent the connection between the spectral parameters k, z, and λ in the form

$$k = ze^{\eta\zeta(\lambda)}, \qquad \lambda = (1 - e^{-D(z)})\sum_i x_i. \tag{3.12}$$

These two equations for three spectral parameters k, z, λ define the spectral curve. Another description of the same spectral curve is obtained below as the spectral curve of the Ruijsenaars–Schneider system (it is given by the characteristic polynomial of the Lax matrix $L(\lambda)$ for the Ruijsenaars–Schneider system). It appears in the form $R(k,\lambda) = 0$, where $R(k,\lambda)$ is a polynomial in k whose coefficients are elliptic functions of λ (see Sec. 5 below). These coefficients are integrals of motion in involution. The spectral curve in the form $R(k,\lambda) = 0$ appears if we eliminate z from Eqs. (3.12). Equivalently, we can represent the spectral curve as a relation connecting z and λ:

$$R(ze^{\eta\zeta(\lambda)}, \lambda) = 0. \tag{3.13}$$

The second equation in (3.12) can be written as an expansion in powers of z:

$$\lambda = \lambda(z) = -\sum_{m\geq 1} z^{-m}\hat{h}_m\mathcal{X}, \qquad \mathcal{X} := \sum_i x_i, \tag{3.14}$$

Here, $\hat{h}_k$ are differential operators of the form

$$\hat{h}_m = -\frac{1}{m}\,\partial_{t_m} + \text{higher order operators in } \partial_{t_1}, \partial_{t_2}, \ldots, \partial_{t_{m-1}}; \tag{3.15}$$

The first few are

$$\hat{h}_1 = -\partial_{t_1}, \qquad \hat{h}_2 = \frac{1}{2}\,(\partial_{t_1}^2 - \partial_{t_2}), \qquad \hat{h}_3 = \frac{1}{6}\,(-\partial_{t_1}^3 + 3\,\partial_{t_1}\,\partial_{t_2} - 2\,\partial_{t_3}).$$

As is mentioned above, the coefficients in expansion (3.14) are integrals of motion, i.e., $\partial_{t_j}\hat{h}_m\mathcal{X} = 0$ for all j and m. It then follows from the explicit form of the operators $\hat{h}_m$ that $\partial_{t_j}\,\partial_{t_1}\mathcal{X} = 0$. A simple inductive argument then shows that $\partial_{t_j}\,\partial_{t_m}\mathcal{X} = 0$ for all j, m. This means that $-\hat{h}_m\,\mathcal{X} = \frac{1}{m}\,\partial_{t_m}\mathcal{X}$ and hence $\mathcal{X}$ is a linear function of the times,

$$\mathcal{X} = \sum_i x_i = \mathcal{X}_0 + \sum_{m\geq 1} V_m t_m \tag{3.16}$$

with some constants V_m (which are velocities of the "center of masses" of the points x_i with respect to the times t_m multiplied by N). Therefore, the second equation in (3.12) can be written as

$$\lambda = D(z)\sum_i x_i = \sum_{j\geq 1}\frac{z^{-j}}{j}V_j. \tag{3.17}$$

We show in what follows that $H_m = V_m/m$ are Hamiltonians for the dynamics of poles in t_m, with H_1 being the standard Ruijsenaars–Schneider Hamiltonian.

4. Dynamics of poles with respect to t_1

The next procedure has become standard after paper [9]. We substitute $v(x)$ in form (3.2) and ψ in form (3.9) into the left-hand side of the linear problem

$$\partial_{t_1}\psi(x) - \psi(x+\eta) - v(x)\psi(x) = 0$$

and cancel the poles at the points $x = x_i$ and $x = x_i - \eta$. The highest poles are of the second order, but it is easy to see that they cancel identically. A simple calculation shows that the cancellation of first-order poles leads to the conditions

$$\begin{aligned} &zc_i + \dot{c}_i = \dot{x}_i\sum_{j\neq i} c_j\Phi(x_i - x_j) + c_i\sum_{j\neq i}\dot{x}_j\zeta(x_i - x_j) - c_i\sum_j \dot{x}_j\zeta(x_i - x_j + \eta),\\ &kc_i - \dot{x}_i\sum_j c_j\Phi(x_i - x_j - \eta) = 0. \end{aligned}$$

Here and below, the dot means the t_1-derivative. These conditions can be written in the matrix form as a system of linear equations for the vector $\mathbf{c} = (c_1, \ldots, c_N)^{\mathrm{T}}$:

$$\begin{aligned} L(\lambda)\mathbf{c} &= k\mathbf{c},\\ \dot{\mathbf{c}} &= M(\lambda)\mathbf{c}, \end{aligned} \tag{4.1}$$

where $(N \times N)$ matrices L and M are

$$L_{ij} = \dot{x}_i \Phi(x_i - x_j - \eta), \tag{4.2}$$

$$M_{ij} = \delta_{ij}\Big(k - z + \sum_{l \neq i} \dot{x}_l \zeta(x_i - x_l) - \sum_l \dot{x}_l \zeta(x_i - x_l + \eta)\Big) + \\ + (1 - \delta_{ij})\dot{x}_i \Phi(x_i - x_j) - \dot{x}_i \Phi(x_i - x_j - \eta). \tag{4.3}$$

They depend on the spectral parameter λ.

Differentiating the first equation in (4.1) with respect to t_1 and substituting the second equation, we obtain the compatibility condition for linear problems (4.1):

$$(\dot{L} + [L, M])\mathbf{c} = 0. \tag{4.4}$$

The Lax equation $\dot{L} + [L, M] = 0$ is equivalent to the equations of motion of the elliptic Ruijsenaars–Schneider system (see [15] for the details of the calculation). We note that our matrix M differs from the standard one by the term $\delta_{ij}(k - z)$, but this does not affect the compatibility condition because it is proportional to the unity matrix. It follows from the Lax representation that the time evolution is an isospectral transformation of the Lax matrix L, and hence all traces $\operatorname{tr} L^m$ and the characteristic polynomial $\det(kI - L)$, where I is the unit matrix, are integrals of motion. We note that the Lax matrix is written in terms of the momenta p_i as

$$L_{ij} = \Phi(x_i - x_j - \eta) e^{p_i} \prod_{l \neq i} \frac{\sigma(x_i - x_l + \eta)}{\sigma(x_i - x_l)}. \tag{4.5}$$

A similar calculation shows that the adjoint linear problem for function (3.10) leads to the equations

$$\begin{aligned} &\mathbf{c}^{*\mathrm{T}} \dot{X}^{-1} L(\lambda) \dot{X} = k \mathbf{c}^{*\mathrm{T}}, \\ &\dot{\mathbf{c}}^{*\mathrm{T}} = -\mathbf{c}^{*\mathrm{T}} M^*(\lambda), \end{aligned} \tag{4.6}$$

where $\dot{X} = \operatorname{diag}(\dot{x}_1, \ldots, \dot{x}_N)$ and

$$M^*_{ij} = \delta_{ij}\Big(k - z - \sum_{l \neq i} \dot{x}_l \zeta(x_i - x_l) + \sum_l \dot{x}_l \zeta(x_i - x_l - \eta)\Big) + \\ + (1 - \delta_{ij})\dot{x}_j \Phi(x_i - x_j) - \dot{x}_j \Phi(x_i - x_j - \eta). \tag{4.7}$$

5. The spectral curve

The first equation in (4.1) defines a connection between the spectral parameters k and λ, which is the equation of the spectral curve:

$$R(k, \lambda) = \det(kI - L(\lambda)) = 0. \tag{5.1}$$

As was mentioned already, the spectral curve is an integral of motion. The matrix $L = L(\lambda)$, which has an essential singularity at $\lambda = 0$, can be represented in the form $L(\lambda) = e^{\eta\zeta(\lambda)} V \widetilde{L}(\lambda) V^{-1}$, where matrix elements of $\widetilde{L}(\lambda)$ do not have essential singularities and V is the diagonal matrix $V_{ij} = \delta_{ij} e^{-\zeta(\lambda) x_i}$. The matrix $\widetilde{L}(\lambda)$ is given by

$$\widetilde{L}_{ij}(\lambda) = \frac{\sigma(x_i - x_j - \eta + \lambda)}{\sigma(\lambda)\sigma(x_i - x_j - \eta)} e^{p_i} \prod_{l \neq i} \frac{\sigma(x_i - x_l + \eta)}{\sigma(x_i - x_l)}. \tag{5.2}$$

Using the connection between k and z in Eq. (3.12), we represent the spectral curve in the form

$$\det(zI - \widetilde{L}(\lambda)) = 0. \tag{5.3}$$

The spectral curve is an N-sheet covering over the λ-plane. Any point of the curve is $P = (z, \lambda)$, where z and λ are connected by Eq. (5.3) and there are N points above each point λ.

To represent the spectral curve in explicit form, we recall the identity

$$\det_{1\le i,j\le N}\left(\frac{\sigma(x_i - y_j + \lambda)}{\sigma(\lambda)\sigma(x_i - y_j)}\right) = \frac{1}{\sigma(\lambda)}\sigma\left(\lambda + \sum_{i=1}^{N}(x_i - y_i)\right)\prod_{i<j}\sigma(x_i - x_j)\sigma(y_j - y_i) \Big/ \prod_{i,j}\sigma(x_i - y_j) \tag{5.4}$$

for the determinant of the elliptic Cauchy matrix. Using this identity, we can represent (5.3) as

$$\sum_{n=0}^{N}\varphi_n(\lambda) I_n z^{N-n} = 0, \tag{5.5}$$

where I_n are the Ruijsenaars–Schneider integrals of motion (1.3) and

$$\varphi_n(\lambda) = \frac{\sigma(\lambda - n\eta)}{\sigma(\lambda)\sigma^n(\eta)}. \tag{5.6}$$

We now fix two distinguished points on the spectral curve. As $\lambda \to 0$, we have

$$\widetilde{L}(\lambda) = \dot{X}E\lambda^{-1} + O(1),$$

where E is the rank-1 matrix with the entries $E_{ij} = 1$ for all i and j. Therefore, using the formula for the determinant of the matrix $I + Y$, where Y is a rank-1 matrix, we can write

$$\det(zI - \widetilde{L}(\lambda)) = z^N - z^{N-1}\lambda^{-1}I_1 + O(z^{N-1}),$$

or

$$\det(zI - \widetilde{L}(\lambda)) = (z - h_N(\lambda)\lambda^{-1})\prod_{j=1}^{N-1}(z - h_j(\lambda)),$$

where the functions h_j are regular functions at $\lambda = 0$ and $h_N(0) = I_1 = H_1$. We see that among the N points above $\lambda = 0$, one is distinguished: it is the point $P_\infty = (\infty, 0)$. Another distinguished point is $P_0 = (0, N\eta)$. Because $\varphi_N(N\eta) = 0$, we see that this point indeed belongs to the curve.

6. Dynamics in higher times

6.1. Positive times. To study the dynamics of poles in the higher positive times, we take advantage of relation (2.23), which, after the substitution of the wave functions for elliptic solutions, acquires the form

$$\sum_{i,j}\operatorname*{res}_{\infty}(z^m k^{-1} c_i c_j^* \Phi(x - x_i, \lambda)\Phi(x - x_j + \eta, -\lambda)) =$$
$$= \sum_n \partial_{t_m} x_n(\zeta(x - x_n) - \zeta(x - x_n + \eta)). \tag{6.1}$$

Equating the residues at $x = x_i - \eta$, we obtain

$$\partial_{t_m} x_i = -\sum_j \operatorname*{res}_{\infty} l(z^m k^{-1} c_i^* c_j \Phi(x_i - x_j - \eta, \lambda)). \tag{6.2}$$

In matrix form, we can write this relation as

$$\partial_{t_m} x_i = -\operatorname*{res}_{\infty}(z^m k^{-1} \mathbf{c}^{*\mathrm{T}} \dot{X}^{-1} (\partial_{p_i} L)\mathbf{c}). \tag{6.3}$$

Summing (6.3) over i and using (3.17), we can write

$$\operatorname*{res}_{\infty}(z^m \lambda'(z)) = \operatorname*{res}_{\infty}(z^m k^{-1} \mathbf{c}^{*\mathrm{T}} \dot{X}^{-1} L\mathbf{c}), \tag{6.4}$$

where we took into account that $\sum_i \partial_{p_i} L = L$. Using the fact that $\mathbf{c} = O(1/z)$, we conclude that

$$k^{-1} \mathbf{c}^{*\mathrm{T}} \dot{X}^{-1} L\mathbf{c} = \lambda'(z),$$

or

$$\mathbf{c}^{*\mathrm{T}} \dot{X}^{-1} \mathbf{c} = \lambda'(z). \tag{6.5}$$

Next, with (6.5), we can continue (6.3) as the following chain of equalities:

$$\begin{aligned}
\partial_{t_m} x_i =& -\operatorname*{res}_{\infty}(z^m k^{-1} \mathbf{c}^{*\mathrm{T}} \dot{X}^{-1} (\partial_{p_i} L)\mathbf{c}) = \\
=& -\operatorname*{res}_{\infty}(z^m\, \partial_{p_i}(k^{-1} \mathbf{c}^{*\mathrm{T}} \dot{X}^{-1} L\mathbf{c})) + \operatorname*{res}_{\infty}(z^m k^{-1}\, \partial_{p_i}(\mathbf{c}^{*\mathrm{T}} \dot{X}^{-1}) L\mathbf{c}) + \\
&+ \operatorname*{res}_{\infty}(z^m k^{-1} \mathbf{c}^{*\mathrm{T}} \dot{X}^{-1} L\, \partial_{p_i}\mathbf{c}) + \operatorname*{res}_{\infty}(z^m\, \partial_{p_i}(k^{-1}) \mathbf{c}^{*\mathrm{T}} \dot{X}^{-1} L\mathbf{c}) = \\
=& -\operatorname*{res}_{\infty}(z^m\, \partial_{p_i}\lambda'(z)) + \operatorname*{res}_{\infty}(z^m\, \partial_{p_i}(\mathbf{c}^{*\mathrm{T}} \dot{X}^{-1})\mathbf{c}) + \\
&+ \operatorname*{res}_{\infty}(z^m \mathbf{c}^{*\mathrm{T}} \dot{X}^{-1}\, \partial_{p_i}\mathbf{c}) - \operatorname*{res}_{\infty}(z^m\, \partial_{p_i} \ln k\, \mathbf{c}^{*\mathrm{T}} \dot{X}^{-1} \mathbf{c}) = \\
=& -\operatorname*{res}_{\infty}(z^m \lambda'(z)\, \partial_{p_i} \ln k).
\end{aligned}$$

Here, $\partial_{p_i} \ln k = \partial_{p_i} \ln k(\lambda, \mathbf{I})|_{\lambda=\mathrm{const}}$, where $\mathbf{I}$ is the full set of integrals of motion. We see from (3.17) that $\partial_{p_i} \ln k = \partial_{p_i} \ln z(\lambda, \mathbf{I})|_{\lambda=\mathrm{const}}$. We regard z as an independent variable, whence

$$0 = \frac{d \ln z}{dp_i} = \partial_{p_i} \ln z|_{\lambda=\mathrm{const}} + \partial_\lambda \ln z|_{\mathbf{I}=\mathrm{const}}\, \partial_{p_i}\lambda$$

or

$$\partial_{p_i} \ln z = -\frac{\partial_{p_i}\lambda}{z\lambda'(z)}. \tag{6.6}$$

Therefore,

$$\partial_{t_m} x_i = \operatorname*{res}_{\infty}(z^{m-1}\, \partial_{p_i}\lambda(z)) = \frac{\partial H_m}{\partial p_i}, \tag{6.7}$$

where the Hamiltonian H_m is given by

$$H_m = \operatorname*{res}_{\infty}(z^{m-1}\lambda(z)). \tag{6.8}$$

This is the first half of the Hamiltonian equations.

The calculation leading to the second half of the Hamiltonian equations is rather involved. We present the main steps of it. First of all, we note that

$$\begin{aligned}
\dot{X}^{-1}\, \partial_{x_i} L = [E_i, B^-] + (\widetilde{D}^+ - \widetilde{D}^0)E_i A^- - \\
- \sum_l E_l \zeta(x_l - x_i + \eta)A^- + \sum_{l \neq i} E_l \zeta(x_l - x_i)A^- := Y_i,
\end{aligned} \tag{6.9}$$

where we introduce the matrices A^- and B^- and diagonal matrices E_i, $\widetilde{D}^0$, and $\widetilde{D}^\pm$ defined by their matrix elements as follows:

$$A^-_{jk} = \Phi(x_j - x_k - \eta), \qquad B^-_{jk} = \Phi'(x_j - x_k - \eta),$$
$$(E_i)_{jk} = \delta_{ij}\delta_{ik}, \qquad \widetilde{D}^0_{jk} = \delta_{jk}\sum_{l\neq j}\zeta(x_j - x_l), \qquad \widetilde{D}^\pm_{jk} = \delta_{jk}\sum_{l}\zeta(x_j - x_l \pm \eta).$$

Now, in order to prove the second half of the Hamiltonian equations, it is enough to obtain the equations

$$\partial_{t_m} p_i = \operatorname*{res}_{\infty}(z^m k^{-1}\mathbf{c}^{*\mathrm{T}} Y_i \mathbf{c}), \tag{6.10}$$

where Y_i is the right-hand side of (6.9). Indeed, repeating the argument given after Eq. (6.5) with the change $\partial_{p_i} \to \partial_{x_i}$, we obtain the Hamiltonian equations

$$\partial_{t_m} p_i = -\frac{\partial H_m}{\partial x_i} \tag{6.11}$$

with the same Hamiltonian given by (6.8).

It is clear that

$$p_i = \ln \dot{x}_i - \sum_{k\neq i}\ln\frac{\sigma(x_i - x_k + \eta)}{\sigma(x_i - x_k)}. \tag{6.12}$$

The t_m-derivative of this equality yields

$$\partial_{t_m} p_i = \dot{x}_i^{-1}\,\partial_{t_m}\dot{x}_i - \sum_{k\neq i}(\partial_{t_m} x_i - \partial_{t_m} x_k)(\zeta(x_i - x_k + \eta) - \zeta(x_i - x_k)). \tag{6.13}$$

We find the first term in the right-hand side. For this, we differentiate Eq. (6.3), which we write here in the form

$$\partial_{t_m} x_i = \operatorname*{res}_{\infty}(z^m k^{-1}\mathbf{c}^{*\mathrm{T}} E_i A^- \mathbf{c}),$$

with respect to t_1 and use Eqs. (4.1), (4.6). In this way we arrive at

$$\partial_{t_m}\dot{x}_i = \operatorname*{res}_{\infty}(z^m k^{-1}\mathbf{c}^{*\mathrm{T}}(M^* E_i A^- - E_i A^- M - E_i \dot{A}^-)\mathbf{c}), \tag{6.14}$$

where the matrices M and M^* are

$$\begin{aligned} M &= (k - z)I + D^0 - D^+ + \dot{X}A - \dot{X}A^-,\\ M^* &= (k - z)I - D^0 + D^- + A\dot{X} - A^-\dot{X}, \end{aligned} \tag{6.15}$$

and we define the off-diagonal matrix A and diagonal matrices D^0 by specifying their matrix elements

$$A_{jk} = (1 - \delta_{jk})\Phi(x_j - x_k),$$
$$D^0_{jk} = \delta_{jk}\sum_{l\neq j}\dot{x}_l\zeta(x_j - x_l), \qquad D^\pm_{jk} = \delta_{jk}\sum_{l}\dot{x}_l\zeta(x_j - x_l \pm \eta).$$

Taking into account that $\mathbf{c}^{*\mathrm{T}}A^- = k\mathbf{c}^{*\mathrm{T}}\dot{X}^{-1}$, $A^-\mathbf{c} = k\dot{X}^{-1}\mathbf{c}$, we have

$$\begin{aligned} &\mathbf{c}^{*\mathrm{T}}(M^* E_i A^- - E_i A^- M - E_i \dot{A}^-)\mathbf{c} =\\ &= \mathbf{c}^{*\mathrm{T}}(k(D^- - D^0)E_i\dot{X}^{-1} + kAE_i + E_iA^-(D^+ - D^0) -\\ &\quad - E_i(A^-\dot{X}A - A\dot{X}A^-) - kE_iA - E_i\dot{X}B^- + E_iB^-\dot{X})\mathbf{c}. \end{aligned}$$

Using identities (3.7) and (3.8), can see after some calculations that the result is

$$\mathbf{c}^{*\mathrm{T}}(M^* E_i A^- - E_i A^- M - E_i \dot{A}^-)\mathbf{c} = k\mathbf{c}^{*\mathrm{T}}[A, E_i]\mathbf{c}. \tag{6.16}$$

It is easy to see that

$$\dot{x}_i^{-1}\mathbf{c}^{*\mathrm{T}} k[A, E_i]\mathbf{c} = \mathbf{c}^{*\mathrm{T}}(AE_iA^- - A^-E_iA)\mathbf{c}.$$

Identities (3.7), (3.8) allow us to prove the relation

$$AE_iA^- - A^-E_iA = [E_i, B^-] + \sum_{l\neq i}\zeta(x_l - x_i)E_lA^- - \sum_l \zeta(x_l - x_i + \eta)A^-E_l + \\ + \sum_{l\neq i}\zeta(x_l - x_i)A^-E_l - \sum_l \zeta(x_l - x_i - \eta)E_lA^-.$$

Therefore, we have

$$\dot{x}_i^{-1}\,\partial_{t_m}\dot{x}_i = \operatorname*{res}_{\infty}\Bigg(z^m k^{-1}\mathbf{c}^{*\mathrm{T}}\Bigg([E_i, B^-] + 2\sum_{l\neq i}\zeta(x_l - x_i)E_lA^- - \\ - \sum_l \zeta(x_l - x_i + \eta)A^-E_l - \sum_l \zeta(x_l - x_i - \eta)A^-E_l\Bigg)\mathbf{c}\Bigg). \tag{6.17}$$

We next transform the remaining part of (6.13):

$$-\sum_{k\neq i}(\partial_{t_m}x_i - \partial_{t_m}x_k)(\zeta(x_i - x_k + \eta) - \zeta(x_i - x_k)) = \\ = \operatorname*{res}_{\infty}\Bigg(z^m k^{-1}\mathbf{c}^{*\mathrm{T}}\Bigg(A^-(\widetilde{D}^+ - \widetilde{D}^0)E_i - \\ - \sum_l \zeta(x_i - x_l + \eta)A^-E_l + \sum_{l\neq i}\zeta(x_i - x_l)A^-E_l\Bigg)\mathbf{c}\Bigg) = \\ = \operatorname*{res}_{\infty}\Bigg(z^m\mathbf{c}^{*\mathrm{T}}\Bigg(\dot{X}^{-1}(\widetilde{D}^+ - \widetilde{D}^0)E_i - \\ - \dot{X}^{-1}\sum_l \zeta(x_i - x_l + \eta)E_l + \dot{X}^{-1}\sum_{l\neq i}\zeta(x_i - x_l)E_l\Bigg)\mathbf{c}\Bigg).$$

Collecting everything together, we finally find

$$\partial_{t_m}p_i = \operatorname*{res}_{\infty}\Bigg(z^m k^{-1}\mathbf{c}^{*\mathrm{T}}\Bigg(k\dot{X}^{-1}(\widetilde{D}^+ - \widetilde{D}^0)E_i - \\ - k\dot{X}^{-1}\sum_l \zeta(x_i - x_l + \eta)E_l + k\dot{X}^{-1}\sum_{l\neq i}\zeta(x_i - x_l)E_l + \\ + [E_i, B^-] + 2k\dot{X}^{-1}\sum_{l\neq i}E_l\zeta(x_l - x_i) - \\ - k\dot{X}^{-1}\sum_l E_l\zeta(x_l - x_i + \eta) - k\dot{X}^{-1}\sum_l E_l\zeta(x_l - x_i - \eta)\Bigg)\mathbf{c}\Bigg) = \\ = \operatorname*{res}_{\infty}\Bigg(z^m k^{-1}\mathbf{c}^{*\mathrm{T}}\Bigg([E_i, B^-] + k\dot{X}^{-1}(\widetilde{D}^+ - \widetilde{D}^0)E_i + \\ + k\dot{X}^{-1}\sum_{l\neq i}\zeta(x_l - x_i)E_l - k\dot{X}^{-1}\sum_l \zeta(x_l - x_i + \eta)E_l\Bigg)\mathbf{c}\Bigg) =$$

$$
\begin{aligned}
&= \operatorname*{res}_{\infty}\Bigg(z^m k^{-1}\mathbf{c}^{*\mathrm{T}}\Bigg([E_i, B^-] + A^-(\widetilde{D}^+ - \widetilde{D}^0)E_i + \\
&\qquad + \sum_{l\neq i}\zeta(x_l - x_i)E_l A^- - \sum_{l}\zeta(x_l - x_i + \eta)E_l A^-\Bigg)\mathbf{c}\Bigg) = \\
&= \operatorname*{res}_{\infty}(z^m k^{-1}\mathbf{c}^{*\mathrm{T}} Y_i \mathbf{c}),
\end{aligned}
$$

where Y_i is the right-hand side of (6.9). Therefore, the second half of the Hamiltonian equations is proved.

We comment on how the result is to be modified if we consider a more general tau function of the form

$$
\tau(x, \mathbf{t}) = e^{Q(x,\mathbf{t})}\prod_{i=1}^{N}\sigma(x - x_i(\mathbf{t})), \tag{6.18}
$$

where

$$
Q(x, \mathbf{t}) = cx^2 + x\sum_{j\geq 1} a_j t_j + b(\mathbf{t}) \tag{6.19}
$$

with some constants c, a_j. (Here, we set $\bar{\mathbf{t}} = 0$ for simplicity.) Repeating the arguments leading to (3.12), we can see that the first equation in (3.12) is modified as

$$
k = ze^{\eta\zeta(\lambda) - \alpha(z)}, \qquad \alpha(z) = \sum_{j\geq 1}\frac{a_j}{j}z^{-j}. \tag{6.20}
$$

Instead of (6.6), we then have

$$
\partial_{p_i}\ln k = -\frac{\partial_{p_i}\lambda(z)}{z\lambda'(z)}(1 - z\alpha'(z)), \tag{6.21}
$$

and hence the Hamiltonian for the mth flow is a linear combination of H_m and the H_j with $1 \leq j < m$.

6.2. Negative times. We first obtain the relation between the velocities $\dot{x}_i = \partial_{t_1} x_i$ and $x_i' = \partial_{\bar{t}_1} x_i$. The relation follows from Toda equation (2.16), where we substitute the tau function (1.1) for elliptic solutions:

$$
-\sum_i \dot{x}_i'\zeta(x - x_i) - \sum_i \dot{x}_i x_i'\wp(x - x_i) = 1 - \prod_i \frac{\sigma(x - x_i + \eta)\sigma(x - x_i - \eta)}{\sigma^2(x - x_i)}.
$$

Equating the coefficients in front of the second-order poles, we obtain

$$
\dot{x}_i x_i' = -\sigma^2(\eta)\prod_{j\neq i}\frac{\sigma(x_i - x_j + \eta)\sigma(x_i - x_j - \eta)}{\sigma^2(x_i - x_j)}. \tag{6.22}
$$

Our strategy is to solve linear problems (2.21) for the complementary wave functions ϕ and $\phi^\dagger$ represented as linear combinations of the elementary double-Bloch functions,

$$
\begin{aligned}
\phi(x) &= \tilde{k}^{x/\eta} e^{\xi(\mathbf{t}, z^{-1})}\sum_i b_i \Phi(x - x_i + \eta, -\mu), \\
\phi^\dagger(x) &= \tilde{k}^{-x/\eta} e^{-\xi(\mathbf{t}, z^{-1})}\sum_i b_i^* \Phi(x - x_i - \eta, \mu),
\end{aligned} \tag{6.23}
$$

where μ and $\tilde{k}$ are new spectral parameters to be connected with λ and k later, and b_i and b_i^* are independent of x. Identifying the monodromy properties of functions (6.23) and (2.20), we obtain the relations between $\tilde{k}$, z, and μ in the same way as relations (3.12) and (3.17) were obtained:

$$
\tilde{k} = ze^{-\eta\zeta(\mu)}, \qquad \mu = -\sum_{m\geq 1}\frac{z^m}{m}\partial_{\bar{t}_m}\sum_i x_i. \tag{6.24}
$$

It then follows that

$$\tilde{k} = ke^{-\eta\zeta(\lambda)-\eta\zeta(\mu)}. \tag{6.25}$$

We note that the spectral parameters $\tilde{k}$ and z here have a different meaning than in (3.12) and (3.17): they are local parameters on the spectral curve in the vicinity of the point P_0, while in (3.12) and (3.17) k^{-1} and z^{-1} are local parameters in the vicinity of P_∞.

The substitution of (6.23) in (2.21) with

$$\bar{v}(x) = \sum_i x_i'(\zeta(x-x_i) - \zeta(x-x_i+\eta))$$

gives, after the cancelation of poles at $x = x_i$, the following conditions:

$$\begin{aligned} x_i' \sum_j b_j \Phi(x_i - x_j + \eta, -\mu) &= \tilde{k}^{-1} b_i, \\ x_i' \sum_j b_j^* \Phi(x_i - x_j - \eta, \mu) &= -\tilde{k}^{-1} b_i^*. \end{aligned} \tag{6.26}$$

Using identity (3.6), we can write them in matrix form as follows:

$$\mathbf{b}^{\mathrm{T}} X'^{-1} \bar{L}(\mu) X' = -\tilde{k}^{-1}\mathbf{b}^{\mathrm{T}}, \qquad \bar{L}(\mu)\mathbf{b}^* = -\tilde{k}^{-1}\mathbf{b}^*. \tag{6.27}$$

Here, the matrix $\bar{L}(\mu)$ is

$$\bar{L}_{ij}(\mu) = x_i'\Phi(x_i - x_j - \eta, \mu) = -\sigma^2(\eta)e^{-p_i}\Phi(x_i - x_j - \eta, \mu)\prod_{l\neq i}\frac{\sigma(x_i - x_l - \eta)}{\sigma(x_i - x_l)}, \tag{6.28}$$

where we use relation (6.22).

Equations (6.27) allow writing the equation of the spectral curve in the form

$$\det(\tilde{k}^{-1}I + \bar{L}(\mu)) = 0. \tag{6.29}$$

We show that it is the same spectral curve as the one given by (5.1) and discussed in Sec. 5. To show this, we find the inverse of the Lax matrix $L(\lambda)$ given by (4.2). We have $(L^{-1}(\lambda))_{kl} = (-1)^{k+l}$ $(\mathrm{minor}_{lk})/\det L(\lambda)$ and use the fact that both the numerator and the denominator here are determinants of the elliptic Cauchy matrices given explicitly by (5.4). In particular,

$$\det L(\lambda) = e^{N\eta\zeta(\lambda)}\left(\prod_i \dot{x}_i\right)\frac{\sigma(\lambda - N\eta)}{\sigma(\lambda)\sigma^N(-\eta)}\prod_{i\neq j}\frac{\sigma(x_i - x_j)}{\sigma(x_i - x_j - \eta)}. \tag{6.30}$$

After some simple transformations, we obtain

$$\begin{aligned} (L^{-1}(\lambda))_{kl} = -e^{(x_l - x_k - \eta)\zeta(\lambda)} x_l' \frac{\sigma(N\eta - \lambda + x_l - x_k - \eta)}{\sigma(N\eta - \lambda)\sigma(x_l - x_k - \eta)} \times \\ \times \prod_{i\neq l}\frac{\sigma(x_l - x_i)}{\sigma(x_l - x_i + \eta)}\prod_{j\neq k}\frac{\sigma(x_k - x_j + \eta)}{\sigma(x_k - x_j)}. \end{aligned} \tag{6.31}$$

Hence,

$$e^{\eta\zeta(\lambda)}(L^{\mathrm{T}}(\lambda))^{-1} \cong -e^{-\eta\zeta(\mu)}\bar{L}(\mu), \qquad \mu = N\eta - \lambda, \tag{6.32}$$

where $\cong$ means equality up to a similarity transformation. Taking Eqs. (6.32) and (6.25) into account, we write spectral curve (6.29) in the form

$$\det(k^{-1}I - L^{-1}(\lambda)) = 0,$$

which is the same as (5.1). However, we expand the spectral curve in the vicinity of different points in the previous sections and here: it was the point $P_\infty = (\infty, 0)$ previously and is $P_0 = (0, N\eta)$ now.

Using Eq. (2.25) and repeating the calculations leading to (6.3) and (6.5), we obtain the relations

$$\partial_{\bar{t}_m} x_i = -\operatorname*{res}_0 (z^{-m} \tilde{k}^{-1} \mathbf{b}^{\mathrm{T}} X'^{-1}\, \partial_{p_i} \bar{L}(\mu) \mathbf{b}^*) \tag{6.33}$$

and

$$\mathbf{b}^{\mathrm{T}} X'^{-1} \mathbf{b}^* = \tilde{k}^2 \mu'(z). \tag{6.34}$$

A similar chain of equalities as the one after (6.35) leads to

$$\partial_{\bar{t}_m} x_i = \operatorname*{res}_0 (z^{-m-1}\, \partial_{p_i} \mu(z)) = \frac{\partial \bar{H}_m}{\partial p_i}, \tag{6.35}$$

where

$$\bar{H}_m = \operatorname*{res}_0 (z^{-m-1} \mu(z)) = -\operatorname*{res}_0 (z^{-m-1} \lambda(z)). \tag{6.36}$$

We see that the Hamiltonians for the negative-time flows are obtained from the expansion of $\lambda(z)$ as $z \to 0$:

$$\lambda(z) = N\eta - \sum_{m \ge 1} \bar{H}_m z^m. \tag{6.37}$$

The other half of the Hamiltonian equations,

$$\partial_{\bar{t}_m} p_i = -\frac{\partial \bar{H}_m}{\partial x_i}, \tag{6.38}$$

can be proved in the same way as Eqs. (6.11) were proved.

7. Degenerations of elliptic solutions

7.1. Rational limit. In the rational limit $\omega, \omega' \to \infty$, $\sigma(x) = x$ and

$$L_{ij}(\lambda) = \left(\frac{\dot{x}_i}{x_i - x_j - \eta} + \frac{\dot{x}_i}{\lambda} \right) e^{-(x_i - x_j)/\lambda + \eta/\lambda}, \tag{7.1}$$

where

$$\dot{x}_i = e^{p_i} \prod_{l \ne i} \frac{x_i - x_l + \eta}{x_i - x_l}.$$

The equation of the spectral curve is

$$\det(kI - e^{\eta/\lambda}(L_{\mathrm{rat}} + \lambda^{-1} \dot{X} E)) = 0,$$

where L_{rat} is the Lax matrix of the rational Ruijsenaars–Schneider system with matrix elements

$$(L_{\mathrm{rat}})_{ij} = \frac{\dot{x}_i}{x_i - x_j - \eta}. \tag{7.2}$$

Recalling the connection between the spectral parameters k, z, and λ, $k = ze^{\eta/\lambda}$, we can write the equation of the spectral curve in the form

$$\det(zI - L_{\mathrm{rat}} - \lambda^{-1}\dot{X}E) = 0. \tag{7.3}$$

Because E is a rank-1 matrix, we have

$$\det\biggl(I - \lambda^{-1}\dot{X}E\,\frac{1}{zI - L_{\mathrm{rat}}}\biggr) = 1 - \lambda^{-1}\operatorname{tr}\biggl(\dot{X}E\frac{1}{zI - L_{\mathrm{rat}}}\biggr) = 0,$$

whence

$$\lambda = \operatorname{tr}\biggl(\dot{X}E\frac{1}{zI - L_{\mathrm{rat}}}\biggr). \tag{7.4}$$

As $z \to \infty$, we expand this as

$$\lambda(z) = \sum_{m\geq 1} z^{-m}\operatorname{tr}(\dot{X}EL_{\mathrm{rat}}^{m-1}). \tag{7.5}$$

It is easy to verify the commutation relation

$$XL_{\mathrm{rat}} - L_{\mathrm{rat}}X = \dot{X}E + \eta L_{\mathrm{rat}}. \tag{7.6}$$

Using it, we have

$$\operatorname{tr}(\dot{X}EL_{\mathrm{rat}}^{m-1}) = \operatorname{tr}(XL_{\mathrm{rat}}^{m} - L_{\mathrm{rat}}XL_{\mathrm{rat}}^{m-1} - \eta L_{\mathrm{rat}}^{m}) = -\eta\operatorname{tr} L_{\mathrm{rat}}^{m}.$$

We hence conclude that

$$\lambda(z) = -\eta\sum_{m\geq 1} z^{-m}\operatorname{tr} L_{\mathrm{rat}}^{m}, \qquad z \to \infty, \tag{7.7}$$

and thus the Hamiltonians for positive-time flows are

$$H_m = -\eta\operatorname{tr} L_{\mathrm{rat}}^{m}. \tag{7.8}$$

This agrees with the result in [18], [19].

To find the Hamiltonians for negative-time flows, we expand (7.4) as $z \to 0$:

$$\lambda(z) = -\sum_{m\geq 0} z^{m}\operatorname{tr}(\dot{X}EL_{\mathrm{rat}}^{-m-1}) = N\eta + \eta\sum_{m\geq 1} z^{m}\operatorname{tr} L_{\mathrm{rat}}^{-m}. \tag{7.9}$$

Therefore,

$$\bar{H}_m = -\eta\operatorname{tr} L_{\mathrm{rat}}^{-m}. \tag{7.10}$$

This agrees with the result in [20].

7.2. Trigonometric limit. We now pass to the trigonometric limit. Let $\pi i/\gamma$ be the period of the trigonometric (or hyperbolic) functions, with the second period tending to infinity. The Weierstrass functions in this limit are

$$\sigma(x) = \gamma^{-1}e^{-\gamma^2x^2/6}\sinh(\gamma x), \qquad \zeta(x) = \gamma\coth(\gamma x) - \frac{1}{3}\gamma^2 x.$$

The tau function for trigonometric solutions is

$$\tau = \prod_{i=1}^{N}(e^{2\gamma x} - e^{2\gamma x_i}), \tag{7.11}$$

and hence we should consider

$$\tau = \prod_{i=1}^{N} \sigma(x - x_i) e^{\gamma^2 (x - x_i)^2/6 + \gamma(x + x_i)}, \tag{7.12}$$

where the exponential factor is inserted to ensure the correct trigonometric limit (7.11) of the elliptic tau function. Similarly to the KP case [13], Eq. (3.12) with this choice acquires the form

$$\ln k = \ln z + \eta\gamma \coth(\gamma\lambda). \tag{7.13}$$

The trigonometric limit of the function $\Phi(x, \lambda)$ is

$$\Phi(x, \lambda) = \gamma(\coth(\gamma x) + \coth(\gamma\lambda)) e^{-\gamma x \coth(\gamma\lambda)},$$

and $L(\lambda)$ takes the form

$$L_{ij}(\lambda) = \gamma e^{\eta\gamma \coth(\gamma\lambda)} e^{-\gamma \coth(\gamma\lambda)(x_i - x_j)} (\dot{x}_i \coth(\gamma(x_i - x_j - \eta)) + \dot{x}_i \coth(\gamma\lambda)).$$

For further calculations, it is convenient to change the variables as

$$w_i = e^{2\gamma x_i}, \qquad q = e^{2\gamma\eta} \tag{7.14}$$

and introduce the diagonal matrix $W = \mathrm{diag}(w_1, w_2, \ldots, w_N)$. In this notation, the equation of the spectral curve acquires the form

$$\det(zI - q^{-1/2} W^{1/2} L_{\mathrm{trig}} W^{-1/2} - \gamma(\coth(\gamma\lambda) - 1)\dot{X}E) = 0, \tag{7.15}$$

where L_{trig} is the Lax matrix of the trigonometric Ruijsenaars–Schneider model:

$$(L_{\mathrm{trig}})_{ij} = 2\gamma q^{1/2} \frac{\dot{x}_i w_i^{1/2} w_j^{1/2}}{w_i - q w_j} \tag{7.16}$$

(see [20]). Again, because E is a rank-1 matrix, we can use (7.15) to obtain

$$(\coth(\gamma\lambda) - 1)^{-1} = \gamma \,\mathrm{tr}\left(\dot{X}E \frac{1}{zI - q^{-1/2} W^{1/2} L_{\mathrm{trig}} W^{-1/2}}\right)$$

or

$$\lambda = \frac{1}{2\gamma} \ln\left(1 + 2\gamma \,\mathrm{tr}\left(\dot{X}E \frac{1}{zI - q^{-1/2} W^{1/2} L_{\mathrm{trig}} W^{-1/2}}\right)\right).$$

Applying the formula $\det(I + Z) = 1 + \mathrm{tr}\, Z$ for any matrix Z of rank 1 in the opposite direction, we have

$$\lambda = \frac{1}{2\gamma} \ln\det\biggl((zI - q^{-1/2} W^{1/2} L_{\mathrm{trig}} W^{-1/2} + 2\gamma\dot{X}E) \times \\ \times \frac{1}{zI - q^{-1/2} W^{1/2} L_{\mathrm{trig}} W^{-1/2}}\biggr). \tag{7.17}$$

Next, we use the trigonometric analogue of relation (7.6),

$$2\gamma\dot{X}E = q^{-1/2} W^{1/2} L_{\mathrm{trig}} W^{-1/2} - q^{1/2} W^{-1/2} L_{\mathrm{trig}} W^{1/2}, \tag{7.18}$$

which can be verified easily. Using this relation, we deduce from (7.17) that

$$\lambda = \frac{1}{2\gamma}\ln\det(zI - q^{1/2}W^{-1/2}L_{\rm trig}W^{1/2}) - \frac{1}{2\gamma}\ln\det(zI - q^{-1/2}W^{1/2}L_{\rm trig}W^{-1/2}) =$$
$$= \frac{1}{2\gamma}\ln\det\frac{1 - z^{-1}q^{1/2}L_{\rm trig}}{1 - z^{-1}q^{-1/2}L_{\rm trig}} = -\sum_{m\geq 1} z^{-m}\frac{q^{m/2} - q^{-m/2}}{2\gamma m}\,{\rm tr}\,L_{\rm trig}^m.$$

Therefore, we finally obtain

$$\lambda(z) = -\sum_{m\geq 1} z^{-m}\frac{\sinh(\gamma\eta m)}{\gamma m}\,{\rm tr}\,L_{\rm trig}^m, \tag{7.19}$$

and hence the Hamiltonians are

$$H_m = -\frac{\sinh(\gamma\eta m)}{\gamma m}\,{\rm tr}\,L_{\rm trig}^m. \tag{7.20}$$

This agrees with the result in [20].

8. Examples of the Hamiltonians

We return to the general elliptic case and introduce renormalized integrals of motion

$$J_m = \frac{\sigma(m\eta)}{\sigma^m(\eta)}I_m, \qquad m = \pm 1, \pm 2, \ldots, \pm N, \tag{8.1}$$

where I_m are the integrals of motion in (1.3) and (1.7). The equation of the spectral curve, Eq. (5.5), is

$$z^N + \sum_{n=1}^{N}\phi_n(\lambda)J_n z^{N-n} = 0, \tag{8.2}$$

where

$$\phi_n(\lambda) = \frac{\sigma(\lambda - n\eta)}{\sigma(\lambda)\sigma(n\eta)}. \tag{8.3}$$

It can be expanded as $\lambda \to 0$ ($z \to \infty$) as follows:

$$\phi_n(\lambda) = -\frac{1}{\sigma(\lambda)} + \zeta(n\eta)\,\frac{\lambda}{\sigma(\lambda)} - (\zeta^2(n\eta) - \wp(n\eta))\,\frac{\lambda^2}{2\sigma(\lambda)} + O(z^{-2}).$$

Expanding the equation of the spectral curve, we have

$$1 + \zeta(\eta)J_1 z^{-1} - \frac{1}{2}(\zeta^2(\eta) - \wp(\eta))J_1^2 z^{-2} + \zeta(2\eta)J_2 z^{-2} = \frac{1}{\lambda z}(J_1 + J_2 z^{-1} + J_3 z^{-2}) + O(z^{-3})$$

or

$$\lambda(z) = \frac{z^{-1}(J_1 + J_2 z^{-1} + J_3 z^{-2} + O(z^{-3}))}{1 + \zeta(\eta)J_1 z^{-1} + \zeta(2\eta)J_2 z^{-2} - (1/2)(\zeta^2(\eta) - \wp(\eta))J_1^2 z^{-2} + O(z^{-3})}.$$

Expanding this in a series, $\lambda(z) = \sum_{m\geq 1} H_m z^{-m}$, we find the first three Hamiltonians:

$$\begin{aligned} H_1 &= J_1, \\ H_2 &= J_2 - \zeta(\eta)J_1^2, \\ H_3 &= J_3 - (\zeta(\eta) + \zeta(2\eta))J_1J_2 + \left(\frac{3}{2}\zeta^2(\eta) - \frac{1}{2}\wp(\eta)\right)J_1^3. \end{aligned} \tag{8.4}$$

In the rational limit, we can obtain a general formula for H_m. In this limit,

$$\phi_n(\lambda) = \frac{1}{n\eta} - \frac{1}{\lambda}$$

and the equation of the spectral curve becomes

$$z^N + \frac{1}{\eta}\sum_{n=1}^{N} \frac{J_n}{n} z^{N-n} = \frac{1}{\lambda}\sum_{n=1}^{N} J_n z^{N-n}, \tag{8.5}$$

or, recalling that $J_n = n\eta^{1-n} I_n$ in the rational limit,

$$\lambda(z) = \eta \sum_{n=1}^{N} nI_n(\eta z)^{-n} \Big/ \Big[1 + \sum_{n=1}^{N} I_n(\eta z)^{-n}\Big]. \tag{8.6}$$

Expanding the right-hand side in powers of z^{-1}, we obtain

$$\begin{aligned} H_1 &= I_1, \\ H_2 &= \eta^{-1}(2I_2 - I_1^2), \\ H_3 &= \eta^{-2}(3I_3 - 3I_1I_2 + I_1^3). \end{aligned} \tag{8.7}$$

In general, it follows from (8.6) that

$$H_m = (-\eta)^{1-m} \sum_{|\nu|=m} C_\nu^m I_{\nu_1} \cdots I_{\nu_\ell}, \tag{8.8}$$

where the sum is taken over all Young diagrams ν with $|\nu| = m$ boxes having nonzero rows $\nu_1, \ldots, \nu_\ell$, and C_ν^m is the matrix of transition from the basis of elementary symmetric polynomials e_j to the power sums p_k. Indeed, it is easy to see that the equality

$$\sum_{n=1}^{N}(-1)^{n-1} p_n z^{-n} = \sum_{m=1}^{N} m e_m z^{N-m} \Big/ \sum_{r=0}^{N} e_r z^{N-r}$$

is equivalent to the well-known Newton identity

$$m e_m = \sum_{n=1}^{m}(-1)^{n-1} p_n e_{m-n}$$

for the symmetric functions. The explicit formula is

$$H_m = -m\eta^{1-m} \sum_{\substack{r_1+2r_2+\cdots+mr_m=m \\ r_1\ge 0,\ldots,r_m\ge 0}} \frac{(r_1+\cdots+r_m-1)!}{r_1!\ldots r_m!} \prod_{i=1}^{m}(-I_i)^{r_i}. \tag{8.9}$$

We now consider Hamiltonians for the negative-time flows. The equation of the spectral curve (5.5) can be rewritten in the form

$$\varphi_N(\lambda) + \sum_{n=1}^{N} \varphi_{N-n}(\lambda) I_{-n} z^n = 0, \tag{8.10}$$

or, equivalently,

$$\sigma(\mu)+\sum_{n=1}^{N}\frac{\sigma(n\eta-\mu)}{\sigma(n\eta)}J_{-n}z^n=0, \qquad \mu\equiv N\eta-\lambda. \tag{8.11}$$

Expanding $\mu \to 0$ in powers of z as $\mu(z)=\sum_{m\geq 1}\bar{H}_m z^m$, we obtain

$$\begin{aligned}
&\bar{H}_1=-J_{-1},\\
&\bar{H}_2=-J_{-2}-\zeta(\eta)J_{-1}^2,\\
&\bar{H}_3=-J_{-3}-(\zeta(\eta)+\zeta(2\eta))J_{-1}J_{-2}-\left(\frac{3}{2}\zeta^2(\eta)-\frac{1}{2}\wp(\eta)\right)J_{-1}^3.
\end{aligned} \tag{8.12}$$

In the rational limit, we have

$$\begin{aligned}
&\bar{H}_1=\eta^2 I_{-1},\\
&\bar{H}_2=\eta^3(2I_{-2}-I_{-1}^2),\\
&\bar{H}_3=\eta^4(3I_{-3}-3I_{-1}I_{-2}+I_{-1}^3).
\end{aligned} \tag{8.13}$$

We can see that the properly arranged limit $\eta \to 0$ in (8.7) of the Hamiltonians for positive-time flows yields Hamiltonians of the Calogero–Moser model, while Hamiltonians (8.13) for negative-time flows vanish in this limit.

9. Conclusion

The main result in this paper is the precise correspondence between elliptic solutions of the 2D Toda lattice hierarchy and the hierarchy of the Hamiltonian equations for the integrable elliptic Ruijsenaars–Schneider model with higher Hamiltonians. The zeros of the tau function move as particles of the Ruijsenaars–Schneider model. We have shown that the mth-time flow t_m of the 2DTL hierarchy gives rise to the flow with the Hamiltonian H_m of the Ruijsenaars–Schneider model, obtained as the mth coefficient of the expansion of the spectral curve $\lambda(z)$ in negative powers of z as $z \to \infty$. Moreover, the mth-time flow $\bar{t}_m$ of the 2DTL hierarchy corresponds to the flow with the Hamiltonian $\bar{H}_m$ of the Ruijsenaars–Schneider model. obtained as the mth coefficient of the expansion of the spectral curve $\lambda(z)$ in positive powers of z as $z \to 0$. The first few Hamiltonians were found explicitly.

For rational and trigonometric degenerations of elliptic solutions, the previous results in [18]–[20], obtained there by a different method, are reproduced here: the t_m time flow of the 2DTL hierarchy gives rise to the flow with the Hamiltonian H_m proportional to $\operatorname{tr} L^m$, where L is the Lax matrix, while the $\bar{t}_m$ time flow gives rise to the Hamiltonian flow with the Hamiltonian $\bar{H}_m$ proportional to $\operatorname{tr} L^{-m}$.

Conflict of interest. The authors declare no conflicts of interest.

REFERENCES

1. K. Ueno and K. Takasaki, "Toda lattice hierarchy," in: *Group Representations and Systems of Differential Equations* (University of Tokyo, December 20–27, 1982), Advanced Studies in Pure Mathematics, Vol. 4 (K. Okamoto, ed.), North-Holland, Amsterdam–New York–Oxford (1984), pp. 1–95.
2. H. Airault, H. P. McKean, and J. Moser, "Rational and elliptic solutions of the Korteweg–de Vries equation and a related many-body problem," *Commun. Pure Appl. Math.*, **30**, 95–148 (1977).
3. F. Calogero, "Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials," *J. Math. Phys.*, **12**, 419–436 (1971).
4. F. Calogero, "Exactly solvable one-dimensional many-body problems," *Lett. Nuovo Cimento*, **13**, 411–415 (1975).

5. J. Moser, "Three integrable Hamiltonian systems connected with isospectral deformations," *Adv. Math.*, **16**, 197–220 (1975).
6. M. A. Olshanetsky and A. M. Perelomov, "Classical integrable finite-dimensional systems related to Lie algebras," *Phys. Rep.*, **71**, 313–400 (1981).
7. I. M. Krichever, "Rational solutions of the Kadomtsev–Petviashvili equation and integrable systems of N particles on a line," *Funct. Anal. Appl.*, **12**, 59–61 (1978).
8. D. V. Chudnovsky and G. V. Chudnovsky, "Pole expansions of nonlinear partial differential equations," *Nuovo Cimento B*, **40**, 339–350 (1977).
9. I. M. Krichever, "Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles," *Funct. Anal. Appl.*, **14**, 282–290 (1980).
10. T. Shiota, "Calogero–Moser hierarchy and KP hierarchy," *J. Math. Phys.*, **35**, 5844–5849 (1994).
11. L. Haine, "KP trigonometric solitons and an adelic flag manifold," *SIGMA*, **3**, 015, 15 pp. (2007), arXiv: nlin.SI/0701054.
12. A. Zabrodin, "KP hierarchy and trigonometric Calogero–Moser hierarchy," *J. Math. Phys.*, **61**, 043502, 13 pp. (2020), arXiv: 1906.09846.
13. V. Prokofev and A. Zabrodin, "Elliptic solutions to the KP hierarchy and elliptic Calogero–Moser model," arXiv: 2102.03784.
14. I. M. Krichever and A. V. Zabrodin, "Spin generalization of the Ruijsenaars–Schneider model, the non-Abelian Toda chain, and representations of the Sklyanin algebra," *Russian Math. Surveys*, **50**, 1101–1150 (1995).
15. A. Zabrodin, "Elliptic solutions to integrable nonlinear equations and many-body systems," *J. Geom. Phys.*, **146**, 103506, 26 pp. (2019), arXiv: 1905.11383.
16. S. N. M. Ruijsenaars and H. Schneider, "A new class of integrable systems and its relation to solitons," *Ann. Phys.*, **170**, 370–405 (1986).
17. S. N. M. Ruijsenaars, "Complete integrability of relativistic Calogero–Moser systems and elliptic function identities," *Commun. Math. Phys.*, **110**, 191–213 (1987).
18. P. Iliev, "Rational Ruijsenaars–Schneider hierarchy and bispectral difference operators," *Phys. D*, **229**, 184–190 (2007), arXiv: math-ph/0609011.
19. A. Zabrodin, "The master T-operator for inhomogeneous XXX spin chain and mKP hierarchy," *SIGMA*, **10**, 006, 18 pp. (2014), arXiv: 1310.6988.
20. V. Prokofev and A. Zabrodin, "Toda lattice hierarchy and trigonometric Ruijsenaars–Schneider hierarchy," *J. Phys. A: Math. Theor.*, **52**, 495202, 21 pp. (2019), arXiv: 1907.06621.
21. A. V. Zabrodin, "Matrix modified Kadomtsev–Petviashvili hierarchy," *Theoret. and Math. Phys.*, **199**, 771–783 (2019).
22. E. Date, M. Kashiwara, M. Jimbo, and T. Miwa, "Transformation groups for soliton equations," in: *Non-Linear Integrable Systems – Classical Theory and Quantum Theory* (Kyoto, Japan, May 13–16, 1981) (M. Satō, M. Jimbo, and T. Miwa, eds.), World Sci., Singapore (1983), pp. 39–119.
23. M. Jimbo and T. Miwa, "Solitons and infinite dimensional Lie algebras," *Publ. Res. Inst. Math. Sci. Kyoto*, **19**, 943–1001 (1983).

Appendix E

V.Prokofev, A Zabrodin "Elliptic solutions to matrix KP hierarchy and spin generalization of elliptic Calogero-Moser model"
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Contribution: I suggested two main ideas in this paper. First one is the same as in KP and 2d Toda lattice cases, the other is that τ-functions $\tau_{\alpha\beta}(x)$ have form (3.3) and (3.4). Using them it became possible to write systems (4.1) (4.2) and obtain final answer for generation function (6.7) as a sum over sheets. Besides that I conducted all calculations in this paper independently.

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V. Prokofev[1,2,a)] and A. Zabrodin[2,3,4,b)]

AFFILIATIONS

[1] Moscow Institute of Physics and Technology, Dolgoprudny, Institutsky per., 9, Moscow Region 141700, Russia
[2] Skolkovo Institute of Science and Technology, 143026 Moscow, Russian Federation
[3] National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russian Federation
[4] Institute of Biochemical Physics, Kosygina Str. 4, Moscow 119334, Russian Federation

[a)] Author to whom correspondence should be addressed: vadim.prokofev@phystech.edu
[b)] zabrodin@itep.ru

ABSTRACT

We consider solutions of the matrix Kadomtsev–Petviashvili (KP) hierarchy that are elliptic functions of the first hierarchical time $t_1 = x$. It is known that poles x_i and matrix residues at the poles $\rho_i^{\alpha\beta} = a_i^\alpha b_i^\beta$ of such solutions as functions of the time t_2 move as particles of spin generalization of the elliptic Calogero–Moser model (elliptic Gibbons–Hermsen model). In this paper, we establish the correspondence with the spin elliptic Calogero–Moser model for the whole matrix KP hierarchy. Namely, we show that the dynamics of poles and matrix residues of the solutions with respect to the kth hierarchical time of the matrix KP hierarchy is Hamiltonian with the Hamiltonian H_k obtained via an expansion of the spectral curve near the marked points. The Hamiltonians are identified with the Hamiltonians of the elliptic spin Calogero–Moser system with coordinates x_i and spin degrees of freedom a_i^α, b_i^β.

I. INTRODUCTION

The Kadomtsev–Petviashvili (KP) hierarchy is an archetypal infinite hierarchy of compatible nonlinear differential equations with infinitely many independent (time) variables $\mathbf{t} = \{t_1, t_2, t_3, \ldots\}$. In the Lax–Sato formalism, the main object is the Lax operator, which is a pseudo-differential operator of the form

$$\mathcal{L} = \partial_x + u_1 \partial_x^{-1} + u_2 \partial_x^{-2} + \cdots. \tag{1.1}$$

The coefficient functions u_i are dependent variables. Among all solutions to equations of the hierarchy, of special interest are solutions that have a finite number of poles in the variable x in a fundamental domain of the complex plane. Most general solutions of this type are those for which the coefficient functions u_i are elliptic (double-periodic in the complex plane) functions of x with poles depending on the times $\mathbf{t}$.

The study of singular solutions to nonlinear integrable equations and dynamics of their poles was initiated in the pioneering papers.[1–4] Now, it is a well-known subject in the theory of integrable systems. The remarkable result is that the poles of solutions to the KP equation as functions of the time t_2 move as particles of the integrable Calogero–Moser many-body system,[5–8] which is known to be integrable, i.e., having a large number of integrals of motion in involution. Elliptic, trigonometric, and rational solutions correspond to the Calogero–Moser systems with elliptic, trigonometric, and rational potentials, respectively.

In the work of Ref. 9, Shiota showed that in the case of rational solutions, the correspondence between the KP equation and the rational Calogero–Moser system can be extended to the whole KP hierarchy. Namely, the evolution of poles with respect to the higher time t_m was considered, and it was shown that it is described by the higher Hamiltonian flow of the rational Calogero–Moser system with the Hamiltonian $H_m = \operatorname{tr} L^m$, where L is the Lax matrix depending on the coordinates and momenta in a special way. Recently, this remarkable correspondence was generalized to trigonometric and elliptic solutions to the KP hierarchy (see Refs. 10–12). However, in the elliptic case, this correspondence is no longer formulated in terms of traces of the Lax matrix (which, in this case, depends on a spectral parameter). Instead, the Hamiltonian H_m that governs the dynamics of poles with respect to t_m is shown to be obtained by the expansion of the Calogero–Moser spectral curve near a distinguished marked point. It was also shown in Ref. 12 that for trigonometric and rational degenerations of elliptic solutions, this construction gives the results, which agree with the previously obtained ones for trigonometric and rational solutions.

There exists a matrix generalization of the KP hierarchy (matrix KP hierarchy). In the matrix KP hierarchy, the coefficient functions u_i in the Lax operator (1.1) are $n \times n$ matrices. Like the KP hierarchy, it is an infinite set of compatible nonlinear differential equations with infinitely many independent variables $\mathbf{t}$ and matrix dependent variables. It is a subhierarchy of a more general multi-component (n-component) KP hierarchy,[13–16] which has an extended set of independent variables $\{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n\}$, $\mathbf{t}_\alpha = \{t_{\alpha,1}, t_{\alpha,2}, t_{\alpha,3}, \ldots\}$, $\alpha = 1, \ldots, n$. The matrix KP hierarchy is obtained by the restriction $t_{\alpha,m} = t_m$ for all α, m.

The elliptic, trigonometric, and rational solutions to the matrix KP equation were investigated in Ref. 17. In the matrix case, the data of singular solutions include not only the positions of poles x_i but also some "internal degrees of freedom," which are matrix residues at the poles (they were fixed in the scalar case). In the work of Ref. 17, it was shown that the dynamics of the data of such solutions with respect to the time t_2 is isomorphic to the dynamics of a spin generalization of the Calogero–Moser system, which is also known as the Gibbons–Hermsen model.[18] It is a system of N particles with coordinates x_i and with internal degrees of freedom represented by n-dimensional column vectors $\mathbf{a}_i, \mathbf{b}_i$ with components a_i^α and b_i^α, $\alpha = 1, \ldots, n$. The rank 1 matrices $\rho_i = \mathbf{a}_i \mathbf{b}_i^T$, where $\mathbf{b}_i^T$ is the row vector obtained from the vector $\mathbf{b}_i$ by transposition, represent matrix residues at the poles x_i. The particles pairwise interact with each other. The Hamiltonian of the elliptic model is

$$H = \sum_{i=1}^{N} p_i^2 - \sum_{i \neq k} (\mathbf{b}_i^T \mathbf{a}_k)(\mathbf{b}_k^T \mathbf{a}_i) \wp(x_i - x_k), \tag{1.2}$$

where $\wp(x)$ is the Weierstrass $\wp$-function, which is the elliptic function with the only second order pole at $x = 0$ in the fundamental domain. The non-vanishing Poisson brackets are $\{x_i, p_k\} = \delta_{ik}$, $\{a_i^\alpha, b_k^\beta\} = \delta_{\alpha\beta}\delta_{ik}$. The model is known to be integrable, possessing the Lax representation with the Lax matrix $L(\lambda)$ depending on a spectral parameter λ lying on an elliptic curve.

The extension of the isomorphism between rational and trigonometric solutions of the matrix KP equation and the Gibbons–Hermsen system to the whole hierarchy was recently made in Ref. 19 for rational solutions and in Ref. 20 for trigonometric ones. In this paper, we generalize these results to elliptic solutions of the matrix KP hierarchy.

Our main result is that the dynamics of poles x_i and vectors $\mathbf{a}_i$ and $\mathbf{b}_i$, which parameterize matrix residues at the poles with respect to all higher times t_m of the matrix KP hierarchy, is Hamiltonian, with the corresponding Hamiltonians being higher Hamiltonians of the spin elliptic Calogero–Moser model. We find them in terms of expansion of the spectral curve

$$\det_{N \times N}\big((z + \zeta(\lambda))I - L(\lambda)\big) = 0 \tag{1.3}$$

[$\zeta(\lambda)$ is the Weierstrass ζ-function] near some distinguished marked points at infinity. The spectral curve is a covering of the elliptic curve where the spectral parameter λ lives. We show that above the point $\lambda = 0$, there are n points at infinity $P_\alpha = (\infty_\alpha, 0)$, where $z = \infty$, so that there are n distinguished sheets of the covering (neighborhoods of the points P_α). In a neighborhood of the point $\lambda = 0n$, different branches of the function $z(\lambda)$ such that $z(\lambda) \to \infty$ as $\lambda \to 0$ are defined by the equation of the spectral curve. Let us denote them by $z_\alpha(\lambda)$, and let $\lambda_\alpha(z)$ be inverse functions. Our main result is that the sum over all branches $\sum_{\alpha=1}^{n} \lambda_\alpha(z)$ is the generating function for the Hamiltonians H_m,

$$\sum_{\alpha=1}^{n} \lambda_\alpha(z) = -Nz^{-1} - \sum_{m \geq 1} z^{-m-1} H_m \tag{1.4}$$

or $H_m = -\sum_{\alpha=1}^{n} \operatorname{res}_{\infty}(z^m \lambda_\alpha(z))$. We also show that the degeneration of this construction to the rational and trigonometric cases allows one to reproduce the results of Refs. 19 and 20.

The organization of this paper is as follows. In Sec. II, we remind the reader the main facts about the multi-component and matrix KP hierarchies. We recall both the Lax–Sato approach based on Lax equations and the bilinear (Hirota) approach based on the bilinear relation for the tau-function. In Sec. III, we introduce elliptic solutions and discuss the corresponding double-Bloch solutions for the wave function. Section IV contains the derivation of the dynamics of poles and residues with respect to time t_2. Following Ref. 17, we derive the equations of motion together with their Lax representation. In Sec. V, we discuss the properties of the spectral curve and define the distinguished branches of the function $z(\lambda)$ around the point $\lambda = 0$. Sections VI and VII are devoted to the derivation of the Hamiltonian dynamics of poles and residues in the higher times t_m, respectively. In Sec. VIII, we find explicitly the first two Hamiltonians using expansion (1.4) and identify them with the Hamiltonians of the spin generalization of the Calogero–Moser system. Finally, in Sec. IX, we consider the rational and trigonometric degenerations of our construction and show that the results of the previous works are reproduced by the new approach.

II. THE MATRIX KP HIERARCHY

Here, we briefly review the main facts about the multi-component and matrix KP hierarchies following.[15,16] We start from the more general multi-component KP hierarchy. The independent variables are n infinite sets of continuous "times"

$$\mathbf{t} = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n\}, \qquad \mathbf{t}_\alpha = \{t_{\alpha,1}, t_{\alpha,2}, t_{\alpha,3}, \ldots\}, \qquad \alpha = 1, \ldots, n.$$

It is also convenient to introduce the variable x such that

$$\partial_x = \sum_{\alpha=1}^{n} \partial_{t_{\alpha,1}}. \tag{2.1}$$

The hierarchy is an infinite set of evolution equations in times $\mathbf{t}$ for matrix functions of the variable x.

In the Lax–Sato formalism, the main object is the Lax operator, which is a pseudo-differential operator of the form

$$\mathcal{L} = \partial_x + u_1\partial_x^{-1} + u_2\partial_x^{-2} + \cdots, \tag{2.2}$$

where the coefficients $u_i = u_i(x, \mathbf{t})$ are $n \times n$ matrices. The coefficient functions u_k depend on x and also on all the times

$$u_k(x, \mathbf{t}) = u_k(x + t_{1,1}, x + t_{2,1}, \ldots, x + t_{n,1}; t_{1,2}, \ldots, t_{n,2}; \ldots).$$

In addition, there are other matrix pseudo-differential operators $\mathcal{R}_1, \ldots, \mathcal{R}_n$ of the form

$$\mathcal{R}_\alpha = E_\alpha + u_{\alpha,1}\partial_x^{-1} + u_{\alpha,2}\partial_x^{-2} + \cdots, \tag{2.3}$$

where E_α is the $n \times n$ matrix with the (α, α) element equal to 1 and all other components equal to 0 and $u_{\alpha,i}$ are also $n \times n$ matrices. The operators $\mathcal{L}, \mathcal{R}_1, \ldots, \mathcal{R}_n$ satisfy the following conditions:

$$\mathcal{L}\mathcal{R}_\alpha = \mathcal{R}_\alpha\mathcal{L}, \quad \mathcal{R}_\alpha\mathcal{R}_\beta = \delta_{\alpha\beta}\mathcal{R}_\alpha, \quad \sum_{\alpha=1}^{n} \mathcal{R}_\alpha = I, \tag{2.4}$$

where I is the unity matrix. The Lax equations of the hierarchy that define evolution in the times read

$$\partial_{t_{\alpha,k}}\mathcal{L} = [B_{\alpha,k}, \mathcal{L}], \quad \partial_{t_{\alpha,k}}\mathcal{R}_\beta = [B_{\alpha,k}, \mathcal{R}_\beta], \quad B_{\alpha,k} = \left(\mathcal{L}^k\mathcal{R}_\alpha\right)_+, \quad k = 1, 2, 3, \ldots, \tag{2.5}$$

where $(\ldots)_+$ means the differential part of a pseudo-differential operator, i.e., the sum of all terms with ∂_x^k, where $k \geq 0$.

Let us introduce the matrix pseudo-differential "wave operator" $\mathcal{W}$ with matrix elements

$$\mathcal{W}_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{k\geq 1} \xi_{\alpha\beta}^{(k)}(x, \mathbf{t})\partial_x^{-k}, \tag{2.6}$$

where $\xi_{\alpha\beta}^{(k)}(x, \mathbf{t})$ are the some matrix functions. The operators $\mathcal{L}$ and $\mathcal{R}_\alpha$ are obtained from the "bare" operators $I\partial_x$ and E_α by "dressing" by means of the wave operator,

$$\mathcal{L} = \mathcal{W}\partial_x\mathcal{W}^{-1}, \qquad \mathcal{R}_\alpha = \mathcal{W}E_\alpha\mathcal{W}^{-1}. \tag{2.7}$$

Clearly, there is an ambiguity in the definition of the dressing operator: it can be multiplied from the right by any pseudo-differential operator with constant coefficients commuting with E_α for any α.

A very important role in the theory is played by the wave function Ψ and its adjoint $\Psi^\dagger$ (hereafter, † does not mean Hermitian conjugation). The wave function is defined as a result of action of the wave operator to the exponential function,

$$\Psi(x, \mathbf{t}; z) = \mathcal{W}\exp\left(xzI + \sum_{\alpha=1}^{n} E_\alpha\xi(\mathbf{t}_\alpha, z)\right), \tag{2.8}$$

where we use the standard notation

$$\xi(\mathbf{t}_\alpha, z) = \sum_{k\ge 1} t_{\alpha,k} z^k.$$

By definition, the operators ∂_x^{-k} with negative powers act to the exponential function as $\partial_x^{-k} e^{xz} = z^{-k} e^{xz}$. The wave function depends on the spectral parameter z, which does not enter the auxiliary linear problems explicitly. The adjoint wave function is introduced by the following formula:

$$\Psi^\dagger(x, \mathbf{t}; z) = \exp\left(-xzI - \sum_{\alpha=1}^{n} E_\alpha \xi(\mathbf{t}_\alpha, z)\right) \mathcal{W}^{-1}. \tag{2.9}$$

Here, we use the convention that the operators ∂_x that enter $\mathcal{W}^{-1}$ act to the left rather than to the right, the left action being defined as $f\overleftarrow{\partial_x} \equiv -\partial_x f$. Clearly, the expansion of the wave function as $z \to \infty$ is as follows:

$$\Psi_{\alpha\beta}(x, \mathbf{t}; z) = e^{xz+\xi(\mathbf{t}_\beta, z)}\left(\delta_{\alpha\beta} + \xi^{(1)}_{\alpha\beta} z^{-1} + \xi^{(2)}_{\alpha\beta} z^{-2} + \cdots\right). \tag{2.10}$$

As is proved in Ref. 16, the wave function satisfies the linear equations

$$\partial_{t_{\alpha,m}} \Psi(x, \mathbf{t}; z) = B_{\alpha,m} \Psi(x, \mathbf{t}; z), \tag{2.11}$$

where $B_{\alpha,m}$ is the differential operator (2.5), i.e., $B_{\alpha,m} = \left(\mathcal{W} E_\alpha \partial_x^m \mathcal{W}^{-1}\right)_+$ and the adjoint wave function satisfies the transposed equations

$$-\partial_{t_{\alpha,m}} \Psi^\dagger(x, \mathbf{t}; z) = \Psi^\dagger(x, \mathbf{t}; z) B_{\alpha,m}. \tag{2.12}$$

Again, the operator $B_{\alpha,m}$ here acts to the left. In particular, it follows from (2.11) and (2.12) at $m = 1$ that

$$\sum_{\alpha=1}^{n} \partial_{t_{\alpha,1}} \Psi(x, \mathbf{t}; z) = \partial_x \Psi(x, \mathbf{t}; z), \qquad \sum_{\alpha=1}^{n} \partial_{t_{\alpha,1}} \Psi^\dagger(x, \mathbf{t}; z) = \partial_x \Psi^\dagger(x, \mathbf{t}; z), \tag{2.13}$$

so the vector field ∂_x can be naturally identified with the vector field $\sum_\alpha \partial_{t_{\alpha,1}}$.

Another approach to the multi-component KP hierarchy is provided by the bilinear formalism. In the bilinear formalism, the dependent variables are the tau-function $\tau(x, \mathbf{t})$ and tau-functions $\tau_{\alpha\beta}(x, \mathbf{t})$ such that $\tau_{\alpha\alpha}(x, \mathbf{t}) = \tau(x, \mathbf{t})$ for any α. The n-component KP hierarchy is the infinite set of bilinear equations for the tau-functions, which are encoded in the basic bilinear relation

$$\sum_{\nu=1}^{n} \epsilon_{\alpha\nu}\epsilon_{\beta\nu} \oint_{C_\infty} \mathrm{d}z\, z^{\delta_{\alpha\nu}+\delta_{\beta\nu}-2} e^{\xi(\mathbf{t}_\nu - \mathbf{t}'_\nu, z)} \tau_{\alpha\nu}\left(x, \mathbf{t} - [z^{-1}]_\nu\right) \tau_{\nu\beta}\left(x, \mathbf{t}' + [z^{-1}]_\nu\right) = 0 \tag{2.14}$$

valid for any $\mathbf{t}, \mathbf{t}'$. Here, $\epsilon_{\alpha\beta}$ is a sign factor: $\epsilon_{\alpha\beta} = 1$ if $\alpha \le \beta$ and $\epsilon_{\alpha\beta} = -1$ if $\alpha > \beta$. In (2.14), we use the following standard notation:

$$\left(\mathbf{t} \pm [z^{-1}]_\gamma\right)_{\alpha k} = t_{\alpha,k} \pm \delta_{\alpha\gamma} \frac{z^{-k}}{k}.$$

The integration contour C_∞ is a big circle around ∞.

The tau-functions are universal dependent variables of the hierarchy. All other objects including the coefficient functions u_i of the Lax operator and the wave functions can be expressed in terms of them. In particular, for the wave function and its adjoint, we have

$$\begin{aligned} \Psi_{\alpha\beta}(x, \mathbf{t}; z) &= \epsilon_{\alpha\beta} \frac{\tau_{\alpha\beta}\left(x, \mathbf{t} - [z^{-1}]_\beta\right)}{\tau(x, \mathbf{t})} z^{\delta_{\alpha\beta}-1} e^{\xi(\mathbf{t}_\beta, z)}, \\ \Psi^\dagger_{\alpha\beta}(x, \mathbf{t}; z) &= \epsilon_{\beta\alpha} \frac{\tau_{\alpha\beta}\left(x, \mathbf{t} + [z^{-1}]_\alpha\right)}{\tau(x, \mathbf{t})} z^{\delta_{\alpha\beta}-1} e^{-\xi(\mathbf{t}_\alpha, z)}. \end{aligned} \tag{2.15}$$

Note that the bilinear relation (2.14) can be written in the form

$$\oint_{C_\infty} \mathrm{d}z\, \Psi(x, \mathbf{t}; z) \Psi^\dagger(x, \mathbf{t}'; z) = 0. \tag{2.16}$$

The coefficient $\xi^{(1)}_{\alpha\beta}(x,\mathbf{t})$ plays an important role in what follows. Equation (2.15) implies that this coefficient is expressed through the tau-functions as

$$\xi^{(1)}_{\alpha\beta}(x,\mathbf{t}) = \begin{cases} \epsilon_{\alpha\beta}\dfrac{\tau_{\alpha\beta}(x,\mathbf{t})}{\tau(x,\mathbf{t})}, & \alpha \neq \beta \\ -\dfrac{\partial_{t_\beta}\tau(x,\mathbf{t})}{\tau(x,\mathbf{t})}, & \alpha = \beta. \end{cases} \tag{2.17}$$

Let us point out a useful corollary of the bilinear relation (2.14). Differentiating it with respect to $t_{\kappa,m}$ and putting $\mathbf{t}' = \mathbf{t}$ after this, we obtain

$$\frac{1}{2\pi i}\oint_{C_\infty} dz\, z^m \Psi_{\alpha\kappa}(x,\mathbf{t};z)\Psi^{\dagger}_{\kappa\beta}(x,\mathbf{t};z) = -\partial_{t_{\kappa,m}}\xi^{(1)}_{\alpha\beta}(x,\mathbf{t}) \tag{2.18}$$

or, equivalently,

$$\operatorname*{res}_{\infty}\Big(z^m \Psi_{\alpha\kappa}\Psi^{\dagger}_{\kappa\beta}\Big) = -\partial_{t_{\kappa,m}}\xi^{(1)}_{\alpha\beta}. \tag{2.19}$$

The integrand in (2.18) should be regarded as a Laurent series in z, and the residue at infinity is defined according to the convention $\operatorname*{res}_{\infty}(z^{-k}) = \delta_{k1}$.

The matrix KP hierarchy is a subhierarchy of the multi-component KP one. It is obtained by the following restriction of the independent variables: $t_{\alpha,m} = t_m$ for each α and m. The corresponding vector fields are related as $\partial_{t_m} = \sum_{\alpha=1}^{n}\partial_{t_{\alpha,m}}$. As is clear from (2.10), the wave function for the matrix KP hierarchy has the expansion

$$\Psi_{\alpha\beta}(x,\mathbf{t};z) = \Big(\delta_{\alpha\beta} + \xi^{(1)}_{\alpha\beta}(\mathbf{t})z^{-1} + O(z^{-2})\Big)e^{xz+\xi(\mathbf{t},z)}, \tag{2.20}$$

where $\xi(\mathbf{t},z) = \sum_{k\geq 1} t_k z^k$. Equation (2.11) implies that the wave function of the matrix KP hierarchy and its adjoint satisfy the linear equations

$$\partial_{t_m}\Psi(\mathbf{t};z) = B_m\Psi(\mathbf{t};z), \qquad -\partial_{t_m}\Psi^{\dagger}(\mathbf{t};z) = \Psi^{\dagger}(\mathbf{t};z)B_m, \qquad m \geq 1, \tag{2.21}$$

where B_m is the differential operator $B_m = \big(\mathcal{W}\partial_x^m\mathcal{W}^{-1}\big)_+$. At $m = 1$, we have $\partial_{t_1}\Psi = \partial_x\Psi$, so we can identify $\partial_x = \partial_{t_1} = \sum_{\alpha=1}^{N}\partial_{t_{\alpha,1}}$. This means that the evolution in the time t_1 is simply a shift of the variable x: $\xi^{(k)}(x,t_1,t_2,\ldots) = \xi^{(k)}(x+t_1,t_2,\ldots)$. At $m = 2$, Eq. (2.21) turns into the linear problems

$$\partial_{t_2}\Psi = \partial_x^2\Psi + 2V(x,\mathbf{t})\Psi, \tag{2.22}$$

$$-\partial_{t_2}\Psi^{\dagger} = \partial_x^2\Psi^{\dagger} + 2\Psi^{\dagger}V(x,\mathbf{t}), \tag{2.23}$$

which have the form of the matrix non-stationary Schrödinger equations with

$$V(x,\mathbf{t}) = -\partial_x\xi^{(1)}(x,\mathbf{t}). \tag{2.24}$$

Summing (2.18) over κ, we obtain an analog of (2.18) for the matrix KP hierarchy,

$$\frac{1}{2\pi i}\sum_{\nu=1}^{n}\oint_{C_\infty} dz\, z^m \Psi_{\alpha\nu}(x,\mathbf{t};z)\Psi^{\dagger}_{\nu\beta}(x,\mathbf{t};z) = -\partial_{t_m}\xi^{(1)}_{\alpha\beta}(x,\mathbf{t}). \tag{2.25}$$

Below, we will use Eqs. (2.18) and (2.25) for the derivation of dynamics of poles and residues of elliptic solutions in higher times.

III. ELLIPTIC SOLUTIONS OF THE MATRIX KP HIERARCHY AND DOUBLE-BLOCH FUNCTIONS

Our aim is to study solutions to the matrix KP hierarchy, which are elliptic functions of the variable x (and, therefore, t_1). For the elliptic solutions, we take the tau-function in the form

$$\tau(x,\mathbf{t}) = C\prod_{i=1}^{\mathcal{N}} \sigma(x - x_i(\mathbf{t})), \tag{3.1}$$

where

$$\sigma(x) = \sigma(x|\,\omega,\omega') = x\prod_{s\neq 0}\left(1 - \frac{x}{s}\right)e^{\frac{x}{s}+\frac{x^2}{2s^2}}, \quad s = 2\omega m + 2\omega' m', \quad \text{with integer } m, m'$$

being the Weierstrass σ-function with quasi-periods 2ω, $2\omega'$ such that $\mathrm{Im}(\omega'/\omega) > 0$. It is connected with the Weierstrass ζ- and $\wp$-functions by the formulas $\zeta(x) = \sigma'(x)/\sigma(x)$ and $\wp(x) = -\zeta'(x) = -\partial_x^2 \log \sigma(x)$. The monodromy properties of the function $\sigma(x)$ are

$$\sigma(x + 2\omega) = -e^{2\zeta(\omega)(x+\omega)}\sigma(x), \quad \sigma(x + 2\omega') = -e^{2\zeta(\omega')(x+\omega')}\sigma(x), \tag{3.2}$$

where the constants $\zeta(\omega)$ and $\zeta(\omega')$ are related by $\zeta(\omega)\omega' - \zeta(\omega')\omega = \pi i/2$. The N zeros x_i of (3.1) are assumed to be all distinct.

We also assume that the tau-functions $\tau_{\alpha\beta}$ at $\alpha \neq \beta$ have the form

$$\tau_{\alpha\beta}(x,\mathbf{t}) = C_{\alpha\beta}\prod_{i=1}^{\mathcal{N}} \sigma(x - x_i^{(\alpha\beta)}(\mathbf{t})), \tag{3.3}$$

with

$$\sum_i x_i(\mathbf{t}) = \sum_i x_i^{(\alpha\beta)}(\mathbf{t}) \quad \text{for all } \alpha,\beta. \tag{3.4}$$

The consistency of this assumption is justified below.

Equation (2.17) together with condition (3.4) implies that $V(x,\mathbf{t}) = -\partial_x\xi^{(1)}$ in the linear problem (2.22) is an elliptic function of x. Therefore, one can find solutions to (2.22), which are *double-Bloch functions*. The double-Bloch function satisfies the monodromy properties $\Psi_{\alpha\beta}(x + 2\omega) = B_\beta\Psi_{\alpha\beta}(x)$, $\Psi_{\alpha\beta}(x + 2\omega') = B'_\beta\Psi_{\alpha\beta}(x)$ with some Bloch multipliers B_β, B'_β. The Bloch multipliers of the wave function (2.15) are

$$\begin{aligned} B_\beta &= \exp\left(2\omega z - 2\zeta(\omega)\sum_i (e^{-D_\beta(z)} - 1)x_i\right), \\ B'_\beta &= \exp\left(2\omega' z - 2\zeta(\omega')\sum_i (e^{-D_\beta(z)} - 1)x_i\right), \end{aligned} \tag{3.5}$$

where the differential operator $D_\beta(z)$ is

$$D_\beta(z) = \sum_{k\geq 1}\frac{z^{-k}}{k}\,\partial_{t_{\beta,k}}. \tag{3.6}$$

Since the right-hand side of (2.25) is an elliptic function of x, the Bloch multipliers of the adjoint wave function should be $1/B_\alpha$, $1/B'_\alpha$: $\Psi^\dagger_{\alpha\beta}(x + 2\omega) = (B_\alpha)^{-1}\Psi^\dagger_{\alpha\beta}(x)$, $\Psi^\dagger_{\alpha\beta}(x + 2\omega') = (B'_\alpha)^{-1}\Psi^\dagger_{\alpha\beta}(x)$.

Any non-trivial double-Bloch function (i.e., the one which is not just an exponential function) must have at least one pole in x in the fundamental domain. Let us introduce the elementary double-Bloch function $\Phi(x,\lambda)$ having just one pole in the fundamental domain and defined as

$$\Phi(x,\lambda) = \frac{\sigma(x+\lambda)}{\sigma(\lambda)\sigma(x)}\,e^{-\zeta(\lambda)x} \tag{3.7}$$

[here, $\zeta(\lambda)$ is the Weierstrass ζ-function]. The monodromy properties of the function Φ follow from (3.2),

$$\Phi(x + 2\omega,\lambda) = e^{2(\zeta(\omega)\lambda - \zeta(\lambda)\omega)}\Phi(x,\lambda),$$

$$\Phi(x + 2\omega',\lambda) = e^{2(\zeta(\omega')\lambda - \zeta(\lambda)\omega')}\Phi(x,\lambda).$$

We see that it is indeed a double-Bloch function. It has a single simple pole in the fundamental domain at $x = 0$ with residue 1,

$$\Phi(x,\lambda) = \frac{1}{x} - \frac{1}{2}\wp(\lambda)x + \cdots, \qquad x \to 0.$$

It is easy to see that $\Phi(x,\lambda)$ is an elliptic function of λ. The expansion as $\lambda \to 0$ is

$$\Phi(x,\lambda) = \Big(\lambda^{-1} + \zeta(x) + \frac{1}{2}(\zeta^2(x) - \wp(x))\lambda + O(\lambda^2)\Big)e^{-x/\lambda}. \tag{3.8}$$

We will also need the x-derivatives $\Phi'(x,\lambda) = \partial_x \Phi(x,\lambda)$ and $\Phi''(x,\lambda) = \partial_x^2 \Phi(x,\lambda)$.

It is clear from (2.15) that the wave functions Ψ, $\Psi^\dagger$ (and thus the coefficient $\xi^{(1)}$), as functions of x, have simple poles at $x = x_i$. It is shown in Ref. 19 that the residues at these poles are matrices of rank 1. We parameterize the residues of $\xi^{(1)}$ through the column vectors $\mathbf{a}_i = (a_i^1, a_i^2, \ldots, a_i^n)^T$ and $\mathbf{b}_i = (b_i^1, b_i^2, \ldots, b_i^n)^T$ (T means transposition),

$$\xi^{(1)}_{\alpha\beta} = S_{\alpha\beta} - \sum_i a_i^\alpha b_i^\beta \zeta(x - x_i), \tag{3.9}$$

where $S_{\alpha\beta}$ does not depend on x. Therefore,

$$V(x,\mathbf{t}) = -\sum_i a_i^\alpha b_i^\beta \wp(x - x_i). \tag{3.10}$$

The components of the vectors $\mathbf{a}_i$, $\mathbf{b}_i$ are going to be spin variables of the elliptic Gibbons–Hermsen model.

One can expand the wave functions using the elementary double-Bloch functions as follows:

$$\Psi_{\alpha\beta} = e^{k_\beta x + \xi(\mathbf{t},z)} \sum_i a_i^\alpha c_i^\beta \Phi(x - x_i, \lambda_\beta), \tag{3.11}$$

$$\Psi^\dagger_{\alpha\beta} = e^{-k_\alpha x - \xi(\mathbf{t},z)} \sum_i c_i^{*\alpha} b_i^\beta \Phi(x - x_i, -\lambda_\alpha), \tag{3.12}$$

where c_i^α and $c_i^{*\alpha}$ are components of some x-independent vectors $\mathbf{c}_i = (c_i^1, \ldots, c_i^n)^T$ and $\mathbf{c}_i^* = (c_i^{*1}, \ldots, c_i^{*n})^T$. This is similar to the expansion of a rational function in a linear combination of simple fractions.

One can see that (3.11) is a double-Bloch function with Bloch multipliers

$$B_\beta = e^{2\omega(k_\beta - \zeta(\lambda_\beta)) + 2\zeta(\omega)\lambda_\beta}, \qquad B'_\beta = e^{2\omega'(k_\beta - \zeta(\lambda_\beta)) + 2\zeta(\omega')\lambda_\beta}, \tag{3.13}$$

and (3.12) has Bloch multipliers $(B_\alpha)^{-1}$ and $(B'_\alpha)^{-1}$. These Bloch multipliers should coincide with (3.5). Therefore, comparing (3.5) with (3.13), we get

$$2\omega\big(k_\beta - z - \zeta(\lambda_\beta)\big) + 2\zeta(\omega)\Big(\lambda_\beta + (e^{-D_\beta(z)} - 1)\sum_i x_i\Big) = 2\pi i n,$$

$$2\omega'\big(k_\beta - z - \zeta(\lambda_\beta)\big) + 2\zeta(\omega')\Big(\lambda_\beta + (e^{-D_\beta(z)} - 1)\sum_i x_i\Big) = 2\pi i n'$$

with some integer n, n'. These equations can be regarded as a linear system. The solution is

$$k_\beta - z - \zeta(\lambda_\beta) = 2n'\zeta(\omega) - 2n\zeta(\omega'),$$

$$\lambda_\beta + (e^{-D_\beta(z)} - 1)\sum_i x_i = 2n\omega' - 2n'\omega.$$

Shifting λ_β by a suitable vector of the lattice spanned by 2ω, $2\omega'$, we can represent the connection between the spectral parameters $k_\beta, z, \lambda_\beta$ in the form

$$\begin{cases} k_\beta = z + \zeta(\lambda_\beta), \\ \lambda_\beta = (1 - e^{-D_\beta(z)})\sum_i x_i. \end{cases} \tag{3.14}$$

These two equations for three spectral parameters $k_\beta, z, \lambda_\beta$ determine the spectral curve, with the index β numbering different sheets of it. Another description of the same spectral curve is obtained below as the spectral curve of the spin generalization of the Calogero–Moser system [it is given by the characteristic polynomial of the Lax matrix $L(\lambda)$ for the spin Calogero–Moser system]. As we shall see below, it has the form $R(k,\lambda) = 0$, where $R(k,\lambda)$ is a polynomial in k whose coefficients are elliptic functions of λ. These coefficients are integrals of motion in involution. The spectral curve in the form $R(k,\lambda) = 0$ appears if one excludes z from Eq. (3.14). Equivalently, one can represent the spectral curve as a relation connecting z and λ_β,

$$R(z+\zeta(\lambda_\beta),\lambda_\beta) = 0. \tag{3.15}$$

The function $z(\lambda)$ defined by this equation is multivalued, $z_\beta(\lambda)$ being different branches of this function. Then, the function $\lambda_\beta(z)$ is the inverse function to $z_\beta(\lambda)$. Using the same arguments as in Ref. 12, one can see that the second equation in (3.14) can be written as

$$\lambda_\beta(z) = D_\beta(z)\sum_i x_i = \sum_{j\ge 1}\frac{z^{-j}}{j}\,V_j^{(\beta)}, \quad V_j^{(\beta)} = \partial_{t_{\beta,j}}\sum_i x_i, \tag{3.16}$$

where $\lambda_\beta(z)$ should be understood as the expansion of the βth branch of the function $\lambda(z)$ in negative powers of z near $z = \infty$.

IV. DYNAMICS OF POLES AND RESIDUES IN t_2

We first consider the dynamics of the poles and residues with respect to time t_2. Following Krichever's approach, we consider the linear problems [(2.22) and (2.23)] and substitute the pole ansatz [(3.11) and (3.12)] for the wave functions.

Consider first the equation for Ψ. After the substitution, we see that the expression has poles at $x = x_i$ up to the third order. Equating coefficients at the poles of different orders at $x = x_i$, we get the following conditions:

- At $\frac{1}{(x-x_i)^3}$, $b_i^\nu a_i^\nu = 1$.
- At $\frac{1}{(x-x_i)^2}$, $-\frac{1}{2}\dot{x}_i c_i^\beta - \sum_{j\ne i} b_i^\nu a_j^\nu c_j^\beta \Phi(x_i - x_j, \lambda_\beta) = k_\beta c_i^\beta$.
- At $\frac{1}{x-x_i}$, $\partial_{t_2}(a_i^\alpha c_i^\beta) = (k_\beta^2 - z^2 + \wp(\lambda_\beta))a_i^\alpha c_i^\beta$

$$-2\sum_{j\ne i} a_i^\alpha b_i^\nu a_j^\nu c_j^\beta \Phi'(x_i - x_j, \lambda_\beta) - 2c_i^\beta \sum_{j\ne i} a_i^\nu b_j^\nu a_j^\alpha \wp(x_i - x_j),$$

where dot means the t_2-derivative. Here and below summation over repeated Greek indices, numbering components of vectors from 1 to n is implied, unless otherwise stated. Similar calculations for the linear problem for $\Psi^\dagger$ lead to the following conditions:

- At $\frac{1}{(x-x_i)^3}$, $b_i^\nu a_i^\nu = 1$ (the same as above).
- At $\frac{1}{(x-x_i)^2}$, $-\frac{1}{2}\dot{x}_i c_i^{*\alpha} - \sum_{j\ne i} c_j^{*\alpha} b_j^\nu a_i^\nu \Phi(x_j - x_i, \lambda_\alpha) = k_\alpha c_i^{*\alpha}$.
- At $\frac{1}{x-x_i}$, $\partial_{t_2}(c_i^{*\alpha} b_i^\beta) = -(k_\beta^2 - z^2 + \wp(\lambda_\alpha))c_i^{*\alpha} b_i^\beta$

$$+2\sum_{j\ne i} c_j^{*\alpha} b_j^\nu a_i^\nu b_i^\beta \Phi'(x_j - x_i, \lambda_\alpha) + 2c_i^{*\alpha}\sum_{j\ne i} b_i^\nu a_j^\nu b_j^\beta \wp(x_i - x_j).$$

Here, we have used the obvious property $\Phi(x,-\lambda) = -\Phi(-x,\lambda)$.

The conditions coming from the third order poles are constraints on the vectors $\mathbf{a}_i$, $\mathbf{b}_i$. The other conditions can be written as

$$\begin{cases} (k_\beta I - L(\lambda_\beta))\mathsf{c}^\beta = 0, \\ \dot{\mathsf{c}}^\beta = M(\lambda_\beta)\mathsf{c}^\beta, \end{cases} \tag{4.1}$$

$$\begin{cases} \mathsf{c}^{*\alpha}(k_\alpha I - L(\lambda_\alpha)) = 0, \\ \dot{\mathsf{c}}^{*\alpha} = \mathsf{c}^{*\alpha} M^*(\lambda_\alpha) \end{cases} \tag{4.2}$$

(no summation over α, β), where $\mathsf{c}^\beta = (c_1^\beta, \ldots c_N^\beta)^T$ and $\mathsf{c}^{*\alpha} = (c_1^{*\alpha}, \ldots c_N^{*\alpha})$ are N-dimensional vectors, I is the unity matrix, and $L(\lambda)$, $M(\lambda)$, and $M * (\lambda)$ are $N \times N$ matrices of the form

$$L_{ij}(\lambda) = -\frac{1}{2}\dot{x}_i\delta_{ij} - (1-\delta_{ij})b_i^\nu a_j^\nu \Phi(x_i - x_j, \lambda), \tag{4.3}$$

$$M_{ij}(\lambda) = (k^2 - z^2 + \wp(\lambda) - \Lambda_i)\delta_{ij} - 2(1-\delta_{ij})b_i^\nu a_j^\nu \Phi'(x_i - x_j, \lambda), \tag{4.4}$$

$$M_{ij}^*(\lambda) = -(k^2 - z^2 + \wp(\lambda) - \Lambda_i^*)\delta_{ij} + 2(1-\delta_{ij})b_i^\nu a_j^\nu \Phi'(x_i - x_j, \lambda). \tag{4.5}$$

Here,

$$\Lambda_i = \frac{\dot{a}_i^\alpha}{a_i^\alpha} + 2\sum_{j\neq i}\frac{a_j^\alpha b_j^\nu a_i^\nu}{a_i^\alpha}\wp(x_i - x_j), \quad -\Lambda_i^* = \frac{\dot{b}_i^\alpha}{b_i^\alpha} - 2\sum_{j\neq i}\frac{b_i^\nu a_j^\nu b_j^\alpha}{b_i^\alpha}\wp(x_i - x_j) \tag{4.6}$$

do not depend on the index α (there is summation over ν but no summation over α). In fact, one can see that $\Lambda_i = \Lambda_i^*$ so that $M^*(\lambda) = -M(\lambda)$. Indeed, multiplying Eq. (4.6) by $a_i^\alpha b_i^\alpha$ (no summation here), summing over α, and summing the two equations, we get $\Lambda_i - \Lambda_i^* = \partial_{t_2}(a_i^\alpha b_i^\alpha) = 0$ by virtue of the constraint $a_i^\alpha b_i^\alpha = 1$.

Differentiating the first equation in (4.1) by t_2, we get the compatibility condition of Eq. (4.1),

$$(\dot{L} + [L, M])\mathrm{c}^\beta = 0. \tag{4.7}$$

One can see, taking into account Eq. (4.6), that we write here in the form

$$\dot{a}_i^\alpha = \Lambda_i a_i^\alpha - 2\sum_{j\neq i} a_j^\alpha b_j^\nu a_i^\nu \wp(x_i - x_j), \tag{4.8}$$

$$\dot{b}_i^\alpha = -\Lambda_i b_i^\alpha + 2\sum_{j\neq i} b_i^\nu a_j^\nu b_j^\alpha \wp(x_i - x_j) \tag{4.9}$$

(in this form, they are equations of motion for the spin degrees of freedom) that the off-diagonal elements of the matrix $\dot{L} + [L, M]$ are equal to zero. Vanishing of the diagonal elements yields equations of motion for the poles x_i,

$$\ddot{x}_i = 4\sum_{j\neq i} b_i^\mu a_k^\mu b_k^\nu a_i^\nu \wp'(x_i - x_j). \tag{4.10}$$

The gauge transformation $a_i^\alpha \to a_i^\alpha q_i$, $b_i^\alpha \to b_i^\alpha q_i^{-1}$ with $q_i = \exp\left(\int^{t_2}\Lambda_i \mathrm{d}t\right)$ eliminates the terms with Λ_i in (4.8) and (4.9), so we can put $\Lambda_i = 0$. This gives the equations of motion

$$\dot{a}_i^\alpha = -2\sum_{j\neq i} a_j^\alpha b_j^\nu a_i^\nu \wp(x_i - x_j), \quad \dot{b}_i^\alpha = 2\sum_{j\neq i} b_i^\nu a_j^\nu b_j^\alpha \wp(x_i - x_j). \tag{4.11}$$

Together with (4.10), they are equations of motion of the elliptic Gibbons–Hermsen model. Their Lax representation is given by the matrix equation $\dot{L} = [M, L]$. It states that the time evolution of the Lax matrix is an isospectral transformation. It follows that the quantities $\mathrm{tr}\, L^m(\lambda)$ are integrals of motion. In particular,

$$H_2 = \sum_{i=1}^N p_i^2 - \sum_{i\neq j} b_i^\mu a_j^\mu b_j^\nu a_i^\nu \wp(x_i - x_j) = \mathrm{tr}\, L^2(\lambda) + \mathrm{const} \tag{4.12}$$

is the Hamiltonian of the elliptic Gibbons–Hermsen model. Equations of motion (4.10) and (4.11) are equivalent to the Hamiltonian equations

$$\dot{x}_i = \frac{\partial H_2}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H_2}{\partial x_i}, \quad \dot{a}_i^\alpha = \frac{\partial H_2}{\partial b_i^\alpha}, \quad \dot{b}_i^\alpha = -\frac{\partial H_2}{\partial a_i^\alpha}. \tag{4.13}$$

We see that $\dot{x}_i = 2p_i$ and the Lax matrix is expressed through the momenta as follows:

$$L_{ij}(\lambda) = -p_i\delta_{ij} - (1 - \delta_{ij}) b_i^\nu a_j^\nu \Phi(x_i - x_j, \lambda). \tag{4.14}$$

As we shall see, the higher time flows are also Hamiltonian with the Hamiltonians being linear combinations of spectral invariants of the Lax matrix, i.e., linear combinations of traces of its powers $\mathrm{tr}\, L^j(\lambda)$. It is not difficult to see that

$$G^{\alpha\beta} = \sum_i a_i^\alpha b_i^\beta \tag{4.15}$$

are integrals of motion for all time flows: $\partial_{t_m} G^{\alpha\beta} = 0$. Indeed, we have

$$\partial_{t_m}\left(\sum_i a_i^\alpha b_i^\beta\right)=\sum_i\left(b_i^\beta\frac{\partial H_m}{\partial b_i^\alpha}-a_i^\alpha\frac{\partial H_m}{\partial a_i^\beta}\right),$$

and this is zero because H_m is a linear combination of $\operatorname{tr} L^j(\lambda)$ and

$$\sum_i\left(b_i^\beta \operatorname{tr}\left(\frac{\partial L}{\partial b_i^\alpha}L^{j-1}\right)-a_i^\alpha \operatorname{tr}\left(\frac{\partial L}{\partial a_i^\beta}L^{j-1}\right)\right)$$

$$=\sum_i\sum_{l,k}\left(b_i^\beta\frac{\partial L_{lk}}{\partial b_i^\alpha}L_{kl}^{j-1}-a_i^\alpha\frac{\partial L_{lk}}{\partial a_i^\beta}L_{kl}^{j-1}\right)$$

$$=\sum_i\sum_{l\neq k}(\delta_{ik}-\delta_{il})b_l^\beta a_k^\alpha\Phi(x_l-x_k)L_{kl}^{j-1}=0.$$

A simple lemma from the linear algebra states that eigenvalues ν_α of the $n\times n$ matrix G (4.15) coincide with nonzero eigenvalues of the rank $nN\times N$ matrix F with matrix elements

$$F_{ij}=b_i^\nu a_j^\nu. \tag{4.16}$$

(We assume that $n\le N$.) Indeed, consider the rectangular $N\times n$ matrix $\mathbf{A}_{i\alpha}=a_i^\alpha$ and $\mathbf{B}_{i\beta}=b_i^\beta$, and then, $G=\mathbf{A}^T\mathbf{B}$, $F=\mathbf{B}\mathbf{A}^T$, and a straightforward verification shows that traces of all powers of these matrices coincide: $\operatorname{tr} G^m=\operatorname{tr} F^m$ for all $m\ge 1$. This means that their nonzero eigenvalues also coincide. Note that $\operatorname{tr} G=\sum_{\alpha=1}^n \nu_\alpha=N$.

V. THE SPECTRAL CURVE

The first of Eq. (4.1) determines a connection between the spectral parameters $k=k_\beta,\lambda=\lambda_\beta$, which is the equation of the spectral curve

$$R(k,\lambda):=\det(kI-L(\lambda))=0. \tag{5.1}$$

As already mentioned, the spectral curve is an integral of motion. The matrix $L=L(\lambda)$ has an essential singularity at $\lambda=0$. It can be represented in the form $L=V\tilde{L}V^{-1}$, where V is the diagonal matrix $V_{ij}=\delta_{ij}e^{-\zeta(\lambda)x_i}$. Matrix elements of $\tilde{L}$ do not have any essential singularity in λ. We conclude that

$$R(k,\lambda)=\sum_{m=0}^N R_m(\lambda)k^m,$$

where the coefficients $R_m(\lambda)$ are elliptic functions of λ with poles at $\lambda=0$. They can be represented as linear combinations of the $\wp$-function and its derivatives, coefficients of this expansion being integrals of motion. Fixing their values, we obtain an algebraic curve Γ, which is an N-sheet covering of the initial elliptic curve $\mathcal{E}$ realized as a factor of the complex plane with respect to the lattice generated by 2ω, $2\omega'$.

In a neighborhood of the point $\lambda=0$, the matrix $\tilde{L}(\lambda)$ can be represented as

$$\tilde{L}(\lambda)=\lambda^{-1}(I-F)+O(1),$$

where F is the rank n matrix (4.16) (recall that $n\le N$). This matrix has $N-n$ vanishing eigenvalues and n nonzero eigenvalues ν_α, $\alpha=1,\ldots,n$. They are time-independent quantities because as we have shown above, they coincide with eigenvalues of the matrix G (4.15), which is an integral of motion. Therefore, we can write

$$\det(kI-L(\lambda))=\prod_{\alpha=1}^n\left(k-(1-\nu_\alpha)\lambda^{-1}-h_\alpha(\lambda)\right)\prod_{j=n+1}^N\left(k-\lambda^{-1}-h_j(\lambda)\right),$$

where h_α, h_j are regular functions of λ near $\lambda=0$. This means that the function k has simple poles on all sheets at the points of the curve Γ located above $\lambda=0$. Now, recalling the connection between k and z given by the first equation in (3.14), we have

$$\det((z+\zeta(\lambda))I-L(\lambda))=\prod_{\alpha=1}^n\left(z+\nu_\alpha\lambda^{-1}-h_\alpha(\lambda)\right)\prod_{j=n+1}^N(z-h_j(\lambda)). \tag{5.2}$$

We see that n sheets of the curve Γ lying above a neighborhood of the point $\lambda = 0$ are distinguished. There are n points at infinity above $\lambda = 0$: $P_1^{(\infty)} = (\infty_1, 0), \dots, P_n^{(\infty)} = (\infty_n, 0)$. In the vicinity of the point $P_\alpha^{(\infty)}$, the function $\lambda = \lambda_\alpha(z)$ has the following expansion:

$$\lambda = \lambda_\alpha(z) = -\nu_\alpha z^{-1} + O(z^{-2}). \tag{5.3}$$

As shown in Ref. 17, the points $P_\alpha^{(\infty)} \in \Gamma$ are the marked points, where the Baker–Akhiezer function on the spectral curve has essential singularities.

With expansion (5.3) at hand, we can make a more detailed identification of wave function (2.15) with expansion (2.20) and wave function (3.11). The expansion of function (3.11) as $\lambda_\beta \to 0$ yields

$$\Psi_{\alpha\beta} = e^{zx+\xi(\mathbf{t},z)} \sum_i \Big(a_i^\alpha b_i^\beta \nu_\beta^{-1} + \lambda_\beta \nu_\beta^{-1} \big(a_i^\alpha d_i^\beta + a_i^\alpha b_i^\beta \zeta(x - x_i) \big) + O(\lambda_\beta^2) \Big),$$

where we took into account that the identification implies the expansion

$$c_i^\beta = \nu_\beta^{-1} \lambda_\beta^{-1} e^{-x_i \zeta(\lambda)} \Big(b_i^\beta + \lambda_\beta d_i^\beta + O(\lambda_\beta^2) \Big), \quad \lambda_\beta \to 0. \tag{5.4}$$

Therefore, taking into account (5.3), we can write

$$\Psi_{\alpha\beta} = e^{zx+\xi(\mathbf{t},z)} \left(\sum_i a_i^\alpha b_i^\beta \nu_\beta^{-1} + z^{-1} \left(S_{\alpha\beta} - \sum_i a_i^\alpha b_i^\beta \zeta(x - x_i) \right) + O(z^{-2}) \right).$$

Comparing with (2.20), we conclude that

$$\sum_i a_i^\alpha b_i^\beta = \nu_\alpha \delta_{\alpha\beta}. \tag{5.5}$$

It is easy to see that the Hamiltonian and the Lax matrix are invariant with respect to the gauge transformation

$$\mathbf{a}_i \to W^{-1} \mathbf{a}_i, \qquad \mathbf{b}_i^T \to \mathbf{b}_i^T W \tag{5.6}$$

with arbitrary non-degenerate $n \times n$ matrix W. Therefore, after the transformation $G \to W^{-1} g W$, the matrix G can always be regarded as a diagonal matrix, as in (5.5), with the eigenvalues being the same as nonzero eigenvalues ν_α of the $N \times N$ matrix F.

VI. DYNAMICS OF POLES IN THE HIGHER TIMES

Our basic tool is Eq. (2.18). Substituting Ψ, $\Psi^\dagger$ in the form (3.11) and (3.12) and $\xi^{(1)}$ in the form (3.9), we have

$$\begin{aligned} &\frac{1}{2\pi i} \oint_{C_\infty} \mathrm{d}z\, z^m \sum_{i,j} a_i^\alpha c_i^\nu c_j^{*\nu} b_j^\beta \Phi(x - x_i, \lambda_\nu) \Phi(x - x_j, -\lambda_\nu) \\ &\quad = \sum_i \partial_{t_{\nu,m}} x_i a_i^\alpha b_i^\beta \wp(x - x_i) + \sum_i \partial_{t_{\nu,m}} (a_i^\alpha b_i^\beta) \zeta(x - x_i) \end{aligned} \tag{6.1}$$

(no summation over ν here). Equating the coefficients in front of the second order poles at $x = x_i$, we get the relation

$$\partial_{t_{\nu,m}} x_i = \operatorname*{res}_\infty \big(z^m c_i^{*\nu} c_i^\nu \big) = \operatorname*{res}_\infty \big(z^m \mathrm{c}^{*\nu} E_i \mathrm{c}^\nu \big), \tag{6.2}$$

where E_i is the diagonal $N \times N$ matrix with matrix elements $(E_i)_{jk} = \delta_{ij}\delta_{ik}$ (again, no summation over ν). Summing over i, we get

$$\partial_{t_{\nu,m}} \sum_i x_i = \operatorname*{res}_\infty \big(z^m \mathrm{c}^{*\nu} \mathrm{c}^\nu \big). \tag{6.3}$$

Comparing with Eq. (3.16), we conclude that

$$c^{*\alpha}c^{\alpha} = -\nu_\alpha z^{-2} + \sum_{m\geq 2} z^{-m-1}\partial_{t_{\alpha,m}}\sum_i x_i = -\lambda'_\alpha(z) \tag{6.4}$$

(no summation over α). Substituting $E_i = -\partial_{p_i}L$ in (6.2) and using (4.1), (4.2), and (6.4), we have:

$$\partial_{t_m}x_i = \sum_\nu \operatorname{res}_\infty\big(z^m c^{*\nu}E_i c^\nu\big) = -\sum_\nu \operatorname{res}_\infty\big(z^m c^{*\nu}\partial_{p_i}L(\lambda_\nu)c^\nu\big)$$

$$= -\sum_\nu \operatorname{res}_\infty\big(z^m\partial_{p_i}\big(c^{*\nu}L(\lambda_\nu)c^\nu\big)\big) + \sum_\nu \operatorname{res}_\infty\big(z^m(\partial_{p_i}c^{*\nu})L(\lambda_\nu)c^\nu\big) + \sum_\nu \operatorname{res}_\infty\big(z^m c^{*\nu}L(\lambda_\nu)\partial_{p_i}c^\nu\big)$$

$$= -\sum_\nu \operatorname{res}_\infty\big(z^m\partial_{p_i}\big(c^{*\nu}L(\lambda_\nu)c^\nu\big)\big) + \sum_\nu \operatorname{res}_\infty\big(z^m(\partial_{p_i}c^{*\nu})k_\nu c^\nu\big) + \sum_\nu \operatorname{res}_\infty\big(z^m c^{*\nu}k_\nu\partial_{p_i}c^\nu\big)$$

$$= -\sum_\nu \operatorname{res}_\infty\big(z^m\partial_{p_i}\big(c^{*\nu}k_\nu c^\nu\big)\big) + \sum_\nu \operatorname{res}_\infty\big(z^m k_\nu\partial_{p_i}\big(c^{*\nu}c^\nu\big)\big)$$

$$= \sum_\nu \operatorname{res}_\infty\big(z^m\lambda'_\nu(z)\partial_{p_i}k_\nu\big).$$

Regarding z as an independent variable, we apply the same argument as in Ref. 12 to obtain

$$\partial_{t_m}x_i = -\sum_\nu \operatorname{res}_\infty\big(z^m\partial_{p_i}\lambda_\nu(z)\big). \tag{6.5}$$

In this way, we obtain the first half of the higher Hamiltonian equations for poles

$$\partial_{t_m}x_i = \frac{\partial H_m}{\partial p_i}, \tag{6.6}$$

with the Hamiltonian

$$H_m = \sum_{\alpha=1}^{n}\operatorname{res}_\infty\big(z^m\lambda_\alpha(z)\big). \tag{6.7}$$

The second half of the Hamiltonian equations for poles can be obtained by taking the t_2-derivative of (6.2) and using (4.1) and (4.2). In this way, we obtain

$$\partial_{t_{\nu,m}}\dot{x}_i = \operatorname{res}_\infty\big(z^m c^{*\nu}[E_i, M(\lambda_\nu)]c^\nu\big). \tag{6.8}$$

A straightforward verification shows that

$$[E_i, M(\lambda)] = 2\partial_{x_i}L(\lambda). \tag{6.9}$$

Recalling also that $\dot{x}_i = 2p_i$, we rewrite (6.8) as

$$\partial_{t_{\nu,m}}p_i = \operatorname{res}_\infty\big(z^m c^{*\nu}\partial_{x_i}L(\lambda_\nu)c^\nu\big) \tag{6.10}$$

(no summation over ν). With relation (6.10) at hand, one can repeat the chain of equalities after Eq. (6.4) with the change $\partial_{p_i}\to\partial_{x_i}$ to obtain

$$\partial_{t_m}p_i = \sum_\nu \operatorname{res}_\infty\big(z^m\partial_{x_i}\lambda_\nu(z)\big) \tag{6.11}$$

so that

$$\partial_{t_m}p_i = -\frac{\partial H_m}{\partial x_i} \tag{6.12}$$

with the same Hamiltonian (6.7).

Let us make some comments on a more general case when the tau-function for elliptic solutions has a slightly more general form

$$\tau(x,\mathbf{t}) = e^{Q(x,\mathbf{t})}\prod_{i=1}^{N}\sigma(x - x_i(\mathbf{t})), \tag{6.13}$$

where

$$Q(x,\mathbf{t}) = c(x+t_1)^2 + (x+t_1)\sum_{j\geq 2} a_j t_j + b(t_2,t_3,\ldots) \tag{6.14}$$

with some constants c, a_j and a function $b(t_2,t_3,\ldots)$. Repeating the arguments leading to (6.5), one can see that now the first equation in (3.14) will be modified as

$$k_\beta = z - \alpha(z) + \zeta(\lambda_\beta), \qquad \alpha(z) = 2cz^{-1} + \sum_{j\geq 2}\frac{a_j}{j} z^{-j}. \tag{6.15}$$

Instead of (6.5), we will have

$$\partial_{t_m} x_i = -\sum_\nu \operatorname*{res}_\infty\big(z^m \partial_{p_i}\lambda_\nu(z)(1-\alpha'(z))\big), \tag{6.16}$$

so the Hamiltonian for the m-th flow will be a linear combination of H_m and H_j with $1 \leq j < m$.

VII. DYNAMICS OF SPIN VARIABLES IN THE HIGHER TIMES

The Hamiltonian dynamics of spin variables in the higher times can be derived by analysis of first order poles in (6.1). Equating coefficients in front of first order poles, we get the relation

$$\partial_{t_{\nu,m}}(a_i^\alpha b_i^\beta) = \operatorname*{res}_\infty\left(z^m\sum_{j\neq i} a_i^\alpha c_i^\nu c_j^{*\nu} b_j^\beta \Phi(x_i - x_j, -\lambda_\nu) + z^m \sum_{j\neq i} a_j^\alpha c_j^\nu c_i^{*\nu} b_i^\beta \Phi(x_i - x_j, \lambda_\nu)\right),$$

which can be rewritten as

$$a_i^\alpha\left[\partial_{t_{\nu,m}} b_i^\beta + \operatorname*{res}_\infty\left(z^m c_i^\nu \sum_{j\neq i} c_j^{*\nu} b_j^\beta \Phi(x_j - x_i, \lambda_\nu)\right)\right]$$

$$+ b_i^\beta\left[\partial_{t_{\nu,m}} a_i^\alpha - \operatorname*{res}_\infty\left(z^m c_i^{*\nu}\sum_{j\neq i} c_j^\nu a_j^\alpha \Phi(x_i - x_j, \lambda_\nu)\right)\right] = 0.$$

Now, we note that

$$\begin{aligned}\frac{\partial L_{jk}(\lambda)}{\partial a_i^\alpha} &= -\delta_{ik}(1-\delta_{jk}) b_j^\alpha \Phi(x_j - x_i, \lambda),\\ \frac{\partial L_{jk}(\lambda)}{\partial b_i^\beta} &= -\delta_{ij}(1-\delta_{jk}) a_k^\beta \Phi(x_i - x_k, \lambda),\end{aligned} \tag{7.1}$$

so the equation mentioned above can be written as

$$a_i^\alpha\left[\partial_{t_{\nu,m}} b_i^\beta - \operatorname*{res}_\infty\left(z^m \mathrm{c}^{*\nu}\frac{\partial L(\lambda_\nu)}{\partial a_i^\beta}\mathrm{c}^\nu\right)\right] + b_i^\beta\left[\partial_{t_{\nu,m}} a_i^\alpha + \operatorname*{res}_\infty\left(z^m \mathrm{c}^{*\nu}\frac{\partial L(\lambda_\nu)}{\partial b_i^\alpha}\mathrm{c}^\nu\right)\right] = 0. \tag{7.2}$$

Having this equation at hand, one can repeat the chain of equalities after Eq. (6.4) with the changes $\partial_{p_i} \to \partial/\partial a_i^\beta$, $\partial_{p_i} \to \partial/\partial b_i^\alpha$ to obtain

$$a_i^\alpha P_i^\beta - b_i^\beta Q_i^\alpha = 0, \tag{7.3}$$

where

$$P_i^\beta = -\partial_{t_m} b_i^\beta + \sum_\nu \operatorname*{res}_\infty\left(z^m \frac{\partial}{\partial a_i^\beta}\lambda_\nu(z)\right) = -\partial_{t_m} b_i^\beta - \frac{\partial H_m}{\partial a_i^\beta}, \tag{7.4}$$

$$Q_i^\alpha = \partial_{t_m} a_i^\alpha + \sum_\nu \operatorname*{res}_\infty\left(z^m \frac{\partial}{\partial b_i^\alpha}\lambda_\nu(z)\right) = \partial_{t_m} a_i^\alpha - \frac{\partial H_m}{\partial b_i^\alpha}. \tag{7.5}$$

It follows from (7.3) that

$$\frac{Q_i^\alpha}{a_i^\alpha} = \frac{P_i^\beta}{b_i^\beta} = \Lambda_i^{(m)},$$

and Eqs. (7.4) and (7.5) acquire the form

$$\partial_{t_m} a_i^\alpha = a_i^\alpha \Lambda_i^{(m)} + \frac{\partial H_m}{\partial b_i^\alpha}, \tag{7.6}$$

$$\partial_{t_m} b_i^\beta = -b_i^\beta \Lambda_i^{(m)} - \frac{\partial H_m}{\partial a_i^\beta}. \tag{7.7}$$

The gauge transformation $a_i^\alpha \to a_i^\alpha q_i^{(m)}$, $b_i^\alpha \to b_i^\alpha (q_i^{(m)})^{-1}$ with $q_i^{(m)} = \exp\Big(\int^{t_m} \Lambda_i^{(m)} dt\Big)$ eliminates the terms with $\Lambda_i^{(m)}$, so we can put $\Lambda_i^{(m)} = 0$. We obtain the Hamiltonian equations of motion for spin variables in the higher times,

$$\partial_{t_m} a_i^\alpha = \frac{\partial H_m}{\partial b_i^\alpha}, \qquad \partial_{t_m} b_i^\alpha = -\frac{\partial H_m}{\partial a_i^\alpha}, \tag{7.8}$$

with H_m given by (6.7).

VIII. HOW TO OBTAIN THE FIRST TWO HAMILTONIANS

In order to find the Hamiltonians, we need to expand the spectral curve near $\lambda = 0$. Using expansion (3.8), we represent the equation of the spectral curve as

$$\det\Big(zI + F\lambda^{-1} + Q + S\lambda + O(\lambda^2)\Big) = 0, \tag{8.1}$$

where the matrices Q, S are

$$Q_{ij} = p_i\delta_{ij} + (1-\delta_{ij})F_{ij}\zeta(x_i - x_j), \tag{8.2}$$

$$S_{ij} = \frac{1}{2}(1-\delta_{ij})F_{ij}\Big(\zeta^2(x_i - x_j) - \wp(x_i - x_j)\Big). \tag{8.3}$$

We set

$$z = -\frac{\omega}{\lambda}, \tag{8.4}$$

and then, Eq. (8.1) acquires the form

$$\det\Big(\omega I - F - Q\lambda - S\lambda^2 + O(\lambda^3)\Big) = 0. \tag{8.5}$$

This equation has n roots ω_α such that

$$\omega_\alpha = \omega_\alpha(\lambda) = \nu_\alpha + \omega_1^{(\alpha)}\lambda + \omega_2^{(\alpha)}\lambda^2 + O(\lambda^3) \tag{8.6}$$

and $N - n$ roots, which are $O(\lambda)$. These roots are eigenvalues of the matrix $F + Q\lambda + S\lambda^2 + O(\lambda^3)$. Expressing λ through z from Eq. (8.4) and expanding in powers of z^{-1}, we have

$$\lambda_\alpha = -\frac{\nu_\alpha}{z} + \nu_\alpha\omega_1^{(\alpha)}z^{-2} - (\nu_\alpha^2\omega_2^{(\alpha)} + \nu_\alpha(\omega_1^{(\alpha)})^2)z^{-3} + O(z^{-4}). \tag{8.7}$$

Then,

$$\begin{aligned} H_1 &= -\sum_\alpha \nu_\alpha\omega_1^{(\alpha)}, \\ H_2 &= \sum_\alpha (\nu_\alpha^2\omega_2^{(\alpha)} + \nu_\alpha(\omega_1^{(\alpha)})^2). \end{aligned} \tag{8.8}$$

We regard the matrix $Q\lambda + S\lambda^2$ as a small variation of the matrix F. The idea is to find the variation of the eigenvalues [the corrections $\omega_1^{(\alpha)}\lambda + \omega_2^{(\alpha)}\lambda^2$ in (8.6)] using first two orders of the perturbation theory.

Let $\psi^{(j)}$ be a basis in the N-dimensional space and $\tilde\psi^{(j)}$ be the dual basis such that $(\tilde\psi^{(i)}\psi^{(j)}) = 0$ at $i \neq j$. We take first n vectors to be

$$\psi_i^{(\alpha)} = b_i^\alpha, \qquad \tilde\psi_i^{(\alpha)} = a_i^\alpha,$$

and then,

$$(\tilde\psi^{(\alpha)}\psi^{(\beta)}) = \sum_i a_i^\alpha b_i^\beta = \nu_\alpha \delta_{\alpha\beta}, \quad \alpha,\beta = 1,\dots,n.$$

These vectors are eigenvectors of the (non-perturbed) matrix F with nonzero eigenvalues,

$$F\psi^{(\alpha)} = \nu_\alpha \psi^{(\alpha)}, \quad \tilde\psi^{(\alpha)}F = \nu_\alpha\tilde\psi^{(\alpha)}. \tag{8.9}$$

The other $N-n$ vectors are chosen to be orthonormal,

$$(\tilde\psi^{(i)}\psi^{(j)}) = \delta_{ij}, \quad i,j = n+1,\dots,N.$$

In the first order of the perturbation theory, we have

$$\omega_1^{(\alpha)} = \frac{(\tilde\psi^{(\alpha)}Q\psi^{(\alpha)})}{(\tilde\psi^{(\alpha)}\psi^{(\alpha)})}. \tag{8.10}$$

The next coefficient, $\omega_2^{(\alpha)}$, is obtained in the second order of the perturbation theory as

$$\omega_2^{(\alpha)} = \frac{(\tilde\psi^{(\alpha)}S\psi^{(\alpha)})}{(\tilde\psi^{(\alpha)}\psi^{(\alpha)})} + \sum_{j\neq\alpha}\frac{(\tilde\psi^{(\alpha)}Q\psi^{(j)})(\tilde\psi^{(j)}Q\psi^{(\alpha)})}{(\tilde\psi^{(\alpha)}\psi^{(\alpha)})(\tilde\psi^{(j)}\psi^{(j)})(\nu_\alpha-\nu_j)}. \tag{8.11}$$

In the denominator of the last term, $\nu_j = \nu_\beta$ at $j=\beta$, $\beta = 1,\dots,n$, and $\nu_j = 0$ at $j = n+1,\dots,N$.

Using these formulas, we have

$$\sum_\alpha \nu_\alpha\omega_1^{(\alpha)} = \sum_\alpha(\tilde\psi^{(\alpha)}Q\psi^{(\alpha)}) = \sum_\alpha\sum_{i,j}a_i^\alpha Q_{ij}b_j^\alpha = \sum_{i,j}F_{ji}Q_{ij} = \mathrm{tr}\,(FQ), \tag{8.12}$$

$$\begin{aligned}\sum_\alpha\Big(\nu_\alpha^2\omega_2^{(\alpha)} + \nu_\alpha(\omega_1^{(\alpha)})^2\Big) &= \sum_{\alpha\neq\beta}\frac{\nu_\alpha(\tilde\psi^{(\alpha)}Q\psi^{(\beta)})(\tilde\psi^{(\beta)}Q\psi^{(\alpha)})}{\nu_\beta(\nu_\alpha-\nu_\beta)} + \sum_{j=n+1}^N\sum_\alpha(\tilde\psi^{(\alpha)}Q\psi^{(j)})(\tilde\psi^{(j)}Q\psi^{(\alpha)})\\ &\quad+\sum_\alpha\nu_\alpha^{-1}(\tilde\psi^{(\alpha)}Q\psi^{(\alpha)})^2 + \sum_\alpha\nu_\alpha(\tilde\psi^{(\alpha)}S\psi^{(\alpha)})\\ &= \sum_{\alpha,\beta}\nu_\alpha^{-1}(\tilde\psi^{(\alpha)}Q\psi^{(\beta)})\tilde\psi^{(\beta)}Q\psi^{(\alpha)}) + \sum_{j=n+1}^N\sum_\alpha(\tilde\psi^{(\alpha)}Q\psi^{(j)})(\tilde\psi^{(j)}Q\psi^{(\alpha)}) + \sum_\alpha\nu_\alpha(\tilde\psi^{(\alpha)}S\psi^{(\alpha)})\\ &= \sum_{ijkl}\left(\sum_\alpha\nu_\alpha^{-1}a_i^\alpha b_l^\alpha + \sum_{r=n+1}^N\tilde\psi_i^{(r)}\psi_l^{(r)}\right)Q_{ij}F_{jk}Q_{kl} + \sum_\alpha\nu_\alpha a_i^\alpha S_{ij}b_j^\alpha.\end{aligned}$$

However,

$$\sum_\alpha\nu_\alpha^{-1}a_i^\alpha b_l^\alpha + \sum_{r=n+1}^N\tilde\psi_i^{(r)}\psi_l^{(r)} = \delta_{il}$$

(the completeness relation), and so, finally, we obtain

$$\sum_\alpha\Big(\nu_\alpha^2\omega_2^{(\alpha)} + \nu_\alpha(\omega_1^{(\alpha)})^2\Big) = \mathrm{tr}\,(QFQ) + \mathrm{tr}\,(FSF). \tag{8.13}$$

From (8.12), we obtain

$$H_1 = -\mathrm{tr}\,(FQ) = -\sum_i p_i F_{ii} - \sum_{i\neq j} F_{ij}F_{ji}\zeta(x_i - x_j) = -\sum_i p_i, \tag{8.14}$$

which is indeed the first Hamiltonian. The calculation of (8.13) is more involved. We have, after some cancellations,

$$\mathrm{tr}\,(QFQ) = \sum_i p_i^2 + \sum_{k\neq i}\sum_{j\neq i} F_{ij}F_{jk}F_{ki}\zeta(x_i - x_j)\zeta(x_k - x_i),$$

$$\mathrm{tr}\,(FSF) = \frac{1}{2}\sum_l \sum_{i\neq j} F_{li}F_{ij}F_{jl}\big(\zeta^2(x_i - x_j) - \wp(x_i - x_j)\big).$$

Therefore,

$$H_2 = \sum_i p_i^2 - \sum_{i\neq j} F_{ij}F_{ji}\wp(x_i - x_j) + \mathcal{F}, \tag{8.15}$$

where

$$\mathcal{F} = \sum{}' F_{ij}F_{jk}F_{ki}\zeta(x_i - x_j)\zeta(x_k - x_i) + \frac{1}{2}\sum{}' F_{ij}F_{jk}F_{ki}\big(\zeta^2(x_i - x_j) - \wp(x_i - x_j)\big) = 0. \tag{8.16}$$

Here, $\sum'$ means summation over all distinct indices ijk. The proof of identity (8.16) is given in the Appendix. To conclude, we have reproduced the correct Hamiltonians H_1 and H_2 within our approach.

IX. RATIONAL AND TRIGONOMETRIC LIMITS

In the rational limit $\omega, \omega' \to \infty$, $\sigma(\lambda) = \lambda$, $\Phi(x,\lambda) = (x^{-1} + \lambda^{-1})e^{-x/\lambda}$, and the equation of the spectral curve becomes

$$\det\big(zI - L_{\mathrm{rat}} + \lambda^{-1}F\big) = 0, \tag{9.1}$$

where

$$(L_{\mathrm{rat}})_{ij} = -\delta_{ij}p_i - (1 - \delta_{ij})\frac{b_i^\nu a_j^\nu}{x_i - x_j} \tag{9.2}$$

is the Lax matrix of the spin generalization of the rational Calogero–Moser model. Let us rewrite the equation of the spectral curve in the form

$$\det\Big(\lambda I + F\frac{1}{zI - L_{\mathrm{rat}}}\Big) = 0. \tag{9.3}$$

Expanding the determinant, we have

$$\lambda^N + \sum_{j=1}^{n} D_j(z)\lambda^{N-j} = 0, \qquad D_1(z) = \mathrm{tr}\Big(F\frac{1}{zI - L_{\mathrm{rat}}}\Big), \tag{9.4}$$

where we took into account that the rank of F is equal to $n \le N$. Let us note that the functions $\lambda_\alpha(z)$ are different nonzero roots of Eq. (9.4) and the sum of these roots is equal to $-D_1(z)$. Therefore, we can write

$$H_m = -\sum_\nu \underset{\infty}{\mathrm{res}}\big(z^m\lambda_\nu(z)\big) = \underset{\infty}{\mathrm{res}}\Big(z^m \mathrm{tr}\Big(F\frac{1}{zI - L_{\mathrm{rat}}}\Big)\Big) = \mathrm{tr}\big(FL_{\mathrm{rat}}^m\big). \tag{9.5}$$

It is straightforward to check the commutation relation

$$[X, L_{\mathrm{rat}}] = F - I, \qquad X = \mathrm{diag}\,(x_1, \ldots, x_N). \tag{9.6}$$

Substituting it into (9.5), we see that

$$H_m = \mathrm{tr}L_{\mathrm{rat}}^m. \tag{9.7}$$

This is the result of Ref. 19 obtained there by another method.

We now pass to the trigonometric limit. We choose the period of the trigonometric (or hyperbolic) functions to be $\pi i/\gamma$, where γ is some complex constant (real for hyperbolic functions and purely imaginary for trigonometric functions). The second period tends to infinity. The Weierstrass functions in this limit become

$$\sigma(x) = \gamma^{-1} e^{-\frac{1}{6}\gamma^2 x^2} \sinh(\gamma x), \quad \zeta(x) = \gamma \coth(\gamma x) - \frac{1}{3}\gamma^2 x.$$

The tau-function for trigonometric solutions is[20]

$$\tau = \prod_{i=1}^{N} \left(e^{2\gamma x} - e^{2\gamma x_i}\right), \tag{9.8}$$

so we should consider

$$\tau = \prod_{i=1}^{N} \sigma(x - x_i) e^{\frac{1}{6}\gamma^2 (x-x_i)^2 + \gamma(x+x_i)}. \tag{9.9}$$

Similarly to the KP case,[12] Eq. (3.14) with this choice acquires the form

$$k_\beta = z + \gamma \coth(\gamma \lambda_\beta). \tag{9.10}$$

The trigonometric limit of the function $\Phi(x, \lambda)$ is

$$\Phi(x,\lambda) = \gamma(\coth(\gamma x) + \coth(\gamma\lambda)) e^{-\gamma x \coth(\gamma\lambda)}.$$

For further calculations, it is convenient to pass to the variables

$$w_i = e^{2\gamma x_i} \tag{9.11}$$

and introduce the diagonal matrix $W = \operatorname{diag}(w_1, w_2, \ldots, w_N)$. In this notation, the equation of the spectral curve acquires the form

$$\det\Big(W^{1/2}(zI - (L_{\rm trig} - \gamma I))W^{-1/2} + \gamma(\coth(\gamma\lambda) - 1)F\Big) = 0, \tag{9.12}$$

where $L_{\rm trig}$ is the Lax matrix of the spin Calogero–Moser model with matrix elements

$$(L_{\rm trig})_{ij} = -p_i \delta_{ij} - \frac{(1-\delta_{ij})\gamma F_{ij}}{\sinh(\gamma(x_i - x_j))} = -p_i\delta_{ij} - 2(1-\delta_{ij}) \frac{\gamma w_i^{1/2} w_j^{1/2} F_{ij}}{w_i - w_j}. \tag{9.13}$$

Some simple transformations allow one to bring the equation of the spectral curve to the form

$$\det\left(\omega I + 2\gamma W^{-1/2} F W^{1/2} \frac{1}{zI - (L_{\rm trig} - \gamma I)}\right) = 0, \qquad \omega = e^{2\gamma\lambda} - 1. \tag{9.14}$$

Expanding the determinant, we have

$$\omega^N + \sum_{j=1}^{n} K_j(z)\omega^{N-j} = 0, \tag{9.15}$$

where we took into account that the rank of F is equal to $n \le N$. In particular,

$$K_1 = \operatorname{tr} Y, \quad K_2 = \frac{1}{2}(\operatorname{tr}^2 Y - \operatorname{tr} Y^2),$$

where Y is the matrix

$$Y = 2\gamma W^{-1/2} F W^{1/2} \frac{1}{zI - (L_{\rm trig} - \gamma I)}.$$

The coefficients K_j are expressed through the elementary symmetric polynomials $e_j = e_j(\omega_1, \ldots, \omega_n)$ of nonzero roots $\omega_\nu = \omega_\nu(z)$ of this equation as $K_j = (-1)^j e_j(\omega_1, \ldots, \omega_n)$. Therefore,

$$\sum_\nu \lambda_\nu(z) = \frac{1}{2\gamma} \sum_\nu \log(1 + \omega_\nu(z)) = \frac{1}{2\gamma} \log \prod_\nu (1 + \omega_\nu(z))$$
$$= \frac{1}{2\gamma} \log\left(\sum_{j=0}^n e_j(\omega_1, \ldots, \omega_n) \right) = \frac{1}{2\gamma} \log\left(\sum_{j=0}^n (-1)^j K_j \right).$$

From this, we conclude that

$$\sum_\nu \lambda_\nu(z) = \frac{1}{2\gamma} \log \det\left[I - 2\gamma W^{-1/2} F W^{1/2} \frac{1}{zI - (L_{\rm trig} - \gamma I)} \right]. \tag{9.16}$$

Starting from this point, one can literally repeat the corresponding calculation from Ref. 12 with the change in the rank 1 matrix E to the rank n matrix F and using the easily proved relation

$$[L_{\rm trig}, W] = 2\gamma(W^{1/2} F W^{1/2} - W). \tag{9.17}$$

The result is

$$\begin{aligned}\sum_\nu \lambda_\nu(z) &= \frac{1}{2\gamma} \mathrm{tr}\Big(\log(I - z^{-1}(L_{\rm trig} + \gamma I)) - \log(I - z^{-1}(L_{\rm trig} - \gamma I)) \Big) \\ &= -\frac{1}{2\gamma} \mathrm{tr} \sum_{m \ge 1} \frac{z^{-m}}{m} \Big((L_{\rm trig} + \gamma I)^m - (L_{\rm trig} - \gamma I)^m \Big)\end{aligned} \tag{9.18}$$

and

$$H_m = \frac{1}{2\gamma(m+1)} \mathrm{tr}\Big((L_{\rm trig} + \gamma I)^{m+1} - (L_{\rm trig} - \gamma I)^{m-1} \Big), \tag{9.19}$$

which agrees with the result of Ref. 20.

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APPENDIX: PROOF OF IDENTITY (8.16)

Here, we prove identity (8.16) $\mathcal{F} = 0$, where

$$\mathcal{F} = \sum{}' F_{ij} F_{jk} F_{ki} \zeta(x_i - x_j)\zeta(x_k - x_i) + \frac{1}{2} \sum{}' F_{ij} F_{jk} F_{ki} \big(\zeta^2(x_i - x_j) - \wp(x_i - x_j) \big),$$

and $\sum'$ means summation over all distinct indices ijk. Using the behavior of the ζ-function under shifts by periods

$$\zeta(x + 2\omega) = \zeta(x) + 2\eta, \quad \zeta(x + 2\omega') = \zeta(x) + 2\eta',$$

one can see that $\mathcal{F}$ is a double-periodic function of any of x_i. Consider, for example, the shift of x_1 by 2ω. The terms in $\mathcal{F}(x_1 + 2\omega) - \mathcal{F}(x_1)$ proportional to η^2 are

$$-(2\eta)^2 \sum_{j \ne k \ne 1} F_{1j} F_{jk} F_{k1} + \frac{1}{2} (2\eta)^2 \sum_{j \ne k \ne 1} F_{1j} F_{jk} F_{k1} + \frac{1}{2} (2\eta)^2 \sum_{i \ne k \ne 1} F_{1k} F_{ki} F_{i1} = 0.$$

The terms proportional to η are

$$-2\eta\sum_{j\neq k\neq 1}F_{1j}F_{jk}F_{k1}\zeta(x_1-x_k)-2\eta\sum_{j\neq k\neq 1}F_{1j}F_{jk}F_{k1}\zeta(x_1-x_j)$$

$$+2\eta\sum_{j\neq k\neq 1}F_{j1}F_{1k}F_{kj}\zeta(x_j-x_k)+2\eta\sum_{j\neq k\neq 1}F_{j1}F_{1k}F_{kj}\zeta(x_k-x_j)$$

$$+2\eta\sum_{j\neq k\neq 1}F_{k1}F_{1j}F_{jk}\zeta(x_1-x_j)+2\eta\sum_{j\neq k\neq 1}F_{j1}F_{1k}F_{kj}\zeta(x_1-x_j)=0.$$

Therefore, we see that $\mathcal{F}(x_1+2\omega)=\mathcal{F}(x_1)$. The double-periodicity in all other arguments is established in the same way.

Next, the function $\mathcal{F}$ as a function of x_1 may have poles only at the points x_i, $i=2,\ldots,N$. The second order poles cancel identically in the obvious way. We find the residue at the simple pole at $x_1=x_2$ as follows:

$$-\sum_{k\neq 1,2}F_{12}F_{2k}F_{k1}\zeta(x_1-x_k)-\sum_{j\neq 1,2}F_{1j}F_{j2}F_{21}\zeta(x_1-x_j)$$

$$+\sum_{k\neq 1,2}F_{21}F_{1k}F_{k2}\zeta(x_1-x_k)+\sum_{j\neq 1,2}F_{2j}F_{j1}F_{12}\zeta(x_1-x_j)=0.$$

Vanishing of the residues in all other points and for all other variables can be proved in the same way. We see that the function $\mathcal{F}$ is a regular elliptic function, and therefore, it must be a constant. To find this constant, we set $x_j=j\varepsilon$ and tend ε to 0. Thanks to the fact that $\zeta(x)=x^{-1}+O(x^3)$, $\wp(x)=x^{-2}+O(x^2)$ as $x\to 0$, we find that

$$\mathcal{F}=\frac{1}{\varepsilon^2}\sum{}'\frac{F_{ij}F_{jk}F_{ki}}{(i-j)(k-i)}+O(\varepsilon^2).$$

Making the cyclic changes in the summation variables $(ijk)\to(jki)$ and $(ijk)\to(kij)$, we have

$$\mathcal{F}=\frac{1}{3\varepsilon^2}\sum{}'F_{ij}F_{jk}F_{ki}\left(\frac{1}{(i-j)(k-i)}+\frac{1}{(j-k)(i-j)}+\frac{1}{(k-i)(j-k)}\right)+O(\varepsilon^2)$$

$$=\frac{1}{3\varepsilon^2}\sum{}'F_{ij}F_{jk}F_{ki}\frac{(j-k)+(k-i)+(i-j)}{(i-j)(j-k)(k-i)}+O(\varepsilon^2)=O(\varepsilon^2).$$

Therefore, we conclude that $\mathcal{F}=0$ and the identity (8.16) is proved.

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

REFERENCES

[1] H. Airault, H. P. McKean, and J. Moser, "Rational and elliptic solutions of the Korteweg–De Vries equation and a related many-body problem," Commun. Pure Appl. Math. **30**, 95–148 (1977).
[2] I. M. Krichever, "Rational solutions of the Kadomtsev–Petviashvili equation and integrable systems of N particles on a line," Funct. Anal. Appl. **12**(1), 59–61 (1978).
[3] I. M. Krichever, "Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles," Funk. Anal. Ego Prilozh. **14**(4), 45–54 (1980) (in Russian) [English translation: Funct. Anal. Appl. 14(4), 282–290 (1980)].
[4] D. V. Chudnovsky and G. V. Chudnovsky, "Pole expansions of non-linear partial differential equations," Nuovo Cimento B **40**, 339–350 (1977).
[5] F. Calogero, "Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials," J. Math. Phys. **12**, 419–436 (1971).
[6] F. Calogero, "Exactly solvable one-dimensional many-body systems," Lett. Nuovo Cimento **13**, 411–415 (1975).
[7] J. Moser, "Three integrable Hamiltonian systems connected with isospectral deformations," Adv. Math. **16**, 197–220 (1975).
[8] M. A. Olshanetsky and A. M. Perelomov, "Classical integrable finite-dimensional systems related to Lie algebras," Phys. Rep. **71**, 313–400 (1981).
[9] T. Shiota, "Calogero–Moser hierarchy and KP hierarchy," J. Math. Phys. **35**, 5844–5849 (1994).
[10] L. Haine, "KP trigonometric solitons and an adelic flag manifold," SIGMA **3**, 015 (2007).
[11] A. Zabrodin, "KP hierarchy and trigonometric Calogero–Moser hierarchy," J. Math. Phys. **61**, 043502 (2020); arXiv:1906.09846.
[12] V. Prokofev and A. Zabrodin, "Elliptic solutions to the KP hierarchy and elliptic Calogero–Moser model," arXiv:2102.03784.
[13] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, "Transformation groups for soliton equations III," J. Phys. Soc. Jpn. **50**, 3806–3812 (1981).
[14] V. Kac and J. van de Leur, "The *n*-component KP hierarchy and representation theory," in *Important Developments in Soliton Theory*, edited by A. S. Fokas and V. E. Zakharov (Springer-Verlag, Berlin, Heidelberg, 1993).

[15] K. Takasaki and T. Takebe, "Integrable hierarchies and dispersionless limit," Physica D **235**, 109–125 (2007).
[16] L. P. Teo, "The multicomponent KP hierarchy: Differential fay identities and Lax equations," J. Phys. A: Math. Theor. **44**, 225201 (2011).
[17] I. Krichever, O. Babelon, E. Billey, and M. Talon, "Spin generalization of the Calogero–Moser system and the matrix KP equation," Am. Math. Soc. Trans. Ser. **2**(170), 83–119 (1995).
[18] J. Gibbons and T. Hermsen, "A generalization of the Calogero–Moser system," Physica D **11**, 337–348 (1984).
[19] V. Pashkov and A. Zabrodin, "Spin generalization of the Calogero–Moser hierarchy and the matrix KP hierarchy," J. Phys. A: Math. Theor. **51**, 215201 (2018).
[20] V. Prokofev and A. Zabrodin, "Matrix KP hierarchy and spin generalization of trigonometric Calogero–Moser hierarchy," Trudy MIAN **309**, 241–256 (2020) [English translation: Proc. Steklov Inst. Math. **309**, 225–239 (2020)]; arXiv:1910.00434.